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ON NEIGHBORHOODS OF ANALYTIC FUNCTIONS DEFINED BY USING HADAMARD PRODUCT

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Abstract. We introduce the classes $S_n(\gamma,\beta,\alpha)$ and $\Re_n(\gamma,\beta,\alpha;\mu)$ of functions defined by $f*S_\alpha(z)$ of $f(z)$ and $S_\alpha(z)=\frac{z}{(1-z)^{2(1-\alpha)}}$. By making use of the familiar concept of neighborhoods of analytic functions, we prove several inclusion relations associated with the $(n,\delta)-$neighborhoods of certain subclasses of analytic functions of complex order, which are introduced by means of the Hadamard product.

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1. Introduction

Let A denote the class of functions f of the form:

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k$$

which are analytic in the open unit disk $\Delta:=\{z:z\in \mathrm{C} \quad \text{and} \quad |z|<1\}$.

And let S denote a subclass of A consisting of analytic and univalent functions $f(z)$ in Δ. A function $f(z)$ from S is said to be starlike of order α if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha \qquad (z\in\Delta)$$

for some α $(0\le\alpha<1)$. We denote the class of all starlike functions of order α by $S^*(\alpha)$. Further, a function $f(z)$ from S is said to be convex of order α if and only if

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha \qquad (z\in\Delta)$$

for some α $(0\le\alpha<1)$. We denote the class of all convex functions of order α by $K(\alpha)$. We note that $f(z)\in K(\alpha)$ if and only if $zf'(z)\in S^*(\alpha)$. The classes $S^*(\alpha)$ and $K(\alpha)$ were first introduced by Robertson [16], and later were studied by Schild [19], MacGregor [11] and Pinchuk [15].

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Now, the function

$$S_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \qquad (0 \le \alpha < 1)$$

is the well-known extremal function for the class $S^*(\alpha)$. Setting

$$C(\alpha, k) = \frac{1}{(k-1)!} \prod_{m=2}^{k} (m - 2\alpha) \qquad (k \ge 2),$$

$S_\alpha(z)$ can be written in the form:

$$S_\alpha(z) = z + \sum_{k=n+1}^{\infty} C(\alpha, k) z^k$$

and also $f * S_\alpha(z)$ can be written in the form:

$$f * S_\alpha(z) = z + \sum_{k=n+1}^{\infty} C(\alpha, k) a_k z^k. \tag{1.1}$$

Then we can see that $C(\alpha, k)$ is a decreasing function in α which satisfies

$$\lim_{k\to\infty} C(\alpha, k) = \begin{cases} \infty, & \alpha < \frac{1}{2}, \\ 0, & \alpha > \frac{1}{2}, \\ 1, & \alpha = \frac{1}{2}. \end{cases}$$

The class $L^*_\alpha(\lambda, \beta)$ of functions defined by $f * S_\alpha(z)$ of $f(z)$ and $S_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}}$ were studied by Aouf *et al.* [7]. Let $f * g(z)$ denote the convulation or Hadamard product of two functions $f(z)$ and $g(z)$, that is, $f(z)$ is given by $g(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k$, and $g(z)$ is given by $g(z) = z + \sum\limits_{k=2}^{\infty} b_k z^k$, then

$$f * g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

Let $A(n)$ denote the subclass of S consisting of the functions f of the form:

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \ge 0;\ k \in \mathrm{N} - \{1\};\ n \in \mathrm{N} := \{1, 2, 3, ...\}), \tag{1.2}$$

which are analytic in the open unit disk Δ.

Following the works of Goodman [8] and Ruscheweyh [17], we define the (n, δ)-neighborhood of a function $f \in A(n)$ by (see [14], [6], [5], [1] and [18]).

(1.3)

$$N_{n,\delta}(f) := \left\{ g \in A(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \text{ and } \sum_{k=n+1}^{\infty} k\,|a_k - b_k| \le \delta \right\}.$$

In particular, for the identity function

$$e(z) = z, \tag{1.4}$$

we immediately have

$$N_{n,\delta}(e) := \left\{ g \in A(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \text{ and } \sum_{k=n+1}^{\infty} k\,|b_k| \leq \delta \right\}. \tag{1.5}$$

The above concept of $(n, \delta)-$neighborhoods was extended and applied recently to families of analytically multivalent functions by Altıntaş *et al.* [4] and families of meromorphically multivalent functions by Liu and Srivastava ([10] and [9]).

The main objective of the present paper is to investigate the (n, δ)-neighborhoods of several subclasses of the class $A(n)$ of normalized analytic functions in Δ with negative coefficients, which are introduced below by making use of the Hadamard product.

