Novi Sad J. Math.
Vol. 37, No. 1, 2007, 39-47

OSCILLATION PROPERTIES FOR ADVANCED DIFFERENCE EQUATIONS

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Abstract. In this paper, we provide some sufficient conditions for the oscillation of every solution of the difference equations

$$x_{n+1} - x_n + p_n x_{n-k} = 0,\ n = 0, 1, 2, ...,$$

whenever $k \in \{..., -3, -2\}$ and $p_n \leq 0$; and also

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_{in} x_{n-k_i} = 0,\ n = 0, 1, 2, ...,$$

whenever $k_i \in \{..., -3, -2, -1\}$ and $p_{in} \leq 0$ for $i = 1, 2, ..., m$. We also obtain some alternative results for the oscillation of all solutions of these equations.

AMS Mathematics Subject Classification (2000): 39A10

Key words and phrases: Difference equation, difference inequality, oscillation, nonoscillation

1. Introduction

The oscillatory behavior of some differential and difference equations have been investigated (see, for instance, [1], [3], [4], [5]). In recent years, the oscillations of discrete analogues of delay differential equations have been given [2], [7]. Furthermore, explicit conditions for the oscillation of difference equations with constant coefficients have been studied [6]. Erbe and Zhang [2] have introduced a sufficient condition for the oscillation of all solutions of the following difference equations:

$$x_{n+1} - x_n + p_n x_{n-k} = 0, \quad n = 0, 1, 2, ..., \tag{1.1}$$

whenever $k \in \mathbb{N}$ and $p_n \geq 0$; and also

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_{in} x_{n-k_i} = 0, \quad n = 0, 1, 2, ..., \tag{1.2}$$

whenever $k_i \in \mathbb{N}$ and $p_{in} \geq 0$ for $i = 1, 2, ..., m$.

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By a solution of equation (1.1) we mean a sequence (x_n) which is defined for $n \geq -k$ and which satisfies equation (1.1) for $n \geq 0$. We recall that a solution (x_n) of equation (1.1) is said to be oscillatory if the terms x_n of the sequence (x_n) are neither eventually positive nor eventually negative. Otherwise, the solution is called nonoscillatory.

The aim of the present paper is to provide some sufficient conditions for the oscillation of every solution of equation (1.1) whenever $k \in \{..., -3, -2\}$ and $p_n \leq 0$ and that of equation (1.2) whenever $k_i \in \{..., -3, -2, -1\}$ and $p_{in} \leq 0$ for $i = 1, 2, ..., m$. We also obtain some alternative results for the oscillation of all solutions of these equations.

2. Sufficient Conditions for the Oscillation of Eq. (1.1)

In this section, we provide a sufficient condition for the oscillation of every solution of equation (1.1). Erbe and Zhang [2] have proved the following result.

Theorem A. *Assume that*

$$\liminf_{n\to\infty} p_n = p > \frac{k^k}{(k+1)^{k+1}}, \quad k \in \mathbb{N}.$$

Then every solution of equation (1.1) *oscillates.*

We first need the following lemma.

Lemma 2.1. *Let* $k \in \{..., -3, -2\}$. *If*

$$\limsup_{n\to\infty} p_n = p < \frac{k^k}{(k+1)^{k+1}}, \tag{2.1}$$

then the following holds:

(i) *the difference inequality*

$$x_{n+1} - x_n + p_n x_{n-k} \geq 0 \tag{2.2}$$

has no eventually positive solution,

(ii) *the difference inequality*

$$x_{n+1} - x_n + p_n x_{n-k} \leq 0 \tag{2.3}$$

has no eventually negative solution.

Proof. (i) Assume, for the sake of contradiction, that inequality (2.2) has an eventually positive solution. Then, there exists a number $N_1 > 0$ such that $x_n > 0$ for all $n \geq N_1$. Also by (2.1) there is a number $N_2 > 0$ such that $p_n < 0$ for all $n \geq N_2$. Let $N = \max\{N_1 - k, N_2\}$. By using (2.1) and (2.2), we have

$$x_{n+1} - x_n \geq -p_n x_{n-k} > 0$$

for all $n \geq N$. This implies that x_n is nondecreasing for $n \geq N$. Now, dividing inequality (2.2) by x_n we have

$$\frac{x_{n+1}}{x_n} - 1 + p_n \frac{x_{n-k}}{x_n} \geq 0$$

for all $n \geq N$. This yields, for the same n's, that

$$\frac{x_{n+1}}{x_n} - 1 + p_n \left\{ \frac{x_{n-k}}{x_{n-k-1}} \frac{x_{n-k-1}}{x_{n-k-2}} ... \frac{x_{n+1}}{x_n} \right\} \geq 0 \tag{2.4}$$

