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ON THE FOURTH ORDER ROOT FINDING METHODS OF EULER'S TYPE[1]

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Abstract. A new iterative method of the fourth order for solving nonlinear equations of the form $f(x) = 0$ is derived. The comparison with other existing methods is performed regarding the computational efficiency and numerical examples. The fourth-order derivative free method for the simultaneous calculation of all polynomial zeros, arising from the proposed method, is also studied.

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Let f be a real single-valued function of a real variable, possessing a certain number of continuous derivatives in the neighborhood $\Lambda(\zeta)$ of a real simple zero ζ. Assuming that f' does not vanish in $\Lambda(\zeta)$, let us define the Newton correction

$$u = u(x) = \frac{f(x)}{f'(x)}$$

appearing in the well-known Newton's method

$$\hat{x} = x - \frac{f(x)}{f'(x)} \tag{1}$$

for approximating the zero ζ. The quantity $\hat{x}$ is a new approximation to ζ which, under suitable convergence conditions, is closer to ζ than the previous approximation x. The same notation will be universally used for some other iterative methods.

Let d be the number of new function evaluations (the values of f and its derivatives) per iteration applying an iterative method IM, and let r be the order of convergence of IM. Ostrowski [3, p. 20] introduced the notion of *computational efficiency* of IM by

$$E(\mathrm{IM}) = r^{1/d}. \tag{2}$$

For example, the computational efficiency of Newton's method (1) is $E(1) = 2^{1/2} \cong 1.414$.

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The aim of this paper is to construct a new iterative method for solving nonlinear equations, possessing a higher computational efficiency than Newton's method.

Let $x \in \Lambda(\zeta)$. Using Taylor's development we obtain

$$0 = f(\zeta) = f(x) + f'(x)(\zeta - x) + \frac{f''(x)}{2}(\zeta - x)^2 + \cdots \tag{3}$$

and

$$\begin{aligned} f(x-u) &= f(x) - f'(x)u + (-1)^k \sum_{k=2}^{\infty} \frac{f^{(k)}(x)u^k}{k!} \\ &= (-1)^k \sum_{k=2}^{\infty} \frac{f^{(k)}(x)u^k}{k!}. \end{aligned} \tag{4}$$

If we neglect higher terms in the development (3), then the zero ζ should be replaced by an approximative value $\hat{x}$. In this way we obtain the quadratic equation in $\hat{x} - x$:

$$f(x) + f'(x)(\hat{x} - x) + \frac{f(x - u(x))}{u(x)^2}(\hat{x} - x)^2 = 0$$

or

$$(x - \hat{x})^2 \frac{f(x - u(x))}{u(x)f(x)} - (x - \hat{x}) + u(x) = 0$$

Solving the last quadratic equation in $\hat{x} - x$, after the rationalization of the nominator one obtains a new iterative root finding method

$$\hat{x} = x - \frac{2u(x)}{1 \pm \sqrt{1 - \dfrac{4f(x - u(x))}{f(x)}}} =: \phi(x), \tag{5}$$

where ϕ is the iterative function.

The iterative function ϕ defines a two-step (or predictor-corrector) method; first one calculates Newton's correction $u(x) = \frac{f(x)}{f'(x)}$ and then Newton's approximation $x - u(x)$, which appears in the second step as the argument of f.

Remark 1. We take the sign in front of the square root in (5) so that the denominator is greater in magnitude. If the current approximation x is reasonably close to the wanted zero ζ, it can be shown that the sign + should be chosen (see Henrici [2, p. 532]).

To find the order of convergence of the iterative method (5), we first recall classical Schröder's theorem [9].

Theorem 1. *The order of convergence of an iterative method defined by its iterative function ψ is r if and only if*

$$\psi(\zeta) = \zeta; \quad \psi^{(k)}(\zeta) = 0, \quad (k = 1, 2, \ldots, r-1); \quad \psi^{(r)}(\zeta) \neq 0.$$

Then the asymptotic error constant is $C_r(\zeta) = \dfrac{\psi^{(r)}(\zeta)}{r!}$.