First of all, a function $f(z) \in A(n)$ is said to be starlike of complex order γ ($\gamma \in \mathrm{C} - \{0\}$), that is, $f \in S_n^*(\gamma)$, if it also satisfies the following inequality:

$$Re\left\{1 + \frac{1}{\gamma}\left[\frac{zf'(z)}{f(z)} - 1\right]\right\} > 0, \; z \in \Delta; \; \gamma \in \mathrm{C}-\{0\}. \tag{1.6}$$

Furthermore, a function $f(z) \in A(n)$ is said to be convex of complex order γ ($\gamma \in \mathrm{C} - \{0\}$), that is, $f \in K_n(\gamma)$, if it also satisfies the following inequality:

$$Re\left\{1 + \frac{1}{\gamma}\frac{zf''(z)}{f'(z)}\right\} > 0, \; z \in \Delta; \; \gamma \in \mathrm{C}-\{0\}. \tag{1.7}$$

The classes $S_n^*(\gamma)$ and $K_n(\gamma)$ stem essentially from the class of starlike and convex functions of complex order, which were considered earlier by Nasr and Aouf [13] and Wiatrowski [20], respectively, (see also [3] and [2]).

Finally, in terms of the Hadamard product $f * S_\alpha(z)$ defined by (1.1), let $S_n(\gamma, \beta, \alpha)$ denote the subclass of $A(n)$ consisting of the functions $f(z)$ which satisfy the inequality:

$$\left|\frac{1}{\gamma}\left\{\frac{z(f * S_\alpha(z))'}{f * S_\alpha(z)} - 1\right\}\right| < \beta, \tag{1.8}$$

$$z \in \Delta; \; \gamma \in \mathrm{C} - \{0\}; \; 0 \leq \alpha < 1; \; 0 < \beta \leq 1.$$

Let $\Re_n(\gamma, \beta, \alpha; \mu)$ denote the subclass of $A(n)$ consisting of the functions $f(z)$ which satisfy the inequality:

$$\left|\frac{1}{\gamma}\left\{(1-\mu)\frac{f * S_\alpha(z)}{z} + \mu(f * S_\alpha(z))' - 1\right\}\right| < \beta, \tag{1.9}$$

$$z \in \Delta; \; \gamma \in \mathrm{C} - \{0\}; \; 0 \leq \alpha < 1; \; 0 < \beta \leq 1; \; 0 \leq \mu \leq 1.$$

2. A set of inclusion relations involving $N_{n,\delta}(e)$

In our investigation of the inclusion relations involving $N_{n,\delta}(e)$, we need the following Lemmas.

Lemma 1. *Let the function $f \in A(n)$ be defined by (1.2). Then $f(z)$ is in the class $S_n(\gamma,\beta,\alpha)$ if and only if*

$$\sum_{k=n+1}^{\infty} \frac{1}{(k-1)!} \prod_{m=2}^{k} (m-2\alpha)(\beta\,|\gamma| + k - 1)a_k \le \beta\,|\gamma|\,,\ k \ge 2. \tag{2.1}$$

Proof. We suppose that $f \in S_n(\gamma,\beta,\alpha)$. Then, by recalling the condition (1.8), we readily get

$$Re\left(\frac{z(f * S_\alpha(z))'}{f * S_\alpha(z)} - 1\right) > -\beta\,|\gamma|\,,\ z \in \Delta. \tag{2.2}$$

or equivalently,

$$Re\left(\frac{-\sum\limits_{k=n+1}^{\infty} \frac{1}{(k-1)!} \prod\limits_{m=2}^{k} (m-2\alpha)(k-1)a_k z^k}{z - \sum\limits_{k=n+1}^{\infty} \frac{1}{(k-1)!} \prod\limits_{m=2}^{k} (m-2\alpha)a_k z^k}\right) > -\beta\,|\gamma|\,,\ z \in \Delta \tag{2.3}$$

where we have made use of Definition (1.1) and Definition (1.2). Now, choose values of z on the real axis and let $z \to 1^-$ through real values. Then Inequality (2.3) immediately yields the desired condition (2.1).

