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ON A COMMON FIXED POINT FOR SEQUENCE OF SELFMAPPINGS IN GENERALIZED METRIC SPACE

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Abstract. We prove the existence and uniqueness of a common fixed point for a sequence of mappings on generalized metric space with a contractive condition.

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1. Introduction

Let $\{f_n\}$ be a sequence of selfmappings on a metric space.

As is known, there are three types of theorems for sequences of mappings. The first assumes that each pair f_i, f_j satisfies the same contractive condition, and concludes that $\{f_n\}$ has a common fixed point. The second assumes that each f_n satisfies the same contractive condition and that $\{f_n\}$ tends pointwise to a limit function f. The conclusion is that f has a fixed point z which is the limit of each of the fixed points z_n of f_n. The third type assumes that each f_n has a fixed point z_n, and that $\{f_n\}$ converges uniformly to a function f which satisfies a particular contractive condition. With z, the fixed point of f, the conclusion is that $z_n \to z$.

In this paper we are going to prove a fixed point results of first type in a generalized metric space - so called $D-$metric space.

Let us recall some basic definitions, examplars and properties of $D-$metric spaces.

In 1992, a new structure of a generalized metric space, so called $D-$metric space was introduced by B.C. Dhage [2] on the lines of the ordinary metric space. Also, some fixed point theorems for the contractive mappings in $D-$metric space

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are proved. In last fifteen years there have been many fixed point results in $D-$metric space (see [1], [4], [5], [6], [8], [9]). In this paper we are going to prove a common fixed point theorem for a sequence of selfmappings defined on $D-$metric space.

Definition [2]. *Let X denote a non-empty set and $\mathbb{R}^+$ the set of all nonnegative real numbers. Then X, together with a function $D : X \times X \times X \to \mathbb{R}^+$, is called a $D-$metric space if it satisfies the following properties:*

(i) $D(x, y, z) = 0 \Leftrightarrow x = y = z$ *(coincidence),*

(ii) $D(x, y, z) = D(p(x, y, z))$ *(symmetry),*

(p denotes the permutation function),

(iii) $D(x, y, z) \le D(x, y, a) + D(x, a, z) + D(a, y, z)$

for $x, y, z, a \in X$ (tetrahedral inequality).

A sequence $\{x_n\} \subset X$ is said to be $D-$convergent and converges to a point x if $\lim\limits_{m,n} D(x_m, x_n, x) = 0$. A sequence $\{x_n\} \subset X$ is called $D-$Cauchy if $\lim\limits_{m,n,p} D(x_m, x_n, x_p) = 0$. A complete $D-$metric space is one in which every $D-$Cauchy sequence converges to a point in it. A set $S \subset X$ is said to be bounded if there exists a constant $M > 0$ such that $D(x, y, z) \le M$ for all $x, y, z \in S$ and the constant M is called a $D-$bound of S.

In a $D-$metric space, if D is continuous in two variables, then the limit of a sequence is unique, if it exists. Throughout this paper the $D-$metric is assumed to be continuous in two variables.

Example [2] *Let (X, d) be a metric space. Define a function $D : X \times X \times X \to [0, \infty)$ by*

$$D(x, y, z) = d(x, y) + d(y, z) + d(z, x)$$

for $x, y, z \in X$.

Clearly, the function D is a $D-$metric on X and consequently (X, D) is a $D-$metric space.

Remark. Thus for every ordinary metric space (X, d) there exists a $D-$metric space (X, D), but the converse may not be true. Therefore $D-$metric spaces are the generalizations of ordinary metric spaces.

The more powerful tool in our considerations is $D-$Cauchy Principle.

Lemma 1.1.[3] *($D-$Cauchy Principle) Let $\{x_n\} \subseteq X$ be a bounded sequence* *with $D-$bound M satisfying*

$$D(x_n, x_{n+1}, x_m) \leq \alpha^n \cdot M$$

for all $m > n \in \mathbb{N}$ and $0 \leq \alpha < 1$. Then $\{x_n\}$ is $D-$Cauchy.