Theorem 2. *If x is sufficiently close to the zero ζ of f, then the order of convergence of the iterative method* (5) *is four.*

Proof. Since we consider reasonably good approximations to the zero ζ, following Remark 1 we will take the sign $+$ in front of the square in (5). Truncating the series (4) we obtain

$$\frac{f(x-u)}{f(x)} = \frac{f''(x)f(x)}{2!f'(x)^2} - \frac{f'''(x)f(x)^2}{3!f'(x)^3} + \frac{f^{(4)}(x)f(x)^3}{4!f'(x)^3} + O(|f(x)|^4) \tag{6}$$

so that

$$\phi(x) = x - \frac{2u(x)}{1 + \sqrt{1 - 4\Big(\frac{f''(x)f(x)}{2!f'(x)^2} - \frac{f'''(x)f(x)^2}{3!f'(x)^3} + \frac{f^{(4)}(x)f(x)^3}{4!f'(x)^3}\Big) + O(|f(x)|^4)}}.$$

From the last relation we immediately obtain $\phi(\zeta) = \zeta$. The derivatives of $\phi(x)$ are rather complicated and we found them using symbolic computation in the programming package *Mathematica* 5. Since the term $O(|f(x)|^4)$ is very small and does not influence the order of convergence > 4, we omit it in this procedure. Substituting $x = \zeta$ in the expressions for the derivatives $\phi^{(k)}(x)$ we obtain

$$\phi'(\zeta) = 0, \quad \phi''(\zeta) = 0, \quad \phi'''(\zeta) = 0, \quad \phi^{(4)}(\zeta) = -\frac{2f''(\zeta)f'''(\zeta)}{f'(\zeta)^2} \neq 0.$$

Therefore, the order of convergence of the iterative method (5) is four, and the asymptotic error constant is

$$C_4(\zeta) = \frac{\phi^{(4)}(\zeta)}{4!} = -\frac{f''(\zeta)f'''(\zeta)}{12f'(\zeta)^2}. \ \square$$

Remark 2. From (6) we find $f(x-u)/f(x) \approx f''(x)f(x)/(2f'(x)^2)$. Substituting this approximation in (5), we obtain classical third-order Euler's method

$$\hat{x} = x - \frac{2f(x)}{f'(x) \pm \sqrt{f'(x)^2 - \dfrac{2f''(x)f(x)}{f'(x)^2}}}.$$

For this resemblance, the iterative method (5) can be regarded as the method of Euler's type. Let us note that the convergence speed of Euler's method is decreased in relation to (5).

For the comparison purpose, we present another two predictor-corrector methods. First of them is a combination of Newton's and secant method and has the form

$$\hat{x} = x - \frac{u(x)f(x)}{f(x) - f(x-u(x))} \tag{7}$$

(see Traub [10, p. 180]). It requires three function evaluations, the same as the new method (5), but its order of convergence is only three. More efficient method with the same number of function evaluations and the convergence order four is Ostrowski's method

$$\hat{x} = x - u(x)\left(1 + \frac{f(x-u(x))}{f(x) - 2f(x-u(x))}\right), \tag{8}$$

proposed in [3, p. 253]. Another two-point methods can be found in Traub's book [10] and in the recent paper [6].

Using (2) we found that the computational efficiency of the new method (5) is $E(5) = 4^{1/3} \cong 1.584$. Since $E(7) = 3^{1/3} \cong 1.442$, we have the following chain of inequalities

$$E(5) = E(8) \cong 1.584 > E(7) \cong 1.442 > E(1) \cong 1.414.$$

Example 1. We tested the iterative methods (1), (5), (7) and (8) employing a hundred algebraic and transcendental functions of various types, many of them taken from the respective books and papers. For demonstration, we select the following four examples:

$$f_1(x) = \arctan x,\ x_0 = 2.3,$$

$$f_2(x) = \tfrac{1}{2}\log(x^2+1) - \frac{1}{x}\sin 100x,\ x_0 = 1.6,$$

$$f_3(x) = (x^{15}+1)e^{x^2-1},\ x_0 = 1.7,$$

$$f_4(x) = x^{10} - 4x^9 + 5x^8 - x^2 + 4x - 5,\ x_0 = 4.$$

The listed examples were tested by the programming package *Mathematica* 5 in multiple-precision arithmetic. The stopping criterion was given by $|f(x_k)| < 10^{-14}$ for every k. The results are given in Table 1, where the notation $(-h)_k$ means that the accuracy $\max|f(x_k)| = O(10^{-h})$ is obtained after k iterations.

Let ζ_k be the zero of the function f_k. Most methods converge to $\zeta_1 = 0$, $\zeta_2 = 1.51937...$, $\zeta_3 = -1$, $\zeta_4 = 1$. However, there are two exceptions stressed in Table 1 by the comment $^{c)}$ and listed below Table 1.

