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ON RECTIFYING CURVES AS CENTRODES AND EXTREMAL CURVES IN THE MINKOWSKI 3-SPACE

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Abstract. In this paper, we characterize the spacelike, the timelike and the null rectifying curves in the Minkowski 3-space in terms of centrodes. In particular, we show that the spacelike and timelike rectifying curves are the extremal curves for which the corresponding function takes its extremal value. On the other hand, we also show that the null rectifying curves are not the extremal curves and give some interesting geometric properties of such curves.

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1. Introduction

The notion of rectifying curves was introduced by B. Y. Chen in [3]. By definition, a regular unit speed space curve $\alpha(s)$ in the Euclidean 3-space E^3 is called a *rectifying curve*, if its position vector always lies in its rectifying plane $\{T, B\}$, spanned by the tangent and the binormal vector field. Therefore, the position vector α of a rectifying curve satisfies the equation

$$\alpha(s) = \lambda(s)T(s) + \mu(s)B(s),$$

for some differentiable functions λ and μ in acrlength function s. There are many characterizations of the rectifying curves, lying fully in the Euclidean 3-space and in the Minkowski 3-space E_1^3. For example, a curve α in E^3 (or in E_1^3) is congruent to a rectifying curve if and only if the ratio of its torsion τ and its curvature κ, is a nonconstant linear function in arclength function s (see [3, 5]). The Euclidean rectifying curves are determined explicitly in [3, 5]. It is shown in [4] that there exists a simple relationship between the Euclidean rectifying curves and *centrodes*, which play important roles in mechanics, kinematics, as well as in differential geometry, in defining the curves of constant precession. Moreover, the Euclidean rectifying curves are the *extremal* curves which satisfy the equality case of a general inequality ([4]).

In this paper, we study the spacelike, the timelike and the null rectifying curves in the Minkowski 3-space. By using similar methods as in [4] we show

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that there is a simple relationship between the rectifying curves and centrodes. In particular, we prove that the non-null rectifying curves are the extremal curves, for which the corresponding function takes its extremal value. On the other hand, we prove that the null rectifying curves are not extremal curves, and give some interesting geometric properties of such curves.

2. Preliminaries

The Minkowski 3-space E_1^3 is a Euclidean 3-space E^3 provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. Since g is indefinite metric, recall that a vector $v \in E_1^3$ can be *spacelike* if $g(v,v) > 0$ or $v = 0$, *timelike* if $g(v,v) < 0$ and *null* if $g(v,v) = 0$ and $v \neq 0$. In particular, the norm (length) of a vector v is given by $||v|| = \sqrt{|g(v,v)|}$, and two vectors v and w are said to be orthogonal, if $g(v,w) = 0$. Next, recall that an arbitrary curve $\alpha(s)$ in E_1^3, can locally be *spacelike*, *timelike* or *null* (*lightlike*), if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null. If $g(\alpha'(s), \alpha'(s)) = \pm 1$, the non-null curve α is said to be of *unit speed* (or parameterized by arclength function s).

The Frenet frame $\{T, N, B\}$ of a unit speed non-null curve $\alpha(s)$ in E_1^3, with $g(\alpha''(s), \alpha''(s)) \neq 0$ for each s, is given by $T(s) = \alpha'(s)$, $N(s) = \alpha''(s)/||\alpha''(s)||$, $B(s) = T(s) \times N(s)$. Let us put $g(T,T) = \epsilon_0 = \pm 1$ and $g(N,N) = \epsilon_1 = \pm 1$. Then $g(B,B) = -\epsilon_0\epsilon_1$ and the following Frenet formulas hold ([6]):

$$\begin{aligned} T'(s) &= \epsilon_1 \kappa(s) N(s), \\ N'(s) &= -\epsilon_0 \kappa(s) T(s) - \epsilon_0 \epsilon_1 \tau(s) B(s), \\ B'(s) &= -\epsilon_1 \tau(s) N(s). \end{aligned} \tag{1}$$

