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A NOTE ON THE QUASI-ANTIORDER IN A SEMIGROUP

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Abstract. Connections between quasi-antiorder on a semigroup with apartness and a naturally defined quasi-antiorder relation on factor semigroup (according to congruence and anti-congruence) are presented.

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1. Preliminaries and Introduction

This short investigation is in Bishop's constructive algebra in sense of the papers [3], [8], [12] and books [6] and [13]. Let $(S, =, \neq)$ be a constructive set (in the sense of Mines ([6]), Mulvey ([8]), Ruitenburg ([12]), Troelstra and van Dalen ([13])). The relation $\neq$ is a binary relation on S which satisfies the following properties:

$$\neg(x \neq x), x \neq y \Rightarrow y \neq x, x \neq z \Rightarrow x \neq y \vee y \neq z, x \neq y \wedge y = z \Rightarrow x \neq z.$$

It is called *apartness* (A. Heyting). Let Y be a subset of S and $x \in S$. The subset Y of S is *strongly extensional* in S if and only if $y \in Y \Rightarrow y \neq x \vee x \in Y$ ([10], [11]). A relation q on S is a *coequality* relation on S if and only if it is consistent, symmetric and cotransitive ([6], [7], [9] and [11]). M. Bozic and D. A. Romano were first to define and study this notion in 1985. Let $(S, =, \neq)$ be a semigroup with apartness [3], [6], [12], [13]). As in [11], a relation q on S is anticongruence (in article [7], [9] we used term: cocongruence) if and only if it is a coequality relation on S compatible with the semigroup operation:

$$(\forall x, y \in S)((x, y) \in q \Rightarrow x \neq y),$$

$$(\forall x, y \in S)((x, y) \in q \Rightarrow (y, x) \in q),$$

$$(\forall x, y, z \in S)((x, z) \in q \Rightarrow (x, y) \in q \vee (y, z) \in q,$$

and

$$(\forall x, y, z \in S)(((xz, yz) \in q \Rightarrow (x, y) \in q) \wedge ((zx, zy) \in q \Rightarrow (x, y) \in q)).$$

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A relation α on S is *antiorder* ([6], [9]) on S if and only if

$$\alpha \subseteq \neq,$$

$$(\forall x, y, z \in S)((x, z) \in \alpha \Rightarrow (x, y) \in \alpha \vee (y, z) \in \alpha,$$

$$(\forall x, y \in S)(x \neq y \Rightarrow (x, y) \in \alpha \vee (y, x) \in \alpha), \text{(linearity)}$$

and

$$(\forall x, y, z \in S)(((xz, yz) \in \alpha \Rightarrow (x, y) \in \alpha) \wedge ((zx, zy) \in \alpha \Rightarrow (x, y) \in \alpha)).$$

A relation s on S is *quasi-antiorder* ([7], [9], [11]) on S if

$$\alpha \subseteq \neq,$$

$$(\forall x, y, z \in S)((x, z) \in s \Rightarrow (x, y) \in s \vee (y, z) \in s,$$

$$(\forall x, y, z \in S)(((xz, yz) \in s \Rightarrow (x, y) \in s) \wedge ((zx, zy) \in s \Rightarrow (x, y) \in s)).$$

Let x be an element of S and A a subset of S. We write $x \rhd\lhd A$ if and only if $(\forall a \in A)(x \neq a)$, and $A^C = \{x \in S : x \rhd\lhd A\}$. If s is a quasi-antiorder on S, then the relation $q = s \cup s^{-1}$ is an anticongruence on S. Firstly, the relation $q^C = \{(x, y) \in S \times S : (x, y) \rhd\lhd q = s \cup s^{-1}\}$ is a congruence on S compatible with q, in the following sense $(\forall a, b, c \in S)((a, b) \in q^C \wedge (b, c) \in q \Rightarrow (a, c) \in q)$ ([11], Theorem 1).

We can construct the semigroup $S/(q^C, q) = \{aq^C : a \in S\}$.

Theorem 1. ([11], Theorem 2) *If q is an anticongruence on a semigroup S with apartness, then the set $S/(q, q^C)$ is a semigroup with*

$$aq^C = bq^C \Leftrightarrow (a, b) \rhd\lhd q, aq^C \neq bq^C \Leftrightarrow (a, b) \in q, aq^C \cdot bq^C = abq^C.$$

We can also construct the semigroup $S/q = \{aq : a \in S\}$:

Theorem 2. ([11], Theorem 3) *Let q be anticongruence on a semigroup S with apartness. Then the set S/q is a semigroup with*

$$aq = bq \Leftrightarrow (a, b) \rhd\lhd q, aq \neq bq \Leftrightarrow (a, b) \in q, aq \cdot bq = abq.$$

For a homomorphism $f : (S, =, \neq) \to (T, =, \neq)$ we say that it is a *strongly extensional homomorphism* if and only if $(\forall a, b \in S)(f(a) \neq f(b) \Rightarrow a \neq b)$.