Conversely, by applying hypothesis (2.1) and letting $|z| = 1$, we find that

$$\left|\frac{z(f * S_\alpha(z))'}{f * S_\alpha(z)} - 1\right| = \left|\frac{\sum\limits_{k=n+1}^{\infty} \frac{1}{(k-1)!} \prod\limits_{m=2}^{k} (m-2\alpha)(k-1)a_k z^k}{z - \sum\limits_{k=n+1}^{\infty} \frac{1}{(k-1)!} \prod\limits_{m=2}^{k} (m-2\alpha)a_k z^k}\right|$$

$$\le \frac{\beta\,|\gamma| \left\{1 - \sum\limits_{k=n+1}^{\infty} \frac{1}{(k-1)!} \prod\limits_{m=2}^{k} (m-2\alpha)a_k\right\}}{1 - \sum\limits_{k=n+1}^{\infty} \frac{1}{(k-1)!} \prod\limits_{m=2}^{k} (m-2\alpha)a_k} \le \beta\,|\gamma|\,. \tag{2.4}$$

Hence, by the maximum modulus theorem we have $f \in S_n(\gamma,\beta,\alpha)$, which evidently completes the proof of Lemma 1.

Similarly, we can prove the following.

Lemma 2. *Let the function $f \in A(n)$ be defined by (1.2). Then $f(z)$ is in the class $\Re(\gamma, \beta, \alpha; \mu)$ if and only if*

$$\sum_{k=n+1}^{\infty} \frac{1}{(k-1)!} \prod_{m=2}^{k} (m-2\alpha)[\mu(k-1)+1]a_k \leq \beta\,|\gamma|\,. \tag{2.5}$$

Remark 1. When we take $\frac{1}{(k-1)!}\prod_{m=2}^{k}(m-2\alpha)$ instead of $\left(\begin{array}{c}\lambda+n\\ n\end{array}\right)$ in Lemma 1 and Lemma 2, we get the corresponding results of Murugusundaramoorthy *et al.* [12].

Theorem 1. *Let*

$$\delta =: \frac{(n+1)\beta\,|\gamma|}{(\beta\,|\gamma|+n)\prod_{m=2}^{n+1}(m-2\alpha)} \qquad (\,|\gamma|<1; \quad 0 \leq \alpha < 1), \tag{2.6}$$

then

$$S_n(\gamma, \beta, \alpha) \subset N_{n,\delta}(e). \tag{2.7}$$

Proof. For a function $f \in S_n(\gamma, \beta, \alpha)$ of the form (1.2), Lemma 1 immediately yields

$$(\beta\,|\gamma|+n)\prod_{m=2}^{n+1}(m-2\alpha)\sum_{k=n+1}^{\infty} a_k \leq \beta\,|\gamma|$$

so that

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{\beta\,|\gamma|}{(\beta\,|\gamma|+n)\prod_{m=2}^{n+1}(m-2\alpha)}. \tag{2.8}$$

On the other hand, we also find from (2.1) and (2.8) that

$$\prod_{m=2}^{n+1}(m-2\alpha)\sum_{k=n+1}^{\infty} k a_k \leq \beta\,|\gamma| + (1-\beta\,|\gamma|)\prod_{m=2}^{n+1}(m-2\alpha)\sum_{k=n+1}^{\infty} a_k$$

$$\leq \beta\,|\gamma| + (1-\beta\,|\gamma|)\prod_{m=2}^{n+1}(m-2\alpha)\frac{\beta\,|\gamma|}{\prod_{m=2}^{n+1}(m-2\alpha)(\beta\,|\gamma|+n)}$$

$$\leq \frac{(n+1)\beta\,|\gamma|}{n+\beta\,|\gamma|} \qquad (|\gamma|<1),$$

that is,

$$\sum_{k=n+1}^{\infty} ka_k \leq \frac{(n+1)\beta\,|\gamma|}{(\beta\,|\gamma|+n)\prod_{m=2}^{n+1}(m-2\alpha)} := \delta, \tag{2.9}$$

which, in view of definition (1.5), proves Theorem 1. □

Similarly, by applying Lemma 2 instead of Lemma 1, we can prove the following.

Theorem 2. *If*

$$\delta =: \frac{(n+1)\beta\,|\gamma|}{(\beta\,|\gamma|+n)\prod_{m=2}^{n+1}(m-2\alpha)} \tag{2.10}$$

then

$$\Re(\gamma,\beta,\alpha;\mu) \subset N_{n,\delta}(e). \tag{2.11}$$

Proof. Suppose that a function $f \in \Re(\gamma,\beta,\alpha;\mu)$ is of the form (1.2). Then we find from the assertion (2.5) of Lemma 2 that

$$\prod_{m=2}^{n+1}(m-2\alpha)(\mu n+1)\sum_{k=n+1}^{\infty} a_k \leq \beta\,|\gamma|$$

which yields the following coefficient inequality:

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{\beta\,|\gamma|}{(\mu n+1)\prod_{m=2}^{n+1}(m-2\alpha)}. \tag{2.12}$$