A hundred experiments, including the four examples displayed in Table 1, demonstrated that the new method (5) is competitive with existing root solvers, and even superior in a number of examples. The convergence rate of the method (5) given in Theorem 2 is fully confirmed by numerical examples when the initial approximation is reasonably close to the sought zero.

Examples →	f_1	f_2	f_3	f_4
Methods ↓	$x_0 = 2.3$	$x_0 = 1.6$	$x_0 = 1.7$	$x_0 = 4$
(1)	*div*	$(-17)_8$	$(-22)_{39}$	$(-17)_{16}$
(5)	$(-20)_5$	$(-45)_4$	$(-37)_{10}$ [1)]	$(-46)_8$ [2)]
(7)	$(-36)_4$	$(-17)_5$	*div*	$(-15)_{10}$
(8)	*div*	$(-15)_4$	$(-53)_{63}$	$(-15)_7$

[1)] $\zeta = 0.97814... + i\,0.20791...$ [2)] $\zeta = 2 + i$

Table 1 Results of numerical experiments; the term *div* points to the divergence.

In general, root finding methods with the square root structure often find a complex zero of a real function having conjugate complex zeros, especially in solving algebraic equations. In that case, some mathematical computation systems, for instance *Mathematica*, automatically continues to run in complex arithmetic. A similar situation occurs when we search for a real zero and face a negative entry under the square root. In most cases the square root methods, including the method (5), produce after a certain number of iterative steps an approximation $\alpha + i\beta$ (in complex arithmetic) to a real zero ζ overcoming the difficulty. Namely, it turns out that $\alpha \approx \zeta$ (to the wanted accuracy) and β is a very small number in magnitude ("parasite" part) that should be rejected.

In recent papers (see, e.g., [4], [5], [8]) it was shown how to construct iterative methods for the simultaneous determination of all zeros of a polynomial starting from methods for finding a single zero of a nonlinear equation $f(x) = 0$. The following question arises: Can we derive some simultaneous method based on the iterative method (5)?

In the subsequent discussion we will consider the construction of an iterative method for the simultaneous determination of all simple (real or complex) zeros of algebraic polynomials, derived from the method (5). Let P be a monic polynomial

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0, \quad (a_i \in \mathbb{C})$$

with simple real or complex zeros $\zeta_1, \ldots, \zeta_n$, and let $z_1, \ldots, z_n$ be n pairwise distinct approximations to these zeros. Introducing

$$W_i = W_i(z_1, \ldots, z_n) = \frac{P(z_i)}{\prod\limits_{j \neq i}(z_i - z_j)}, \quad (i \in I_n := \{1, \ldots, n\})$$

and applying Lagrange's interpolation formula to the polynomial

$$P(z) - \prod_{j=1}^{n}(z - z_j)$$

of degree $n-1$, one obtains for any $z \in \mathbb{C}$,

$$P(z) = \prod_{j=1}^{n} (z - z_j) + \sum_{k=1}^{n} W_k \prod_{\substack{j=1 \\ j \neq k}}^{n} (z - z_j). \tag{9}$$

In this paper we will use the following abbreviations

$$G_{k,i} = \sum_{j \neq i} \frac{W_j}{(z_i - z_j)^k} \quad (k = 1, 2), \quad B_i = \frac{W_i}{1 + G_{1,i}}.$$

Let us define the rational function $z \mapsto h_i(z)$ $(i \in I_n)$ by

$$h_i(z) := W_i(z_1, \dots, z_{i-1}, z, z_{i+1}, \dots, z_n) = \frac{P(z)}{\prod_{j \neq i} (z - z_j)}.$$

Then from (9) we obtain

$$h_i(z) = W_i + (z - z_i)\Big(1 + \sum_{j \neq i} \frac{W_j}{z - z_j}\Big). \tag{10}$$

For any $z \in \mathbb{C} \setminus \{z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n\}$ we obtain

$$h_i'(z) = 1 + \sum_{j \neq i} W_j \frac{z_i - z_j}{(z - z_j)^2}, \qquad h_i''(z) = -2 \sum_{j \neq i} W_j \frac{z_i - z_j}{(z - z_j)^3}. \tag{11}$$