Let S be a semigroup with apartness. A relation ρ on S is a quasi-order if it is reflexive and transitive. It is well known that if a quasi-order is compatible with the semigroup operation, then the relation C on S defined by $C = \rho \cap \rho^{-1}$ is a congruence on S (see e. g. [1], [2]).

In the article [4], N. Kehayopulu and M. Tsingelis gave the example of an ordered semigroup $(S, \cdot, \leq)$ and a congruence θ on S such that the relation $\leq$ on set S/θ, defined by

$$\leq = \{(t,z) \in S/\theta \times S/\theta : (\exists (a,b) \in \leq)(t = a\theta \wedge z = b\theta)\} =$$

$$= \{(x\theta, y\theta) \in S/\theta \times S/\theta : (\exists a \in x\theta)(\exists b \in y\theta)((a,b) \in \leq)\},$$

is not an order relation on S/θ, in general. In articles [4] and [5] they developed the theory of pseudo-order (quasi-order [1], [2]) in ordered semigroup. Constructive notion of quasi-antiorder relation is a notion parallel to the classical notion of quasi-order relation. In this paper and some other papers we try to investigate the properties of quasi-antiorder.

Let $(S, =, \neq, \cdot)$ be a semigroup with apartness, σ a quasi-order on S. In this article we will give a connection between the family $A = \{\alpha : \alpha$ is a quasi-antiorder on S such that $\alpha \subseteq \sigma\}$ and the family $\mathbf{B}$ of all quasi-antiorders on S/q, where $q = \sigma \cup \sigma^{-1}$.

2. Results

Let $(S, =, \neq, \cdot)$ be a semigroup with apartness and σ be a quasi-antiorder relation on S. Our first proposition shows the existence of the quasi-antiorder Q on S/q, where $q = \sigma \cup \sigma^{-1}$.

Lemma 1. *Let $(S, =, \neq, \cdot)$ be a semigroup with apartness and σ be a quasi-antiorder relation on S. The relation Q on S/q, where $q = \sigma \cup \sigma^{-1}$, defined by $(aq, bq) \in Q \Leftrightarrow (a,b) \in \sigma$, is a consistent, cotransitive and linear relation on semigroup S/q compatible with the semigroup operation on S/q.*

Proof. Let a, b and c be elements of S.

(i) Let $(aq, bq) \in Q$ i. e. let $(a,b) \in \sigma \subseteq a$, So, $aq \neq bq$.

(ii) Let $(aq, cq) \in Q$, i. e. let $(a,c) \in \sigma$. Therefore, $(a,b) \in \sigma$ or $(b,c) \in \sigma$. Finally, we have $(aq, bq) \in Q$ or $(bq, cq) \in Q$, which means that Q is a cotransitive relation.

(iii) Let $(axbq, aybq) \in Q$, i. e. let $(axb, ayb) \in \sigma$. Hence, $(x,y) \in \sigma$, because the relation σ is compatible with the semigroup operation in S. Therefore $(xq, yq) \in Q$.

(iv) Let $aq \neq bq$, i. e. let $(a,b) \in q = \sigma \cup \sigma^{-1}$. Then $(aq, bq) \in Q$ or $(bq, aq) \in Q$. So, the relation Q is linear. □

Let $\varphi : S \to T$ be a strongly extensional homomorphism and σ a quasi-antiorder on S. Then $\varphi(\sigma)$ is not quasi-antiorder on T, in general case. In the following proposition we prove the following: if t is a quasi-antiorder on the semigroup T, then $\varphi^{-1}(t)$ is a quasi-antiorder on S.