Let $z_n = \dfrac{x_{n+1}}{x_n}$. Then $z_n \geq 1$ for $n \geq N$. By (2.4) we get

$$z_n \geq 1 - p_n (z_{n-k-1}\, z_{n-k-2} ... z_n). \tag{2.5}$$

Setting $\liminf\limits_{n\to\infty} z_n = q$, it is easy to see that $q \geq 1$, and also taking into consideration (2.5) we have

$$\begin{aligned} q &\geq 1 + \liminf_{n\to\infty} \{(-p_n) z_{n-k-1}\, z_{n-k-2} ... z_n\} \\ &\geq 1 + \liminf_{n\to\infty} (-p_n) \liminf_{n\to\infty} z_{n-k-1} ... \liminf_{n\to\infty} z_n \\ &= 1 - \limsup_{n\to\infty} (p_n) \liminf_{n\to\infty} z_{n-k-1} ... \liminf_{n\to\infty} z_n \\ &= 1 - pq^{-k}. \end{aligned}$$

So, we conclude that

$$p \geq (1-q) q^k. \tag{2.6}$$

Consider the function f defined by $f(q) = (1-q)q^k$. Then observe that $f'\left(\frac{k}{k+1}\right) = 0$ and $f''\left(\frac{k}{k+1}\right) > 0$. Therefore, by (2.6) we obtain

$$p \geq f\left(\frac{k}{k+1}\right) = \frac{k^k}{(k+1)^{k+1}},$$

which contradicts condition (2.1).

(ii) It is easily shown that, under condition (2.1), inequality (2.3) has no eventually negative solution by using similar method as in (i). □

By using Lemma 2.1 one can deduce the following main result immediately.

Theorem 2.2. (Main Theorem) *Let* $k \in \{..., -3, -2\}$. *If condition* (2.1) *holds, then every solution of the difference equation* (1.1) *oscillates.*

Proof. Combining (i) and (ii) in Lemma 2.1 we conclude that under condition (2.1) every solution of (1.1) oscillates. □

Remarks. We should note that by choosing $p_n = p$ in condition (2.1), Theorem 2.2 reduces to Theorem 2.1 in [6]. Furthermore, replacing condition (2.1) by $\liminf\limits_{n\to\infty} p_n = p > \dfrac{k^k}{(k+1)^{k+1}}$ and taking $k \in \mathbb{N}$ we have Theorem A (see [2]).

Theorem 2.2 contains the following result.

Corollary 2.3. *Let* $k \in \{..., -3, -2\}$. *If*

$$\sup_{n\in\mathbb{N}} p_n < \frac{k^k}{(k+1)^{k+1}}, \tag{2.7}$$

then every solution of equation (1.1) *oscillates.*

Proof. Assume that (2.7) holds. Since $\limsup_{n\to\infty} p_n \leq \sup_{n\in\mathbb{N}} p_n$, we obtain that $\limsup_{n\to\infty} p_n < \frac{k^k}{(k+1)^{k+1}}$. Hence, the proof follows from Theorem 2.2 at once. □

Corollary 2.4. *Let* $k \in \mathbb{N}$. *If*

$$\inf_{n\in\mathbb{N}} p_n > \frac{k^k}{(k+1)^{k+1}}, \tag{2.8}$$

then every solution of equation (1.1) *oscillates.*

Proof. Suppose that (2.8) holds. Since $\inf_{n\in\mathbb{N}} p_n \leq \liminf_{n\to\infty} p_n$, we may write $\liminf_{n\to\infty} p_n > \frac{k^k}{(k+1)^{k+1}}$, which completes the proof by Theorem A. □

Before closing this section, we will recall the following theorem.

Theorem 2.5. *Let* $k \in \{..., -3, -2\}$. *If* $p_n \leq 0$ *and*

$$\liminf_{n\to\infty} p_n > \frac{k^k}{(k+1)^{k+1}}, \tag{2.9}$$

then equation (1.1) *has a nonoscillatory solution.*

Proof. Condition (2.9) implies that there is a number $N_1 > 0$ such that

$$p_n \geq \frac{k^k}{(k+1)^{k+1}} \tag{2.10}$$

for all $n \geq N_1$. Taking $z_n = \frac{x_{n+1}}{x_n}$ in equation (1.1), we may write

$$z_n = 1 - p_n\, z_{n-k-1} ... z_{n+1} z_n.$$

This yields to

$$z_n = (1 + p_n\, z_{n-k-1} ... z_{n+1})^{-1}. \tag{2.11}$$

To complete the proof it suffices to show that equation (2.11) has a positive solution. Indeed, with $N \geq N_1$ define

$$S_{N-k-1} = ... = S_{N+1} = \frac{k}{k+1} = q > 1, \tag{2.12}$$

and

$$S_N = (1 + p_N\, S_{N-k-1} ... S_{N+1})^{-1} > 1. \tag{2.13}$$

By (2.10), (2.12) and (2.13) we have

$$p_N\, S_{N-k-1} ... S_{N+1} > \frac{1}{k}.$$

So, it is obvious that

$$1 < S_N < q.$$

By induction we get

$$1 < S_{N-k} < q, \text{ for } k = ..., -3, -2.$$

Hence, we conclude that (s_n) $(n \geq N)$ is a solution of equation (2.11). Now, defining $x_N = 1$, $x_{N+1} = x_N S_N$ and so on, it follows that (x_n) $(n \geq N)$ is a positive solution of (1.1). □

The fact that $\liminf_{n\to\infty} p_n \geq \inf_{n\in\mathbb{N}} p_n$ leads us to the following result.