From (10) and (11) we evaluate at the point $z = z_i$:

$$h_i(z_i) = W_i, \quad h_i'(z_i) = 1 + G_{1,i}, \quad h_i''(z_i) = -2G_{2,i}. \tag{12}$$

Let $u(z) = P(z)/P'(z)$. For a fixed z_i the iterative formula (5) reads

$$\hat{z}_i = z_i - \frac{2u(z_i)}{1 \pm \sqrt{1 - \dfrac{4P(z_i - u(z_i))}{P(z_i)}}}. \quad (i \in I_n) \tag{13}$$

According to the definition of the function $h_i(z)$, we note that $h_i(z)$ and the polynomial P have the same zeros. Let us put $h_i(z)$ instead of P in (13), then $u(z_i)$ is replaced by $h_i/h_i' = B_i$. In this way (13) becomes

$$\hat{z}_i = z_i - \frac{2B_i}{1 \pm \sqrt{1 - \dfrac{4h_i(z_i - B_i)}{h_i}}}. \tag{14}$$

Using Taylor's development and (12), we find

$$\frac{h_i(z_i - B_i)}{h_i(z_i)} \cong \frac{h_i(z_i) - h_i'(z_i)B_i + \frac{1}{2}h_i''(z_i)B_i^2}{h_i(z_i)} = -\frac{W_i G_{2,i}}{\big(1 + G_{1,i}\big)^2}.$$

Substituting this in (14) we obtain the following iterative method for the simultaneous determination of polynomial zeros

$$\hat{z}_i = z_i - \frac{2W_i}{1 + G_{1,i} \pm \sqrt{(1+G_{1,i})^2 + 4W_i G_{2,i}}} \quad (i \in I_n). \tag{15}$$

The iterative formula (15) was derived [7] starting from the Euler's method. Its order of convergence is four. If the approximations $z_1, \ldots, z_n$ are reasonably close to the zeros $\zeta_1, \ldots, \zeta_n$, then the sign $+$ is to be chosen in (15), see [7].

Example 2. The simultaneous method (15) was applied for the determination of all zeros of the polynomial equation $f_4(x) = 0$ given above. The exact zeros of this polynomial are $\pm 1,\ \pm i,\ 2 \pm i,\ \pm\sqrt{2}/2 \pm i\sqrt{2}/2$. As initial approximations we have taken $n = 10$ complex numbers equidistantly spaced on the circle with radius R, that is

$$z_m^{(0)} = R\exp(i\theta_m),\quad i = \sqrt{-1},\quad \theta_m = \frac{\pi}{n}\Big(2m - \frac{3}{2}\Big) \quad (m = 1, \ldots, 10)$$

(see Aberth [1]). We have experimented with various values of R to demonstrate good convergence behavior of the simultaneous method (15). In practice, we take the radius given by

$$R = 2 \max_{1 \le \lambda \le n} |a_\lambda|^{1/\lambda} \tag{16}$$

following Henrici's well-known result [2, Corollary 6.4k] that all zeros of the monic polynomial $P(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$ lie inside the disk centered at the origin, with the radius R given by (16).

R	100	50	20	8	4	2
$\max_{1 \le i \le 10} \lvert f_4(z_i^{(k)}) \rvert$	$(-37)_{21}$	$(-16)_{17}$	$(-26)_{15}$	$(-27)_{11}$	$(-19)_8$	$(-14)_5$

Table 2

The maximal values of the polynomial evaluated at the approximations obtained by the simultaneous process (15) for different initial approximations.

The results of our numerical experiment are given in Table 2. Similarly as in Table 1, the superscript index k in $(-h)_k$ indicates the number of iterations necessary to satisfy the stopping criterion

$$\max_{1 \le i \le 10} |f_4(z_i^{(k)})| \Big(= O(10^{-h}\Big) < 10^{-14}.$$

From Table 2 we conclude that the method (15) converged for all initial approximations, some of them being very far from the exact zeros. This is a significant advantage since user is not obliged to take care about the choice of initial approximations, which is one of the most difficult problems in solving nonlinear equations.

Fig. 1 Trajectories of approximations generated by the method (15).

The convergence of initial approximations equidistantly spaced on the circle $\{z : |z| = 8\}$ (found by (16)) is shown graphically in Fig. 1, where the exact zeros are marked by small circles. With few exceptions, the approximations are approaching the sought zeros with small variations. In fact, they permanently aim at the targets - desired zeros, in the course of iterative procedure.

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