Lemma 2. *If $(S, =, \neq, \cdot)$ and $(T, =, \neq, \cdot)$ are semigroups, t is a quasi-antiorder on T, and $\varphi : S \to T$ a strongly extensional homomorphism, then the relation $\varphi^{-1}(t) = \{(a,b) \in S \times S : (\varphi(a), \varphi(b)) \in t\}$ is a quasi-antiorder on S, the relation*

$Coker\varphi = \{(a,b) \in S \times S : \varphi(a) \neq \varphi(b)\}$ is anticongruence on S compatible with congruence $Ker\varphi = \varphi \cdot \varphi^{-1}$, and $Coker\varphi \supseteq \varphi^{-1}(t) \cdot (\varphi^{-1}(t))^{-1}$ holds. Also, if the relation t is linear in T we have $Coker\varphi = \varphi^{-1}(t) \cdot (\varphi^{-1}(t))^{-1}$.

Proof.

(i) $(a,b) \in \varphi^{-1}(t)$ $\Leftrightarrow (\varphi(a), \varphi(b)) \in t \subseteq \neq$ (by definition of the relation $\varphi^{-1}(t)$)
$\Leftrightarrow \varphi(a) \neq \varphi(b)$ (φ is strongly extensional homomorphism)
$\Rightarrow a \neq b$;

(ii) $(a,c) \in \varphi^{-1}(t)$ $\Leftrightarrow (\varphi(a), \varphi(c)) \in t$ (by cotransitivity of ρ)
$\Rightarrow (\forall b \in S)((\varphi(a), \varphi(b)) \in t \vee (\varphi(b), \varphi(c)) \in t)$
$\Rightarrow (\forall b \in S)((a,b) \in \varphi^{-1}(t) \vee (b,c) \in \varphi^{-1}(t))$;

(iii) $(xay, xby) \in \varphi^{-1}(t)$ $\Leftrightarrow (\varphi(xay), \varphi(xby)) \in t$
$\Rightarrow (\varphi(x)\varphi(a)\varphi(y), \varphi(x)\varphi(b)\varphi(y)) \in t$ (by compatibility of t with the operation in T)
$\Rightarrow (\varphi(a), \varphi(b)) \in t$
$\Leftrightarrow (a,b) \in \varphi^{-1}(t)$;

(iv) Suppose that the relation t is linear. Then we will have
$(a,b) \in Coker\varphi$ $\Leftrightarrow \varphi(a) \neq \varphi(b)$ (by linearity of t)
$\Rightarrow (\varphi(a), \varphi(b)) \in t \vee (\varphi(b), \varphi(a)) \in t$
$\Leftrightarrow (a,b) \in \varphi^{-1}(t) \vee (b,a) \in \varphi^{-1}(t)$. □

In the following theorem we prove that there exists bijective mapping between quasi-antiorder T on S/q and quasi-antiorder t on S included in s.

Theorem 3. *Let $(S, =, \neq, \cdot)$ be a semigroup with apartness, σ a quasi-antiorder on S. Let $\mathbf{A} = \{\alpha : \alpha$ is quasi-antiorder on S such that $\alpha \subseteq \sigma\}$. Let $\mathbf{B}$ be the set of all quasi-antiorders on S/q, where $q = \sigma \cup \sigma^{-1}$. For $\alpha \in \mathbf{A}$, we define a relation $\alpha'' = \{(aq, bq) \in S/q \times S/q : (a,b) \in \alpha$. The mapping $f : \mathbf{A} \to \mathbf{B}$ defined by $f(\alpha) = \alpha''$ is strongly extensional, injective and surjective mapping from $\mathbf{A}$ onto $\mathbf{B}$ and for $\alpha, \beta \in \mathbf{A}$ we have $\alpha \subseteq \beta$ if and only if $\alpha'' \subseteq \beta''$.*

Proof.

(1) f is a well defined function. Let $\alpha \in \mathbf{A}$. Then α'' is a quasi-antiorder on S/q. Indeed: let $(aq, bq) \in \alpha''$ i. e. let $(a,b) \in \alpha \subseteq \sigma \subseteq \sigma \cup \sigma^{-1} = q$. Then $aq \neq bq$. This means that $\alpha'' \subseteq \neq$ on S/q. Let $(aq, cq) \in \alpha''$ and let bq be an arbitrary element of S/q. Then $(a,c) \in \alpha$, and b is an arbitrary element of S. Since $(a,b) \in \alpha \vee (b,c) \in \alpha$, we have $(aq, bq) \in \alpha'' \vee (bq, cq) \in \alpha''$. Let $(aqxq, bqxq) \in \alpha''$, i. e. let $(axq, bxq) \in \alpha''$. This means that $(ax, bx) \in \alpha$. From this we conclude $(a,b) \in \alpha$. Thus $(aq, bq) \in \alpha''$, i. e. the relation α'' is compatible with the semigroup operation on S/q. Let $\alpha, \beta \in \mathbf{A}$ with $\alpha = \beta$. If $(aq, bq) \in \alpha''$, then $(a,b) \in \alpha = \beta$, so $(aq, bq) \in \beta''$. Similarly, $\beta'' \subseteq \alpha''$. Therefore, $\beta'' = \alpha''$.