2. Main result

Theorem 2.1. *Let (X, D) be a complete $D-$metric space $f_n : X \to X,\ n \in \mathbb{N}$, be a sequence of mappings with property that for each $x, y, z \in X$ and any $i, j, k \in \mathbb{N} \backslash \triangle,\ \triangle = \{(n, n, n) | n \in \mathbb{N}\}$,*

(1) $$D(f_i(x), f_j(y), f_k(z)) \leq q \cdot D(x, y, z)$$

for some $q < 1$.

If there exists $x_0 \in X$ such that $\sup\limits_{y \in X} D(x_0, f_1(x_0), y) = M$, for some $M > 0$, then there exists a unique common fixed point for the family $\{f_n\}$.

Proof. For $x_0 \in X$ define a sequence

$$x_n = f_n(x_{n-1}), \quad n \in \mathbb{N}.$$

Let us prove that $\{x_n\}$ is a $D-$Cauchy sequence.

For any $n, p \in \mathbb{N}$

$$D(x_n, x_{n+1}, x_{n+p}) = D(f_n(x_{n-1}), f_{n+1}(x_n), f_{n+p}(x_{n+p-1}))$$
$$\leq q \cdot D(x_{n-1}, x_n, x_{n+p-1}) = q \cdot D(f_{n-1}(x_{n-2}), f_n(x_{n-1}), f_{n+p-1}(x_{n+p-2}))$$
$$\leq q^2 \cdot D(x_{n-2}, x_{n-1}, x_{n+p-2}) \leq \cdots \leq q^n \cdot D(x_0, x_1, x_p)$$
$$\leq q^n \cdot \sup_{y \in X} D(x_0, f_1(x_0), y) = q^n \cdot M.$$

So conditions of Lemma 1.1 are satisfied and $\{x_n\}$ is $D-$Cauchy. Since X is complete, there exists $z \in Z$ such that $z = \lim\limits_n x_n$.

We are going to prove that z is the unique fixed point for the sequence $\{f_n\}$.

Fixed $k \in \mathbb{N}$. For any $m \in \mathbb{N},\ m > k$,

$$\begin{aligned} D(x_m, f_k(z), f_k(z)) &= D(f_m(x_{m-1}), f_k(z), f_k(z)) \\ &\leq q \cdot D(x_{m-1}, z, z). \end{aligned}$$

Since D is continuous in two variables, it follows that

$$D(z, f_k(z), f_k(z)) \leq q \cdot D(z, z, z) = 0.$$

Consequently, $z = f_k(z)$.

If we suppose that for some $y \in X$ $f_k(y) = y$, for all $k \in \mathbb{N}$, as $q < 1$ and

$D(z,z,y) = D(f_k(z), f_{k+1}(z), f_{k+2}(y)) \leq q \cdot D(z,z,y)$

it follows that $z = y$. So the uniqueness is proved and the proof is completed.

Corollary 2.1 *Let* (X, D) *be a complete bounded* $D-$*metric space* $f_n : X \to X$, $n \in \mathbb{N}$, *be a sequence of mappings with the property that for some* $m \in \mathbb{N}$, *each* $x, y, z \in X$ *and any* $i, j, k \in \mathbb{N} \setminus \triangle$, $\triangle = \{(n,n,n) | n \in \mathbb{N}\}$,

$$(2) \qquad D(f_i^m(x), f_j^m(y), f_k^m(z)) \leq q \cdot D(x,y,z)$$

for some $q < 1$.

Then there exists a unique common fixed point for the family $\{f_n\}$.

Proof. Theorem 2.1 implies that there exists the unique common fixed point for the sequence $\{f_k^m\}$. But, the fixed point for f_k^m by uniqueness is a fixed point for f_k, so, the proof is completed.

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