Corollary 2.6. *Let* $k \in \{..., -3, -2\}$. *If* $p_n \leq 0$ *and*

$$\inf_{n\in\mathbb{N}} p_n > \frac{k^k}{(k+1)^{k+1}},$$

then equation (1.1) *has a nonoscillatory solution.*

3. Sufficient Conditions for the Oscillation of Eq. (1.2)

In this section we extend the results from Section 2 to equation (1.2). We remark that throughout this paper we will use the convention that $0^0 = 1$. We first recall the following theorem [2]:

Theorem B. *Assume that* $p_{in} \geq 0$ *and*

$$\sum_{i=1}^{m} (\liminf_{n\to\infty} p_{in}) \frac{(k_i+1)^{k_i+1}}{{k_i}^{k_i}} > 1, \quad k_i \in \mathbb{N},\ i = 1, 2, ..., m.$$

Then every solution of (1.2) *oscillates.*

Note that Yan and Qian [7] proved Theorem B by using a different method from that used in [2].

Lemma 3.1. *Let* $k_i \in \{..., -3, -2, -1\}$ *and* $\limsup_{n\to\infty} p_{in} = p_i$ *for* $i = 1, 2, ..., m$. *If* $p_{in} \leq 0$ *and*

$$\sum_{i=1}^{m} p_i \frac{(k_i+1)^{k_i+1}}{{k_i}^{k_i}} > 1, \tag{3.1}$$

then the following holds:

(*i*) *the difference inequality*

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_{in} x_{n-k_i} \geq 0 \tag{3.2}$$

has no eventually positive solution,

(*ii*) *the difference inequality*

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_{in} x_{n-k_i} \leq 0 \tag{3.3}$$

has no eventually negative solution.

Proof. (*i*) Assume that x_n is an eventually positive solution of (3.2). So, there is a number $N_1 > 0$ such that $x_n > 0$ for all $n \geq N_1$. Let $z_n = \frac{x_{n+1}}{x_n}$. Then it is clear that x_n is nondecreasing and $z_n \geq 1$ for $n \geq N_1$. On the other hand, dividing the inequality (3.2) by x_n we have

$$z_n \geq 1 - \sum_{i=1}^{m} p_{in}\, z_{n-k_i-1} ... z_n \tag{3.4}$$

for all $n \geq N_1$, where $N = \max\{N_1, N_1 - k_1, ..., N_1 - k_m\}$. Let $\liminf_{n\to\infty} z_n = q$. Of course, $q \geq 1$. Taking $\liminf$ as $n \to \infty$ on both sides of (3.4) we may write

$$\begin{aligned} q &\geq 1 + \sum_{i=1}^{m} \liminf_{n\to\infty}(-p_{in}) \liminf_{n\to\infty} z_{n-k_i-1} ... \liminf_{n\to\infty} z_n \\ &= 1 - \sum_{i=1}^{m} \limsup_{n\to\infty} p_{in} \liminf_{n\to\infty} z_{n-k_i-1} ... \liminf_{n\to\infty} z_n \\ &= 1 - \sum_{i=1}^{m} p_i q^{-k_i}. \end{aligned}$$

Therefore,

$$\sum_{i=1}^{m} p_i q^{-k_i} \geq 1 - q,$$

which implies that $q \neq 1$ and that

$$\sum_{i=1}^{m} p_i \frac{q^{-k_i}}{1-q} \leq 1. \tag{3.5}$$

Now consider the function f defined by $f(q) = \frac{q^{-k_i}}{1-q}$. Then, observe that

$f'\left(\frac{k_i}{k_i+1}\right) = 0$ and $f''\left(\frac{k_i}{k_i+1}\right) < 0$. It follows that

$$\begin{aligned} \sum_{i=1}^{m} p_i \frac{(k_i+1)^{k_i+1}}{{k_i}^{k_i}} &= \sum_{i=1}^{m} p_i f\left(\frac{k_i}{k_i+1}\right) \\ &\le \sum_{i=1}^{m} p_i \frac{q^{-k_i}}{1-q}. \end{aligned}$$

Hence by (3.5)

$$\sum_{i=1}^{m} p_i \frac{(k_i+1)^{k_i+1}}{{k_i}^{k_i}} \le 1, \tag{3.6}$$

which contradicts condition (3.1).