(2) f is an injection. Let $\alpha, \beta \in \mathbf{A}$, $\alpha'' = \beta''$. Let $(a,b) \in \alpha$. Since $(aq, bq) \in \alpha'' = \beta''$, we have $(a,b) \in \beta$. Similarly, we conclude $\beta \subseteq \alpha$. So, $\beta = \alpha$.

(3) f is strongly extensional. Let $\alpha, \beta \in \mathbf{A}$, $\alpha'' \neq \beta''$, i. e. let there exist an element $(aq, bq) \in \alpha''$ and $(aq, bq)\#\beta''$. Then $(a,b) \in \alpha$. Let (x,y) be an arbitrary element of β. Then $(xq, yq) \in \beta''$ and $(xq, yq) \neq (aq, bq)$. This means $xq \neq aq \vee yq \neq bq$, i. e. $(x,a) \in q \vee (y,b) \in q$. Therefore, from $x \neq a \vee y \neq b$ we have $(a,b) \in \alpha$ and $(a,b) \neq (x,y) \in \beta$. Thus, we have $\alpha \neq \beta$. Similarly, from $(aq, bq)\#\alpha''$ and $(aq, bq) \in \beta''$ we conclude $\alpha \neq \beta$.

(4) f is onto. Let $\delta \in \mathbf{B}$. We define a relation μ on S as follows:

$$\mu = \{(x,y) \in S \times S : (xq, yq) \in \delta\}.$$

μ is a quasi-antiorder. In fact:

(I) Let $(x,y) \in \mu$. Since $(xq, yq) \in \delta \subseteq\neq$ on S/q, we conclude that $xq \neq yq$, i. e. $(x,y) \in q = \sigma \cup \sigma^{-1}$. Hence, $(x,y) \in \sigma \subseteq\neq$ or $(y,x) \in \sigma \subseteq\neq$. Therefore, we have $x \neq y$. Let $(x,z) \in \mu$, i. e. let $(xq, zq) \in \delta$. Then $(xq, yq) \in \delta$ or $(yq, zq) \in \delta$ for arbitrary $yq \in S/q$ by cotransitivity of δ. Thus, $(x,y) \in \mu$ or $(y,z) \in \mu$. Let $(ax, ay) \in \mu$, i. e. let $(axq, ayq) \in \delta$. Then from $(aqxq, aqyq) \in \delta$ follows $(xq, yq) \in \delta$. So, we have $(x,y) \in \mu$. Similarly, we conclude $(x,y) \in \mu$ from $(xa, ya) \in \mu$. Therefore, the relation μ is a compatible relation on S.

(II) $\mu'' = \delta$. Indeed:

$$(xq, yq) \in \mu'' \Leftrightarrow (x,y) \in \mu \Leftrightarrow (xq, yq) \in \delta.$$

(III) $\mu \subseteq \sigma$. In the matter of fact, we have the sequence
$(a,b) \in \mu \Leftrightarrow (f(a), f(b)) \in \mu'' = \delta$
$\Leftrightarrow (f \cdot \pi(q)(a), f \cdot \pi(q)(b)) \in \mu^{'} = \delta$ ($\pi(q) : S \to S/q$ is a strongly extensional epimorphism)
$\Leftrightarrow (\pi(q)(a), \pi(q)(b)) \in f^{-1}(\mu^{'}) = f^{-1}(\delta)$ (by $f^{-1}(\delta) \subseteq Coker(f)$)
$\Rightarrow (\pi(q)(a), \pi(q)(b)) \in Q$
$\Leftrightarrow (a,b) \in \rho$.

(5) Let $\alpha, \beta \in \mathbf{A}$. We have $\alpha \subseteq \beta$ if and only if $\alpha'' \subseteq \beta''$. Indeed: Let $\alpha \subseteq \beta$ and $(xq, yq) \in \alpha''$. Since $(x,y) \in \alpha \subseteq \beta$, we have $(xq, yq) \in \beta''$. Oppositely, let $\alpha'' \subseteq \beta''$ and $(x,y) \in \alpha$. Since $(xq, yq) \in \alpha'' \subseteq \beta''$, we conclude that $(x,y) \in \beta$. □

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