The *vector product* $v \times w$ of two vectors v and w in E_1^3 is defined in [6] by the validity of the equation

$$g(v \times w, z) = det(v, w, z),$$

for all $z \in E_1^3$. Accordingly, the Frenet frame of α satisfies the equations

$$\begin{aligned} T \times N &= B, \\ N \times B &= -\epsilon_1 T, \\ B \times T &= -\epsilon_0 N. \end{aligned}$$

Recall that the *arclength function* s of a null curve β in E_1^3 is defined in [1] by $s(t) = \int_0^t \sqrt{||\beta''(u)||}\, du$. In particular, if $g(\beta''(s), \beta''(s)) = 1$, the null curve β is said to be *parameterized by the arclength function* s. The Frenet frame $\{T, N, B\}$

of a unit speed null curve $\beta(s)$ is given by $T(s) = \beta'(s)$, $N(s) = \beta''(s)$, and $B(s)$ is the unique null vector field such that $g(T, B) = 1$, $g(N, B) = 0$. This frame satisfies the equations

$$T \times N = T,$$
$$N \times B = B,$$
$$B \times T = N,$$

and the Frenet equations of β are given by ([8]):

$$\begin{aligned} T'(s) &= \kappa(s)N(s), \\ N'(s) &= \tau(s)T(s) - \kappa(s)B(s), \\ B'(s) &= -\tau(s)N(s), \end{aligned} \tag{2}$$

where the curvature $\kappa(s)$ takes only two values: $\kappa(s) = 0$ if β is a straight line, or $\kappa(s) = 1$ in all other cases.

When the Frenet frame moves along a curve in E^3_1, there exist an axis of instantaneaus frame's rotation. The direction of such axis is given by the *Darboux (rotation) vector*. If α is a unit speed non-null curve, the Darboux vector of α is given by

$$D_\alpha(s) = -\epsilon_0\epsilon_1\tau_\alpha(s)T_\alpha(s) - \epsilon_0\epsilon_1\kappa_\alpha(s)B_\alpha(s), \tag{3}$$

Moreover, if β is a unit speed null curve, the Darboux vector of β has the equation

$$D_\beta(s) = \tau_\beta(s)T_\beta(s) + \kappa_\beta(s)B_\beta(s), \tag{4}$$

A curve given by (3) or (4) is called respectively the *centrode* of α or β. Hence the *Darboux equations* ([6])

$$T' = D \times T,$$
$$N' = D \times N,$$
$$B' = D \times B,$$

are just a variant of the Frenet equations (1) and (2).

Recall that the *pseudosphere* and the *pseudohyperbolical space* with center at the origin and of radius 1 are hyperquadrics in E^3_1, respectively defined by ([7])

$$S^2_1(1) = \{v \in E^3_1 : g(v, v) = 1\},$$
$$H^2_0(1) = \{w \in E^3_1 : g(w, w) = -1\}.$$

3. Some known results

In this section we recall some theorems from [5], which are important for the proofs of theorems which follow.

Theorem A. *Let $\alpha = \alpha(s)$ be a unit speed non-null curve in E_1^3. Then the following statements hold:*

(i) α is a rectifying curve with a spacelike rectifying plane if and only if, up to the parametrization, α is given by

$$\alpha(t) = \frac{a}{\cos(t)} y(t), \qquad a \in R_0^+,$$

where $y(t)$ is a unit speed spacelike curve lying in $S_1^2(1)$.

(ii) α is a spacelike (timelike) rectifying curve with a timelike rectifying plane and a spacelike (timelike) position vector, if and only if up to the parametrization, α is given by

$$\alpha(t) = \frac{a}{\sinh(t)} y(t), \qquad a \in R_0^+,$$

where $y(t)$ is a unit speed timelike (spacelike) curve lying in $S_1^2(1)$ $(H_0^2(1))$.

(iii) α is a spacelike (timelike) rectifying curve with a timelike rectifying plane and a timelike (spacelike) position vector, if and only if up to the parametrization, α is given by

$$\alpha(t) = \frac{a}{\cosh(t)} y(t), \qquad a \in R_0^+,$$

where $y(t)$ is a unit speed spacelike (timelike) curve lying in $H_0^2(1)(S_1^2(1))$.