Making use of (2.5) in conjunction with (2.12), we also have

$$\mu\prod_{m=2}^{n+1}(m-2\alpha)\sum_{k=n+1}^{\infty} ka_k \leq \beta\,|\gamma| + (\mu-1)\prod_{m=2}^{n+1}(m-2\alpha)\sum_{k=n+1}^{\infty} a_k$$

$$\leq \beta\,|\gamma| + (\mu-1)\prod_{m=2}^{n+1}(m-2\alpha)\frac{\beta\,|\gamma|}{(\mu n+1)\prod_{m=2}^{n+1}(m-2\alpha)}, \tag{2.13}$$

that is

$$\sum_{k=n+1}^{\infty} k a_k \leq \frac{(n+1)\beta\,|\gamma|}{(\mu n+1)\prod\limits_{m=2}^{n+1}(m-2\alpha)} =: \delta,$$

which, in light of the definition (1.5), completes the proof of Theorem 2. □

3. Neighborhoods for the classes $S_n^{(\rho)}(\gamma,\beta,\alpha)$ and $\Re_n^{(\rho)}(\gamma,\beta,\alpha;\mu)$

In this section we determine the neighborhood for each of the classes $S_n^{(\rho)}(\gamma,\beta,\alpha)$ and $\Re_n^{(\rho)}(\gamma,\beta,\alpha;\mu)$, which we define as follows. A function $f \in A(n)$ defined by (1.2) is said to be in the class $S_n^{(\rho)}(\gamma,\beta,\alpha)$ if there exists a function $g \in S_n(\gamma,\beta,\alpha)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1-\rho,\ \ z \in \Delta;\ 0 \leq \rho < 1. \tag{3.1}$$

Analogously, a function $f \in A(n)$ defined by (1.2) is said to be in the class $\Re_n^{(\rho)}(\gamma,\beta,\alpha;\mu)$ if there exists a function $g \in \Re_n(\gamma,\beta,\alpha;\mu)$ such that inequality (3.1) holds true.

Theorem 3. *If $g \in S_n(\gamma,\beta,\alpha)$ and*

$$\rho = 1 - \frac{\delta(\beta\,|\gamma|+n)\prod\limits_{m=2}^{n+1}(m-2\alpha)}{(n+1)\left[(\beta\,|\gamma|+n)\prod\limits_{m=2}^{n+1}(m-2\alpha) - \beta\,|\gamma|\right]} \tag{3.2}$$

then

$$N_{n,\delta}(g) \subset S_n^{(\rho)}(\gamma,\beta,\alpha). \tag{3.3}$$

Proof. Suppose that $f \in N_{n,\delta}(g)$. We then find from (1.3) that

$$\sum_{k=n+1}^{\infty} k\,|a_k - b_k| \leq \delta, \tag{3.4}$$

which readily implies the coefficient inequality:

$$\sum_{k=n+1}^{\infty} |a_k - b_k| \leq \frac{\delta}{n+1} (n \in \mathrm{N}). \tag{3.5}$$

Next, since $g \in S_n(\gamma,\beta,\alpha)$, we have [*cf.* Equation (2.8)]

$$\sum_{k=n+1}^{\infty} b_k \leq \frac{\beta|\gamma|}{(\beta|\gamma|+n)\prod_{m=2}^{n+1}(m-2\alpha)}, \tag{3.6}$$

so that

$$\left|\frac{f(z)}{g(z)}-1\right| < \frac{\sum_{k=n+1}^{\infty}|a_k-b_k|}{1-\sum_{k=n+1}^{\infty}b_k}$$

$$\leq \frac{\delta}{n+1}\cdot\frac{(\beta|\gamma|+n)\prod_{m=2}^{n+1}(m-2\alpha)}{(\beta|\gamma|+n)\prod_{m=2}^{n+1}(m-2\alpha)-\beta|\gamma|} \tag{3.7}$$

$$= 1-\rho,$$

provided that ρ is given precisely by (3.2). Thus, by definition, $f \in S_n^{(\rho)}(\gamma,\beta,\alpha)$ for ρ given by (3.2), which evidently completes our proof of Theorem 3. □

Our proof of Theorem 4 below is much similar to that of Theorem 3.

Theorem 4. *If $g \in \Re_n(\gamma,\beta,\alpha;\mu)$ and*

$$\rho = 1-\frac{\delta(\mu n+1)\prod_{m=2}^{n+1}(m-2\alpha)}{(n+1)\left[(\mu n+1)\prod_{m=2}^{n+1}(m-2\alpha)-\beta|\gamma|\right]} \tag{3.8}$$

then

$$N_{n,\delta}(g) \subset \Re_n^{(\rho)}(\gamma,\beta,\alpha;\mu). \tag{3.9}$$

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