(*ii*) By using similar method as in (*i*), the fact that (3.3) has no eventually negative solution is clear under condition (3.1). □

One can now deduce the following result.

Theorem 3.2. *Let* $k_i \in \{\ldots, -3, -2, -1\}$ *and* $\limsup_{n\to\infty} p_{in} = p_i$ *for* $i = 1, 2, \ldots, m$. *If* $p_{in} \le 0$ *and condition* (3.1) *holds, then every solution of equation* (1.2) *oscillates.*

Proof. Lemma 3.1 yields the result immediately. □

Theorem 3.2 and Theorem B contain the next results, respectively.

Corollary 3.3. *Let* $k_i \in \{..., -3, -2, -1\}$ *for* $i = 1, 2, ..., m$. *If* $p_{in} \le 0$ *and*

$$\sum_{i=1}^{m} \left(\sup_{n\in\mathbb{N}} p_{in}\right) \frac{(k_i+1)^{k_i+1}}{{k_i}^{k_i}} > 1, \tag{3.7}$$

then every solution of equation (1.2) *oscillates.*

Proof. Assume that (3.7) holds. Since $\limsup_{n\to\infty} p_{in} \le \sup_{n\in\mathbb{N}} p_{in}$ and $\frac{(k_i+1)^{k_i+1}}{{k_i}^{k_i}} < 0$ for $i = 1, 2, ..., m$, then, by (3.7), we may write

$$\sum_{i=1}^{m} \limsup_{n\to\infty} p_{in} \frac{(k_i+1)^{k_i+1}}{{k_i}^{k_i}} \ge \sum_{i=1}^{m} \left(\sup_{n\in\mathbb{N}} p_{in}\right) \frac{(k_i+1)^{k_i+1}}{{k_i}^{k_i}} > 1.$$

Therefore, the proof follows from Theorem 3.2. □

Corollary 3.4. *Let $k_i \in \mathbb{N}$ for $i = 1, 2, ..., m$. If $p_{in} \geq 0$ and*

$$\sum_{i=1}^{m} \left(\inf_{n\in\mathbb{N}} p_{in} \right) \frac{(k_i + 1)^{k_i+1}}{{k_i}^{k_i}} > 1, \tag{3.8}$$

then every solution of equation (1.2) *oscillates.*

Proof. Assume now that (3.8) holds. Since $\inf_{n\in\mathbb{N}} p_{in} \leq \liminf_{n\to\infty} p_{in}$ and also $\frac{(k_i + 1)^{k_i+1}}{{k_i}^{k_i}} > 0$, we obtain from (3.8) that

$$\sum_{i=1}^{m} \liminf_{n\to\infty} p_{in} \frac{(k_i + 1)^{k_i+1}}{{k_i}^{k_i}} > \sum_{i=1}^{m} \left(\inf_{n\in\mathbb{N}} p_{in} \right) \frac{(k_i + 1)^{k_i+1}}{{k_i}^{k_i}} > 1.$$

Combining this inequality with Theorem B the proof is completed. □

We now obtain the next results.

Theorem 3.5. *Let $k_i \in \{\ldots, -3, -2, -1\}$ and $\limsup_{n\to\infty} p_{in} = p_i$ for $i = 1, 2, \ldots, m$. If $p_{in} \leq 0$ and*

$$m \left(\prod_{i=1}^{m} |p_i| \right)^{1/m} > \left| \frac{(\bar{k})^{\bar{k}}}{(\bar{k} + 1)^{\bar{k}+1}} \right|, \tag{3.9}$$

where $\bar{k} = \frac{1}{m} \sum_{i=1}^{m} k_i$. Then every solution of (1.2) *oscillates.*

Proof. Assume that (y_n) is an eventually positive solution of equation (1.2). Then, by using (3.5) and (3.6), and also applying the arithmetic-geometric mean inequality, we conclude that

$$\begin{aligned} 1 &\geq \sum_{i=1}^{m} p_i \frac{q^{-k_i}}{1-q} \\ &\geq m \left[\prod_{i=1}^{m} p_i \frac{q^{-k_i}}{1-q} \right]^{1/m} \\ &= m \frac{q^{-(\bar{k})}}{q-1} \left[\prod_{i=1}^{m} (-p_i) \right]^{1/m} \\ &\geq m \left| \frac{(\bar{k}+1)^{\bar{k}+1}}{(\bar{k})^{\bar{k}}} \right| \left(\prod_{i=1}^{m} |p_i| \right)^{1/m}, \end{aligned}$$

which contradicts (3.9). In a similar way one can obtain that equation (1.2) has no eventually negative solution. □

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Received by the editors February 15, 2006