Theorem B. *Let $\alpha = \alpha(s)$ be a unit speed null curve in E_1^3, with the first curvature $k(s) = 1$. Then α is a rectifying curve with a spacelike (timelike) position vector if and only if , up to the parametrization, α is given by*

$$\alpha(t) = e^t y(t),$$

where $y(t)$ is a unit speed timelike (spacelike) curve lying in $S_1^2(1)$ $(H_0^2(1))$.

Theorem C. *Let $\alpha = \alpha(s)$ be a unit speed non-null curve in E_1^3, with a spacelike or a timelike rectifying plane and with the curvature $\kappa(s) > 0$. Then up to the isometries of E_1^3, the curve α is a rectifying if and only if there holds $\tau(s)/\kappa(s) = c_1 s + c_2$, where $c_1 \in R_0$, $c_2 \in R$.*

Theorem D. *Let $\alpha = \alpha(s)$ be a unit speed null curve in E_1^3 with the first curvature $k(s) = 1$. Then up to the isometries of E_1^3, the curve α is a rectifying if and only if there holds $\tau(s)/\kappa(s) = c_1 s + c_2$, where $c_1 \in R_0$, $c_2 \in R$.*

4. The rectifying curves as centrodes in $\mathbf{E_1^3}$

It is shown in [5] that there are no spacelike rectifying curves with the null principal normals in E_1^3. Consequently, in this section we show that the timelike, the null and the spacelike rectifying curves with the non-null principal normals in the Minkowski 3-space, are characterized in terms of centrodes, in a similar way as in E^3 (see [4]).

Theorem 4.1. *The centrode of a unit speed non-null curve $\alpha(s)$ in E_1^3, with constant curvature $\kappa_\alpha \neq 0$, nonconstant torsion and $g(\alpha''(s), \alpha''(s)) \neq 0$ is a* *non-null rectifying curve. Conversely, every unit speed non-null rectifying curve in E^3_1, is the centrode of some unit speed non-null curve with constant curvature $\kappa \neq 0$ and nonconstant torsion.*

Proof. First assume that $\alpha(s)$ is a unit speed non-null curve in $\mathbb{E}^3_1$ with the curvature $\kappa_\alpha(s) = c \in R_0$, the torsion $\tau_\alpha(s) \neq$ constant and $g(\alpha''(s), \alpha''(s)) \neq 0$. Consider the centrode of α given by

$$D_\alpha(s) = -\epsilon_0\epsilon_1\tau_\alpha(s)T_\alpha(s) - \epsilon_0\epsilon_1\kappa_\alpha(s)B_\alpha(s).$$

By taking the derivative of D_α with respect to s and applying the Frenet equations (1), we get

$$D'_\alpha(s) = -\epsilon_0\epsilon_1\tau'_\alpha(s)T_\alpha(s). \tag{5}$$

It follows that $g(D'_\alpha, D'_\alpha) \neq 0$, which means that D_α is a non-null curve. Denote by $\{T_D, N_D, B_D\}$ the Frenet frame of D_α. Then $T_D = D'_\alpha/||D'_\alpha||$, so relation (5) implies $T_D = \pm T_\alpha$. Consequently, $T'_D = \pm T'_\alpha$, which means that N_D and N_α are parallel vectors. This implies that B_D and B_α are also parallel vectors. Hence D_α lies in the plane $\{T_D, B_D\}$, so D_α is a rectifying curve.

Conversely, if $\alpha(s)$ is a unit speed non-null rectifying curve in E^3_1, Theorem C implies $\tau_\alpha(s)/\kappa_\alpha(s) = c_1 s + c_2$, $c_1 \in R_0$, $c_2 \in R$. We may assume that $\kappa_\alpha(s) = (c_0/s)\tau_\alpha(s)$, $c_0 \in R_0$. Next we define functions f and g by $f(s) = (1/c_0)\int_{s_0}^s \kappa_\alpha(u)\,du$ and $g(s) = f^{-1}(s)$. Up to the isometries of $\mathbb{E}^3_1$, there exist a unit speed non-null curve $\beta(t)$ such that $\kappa_\beta(t) = c_0$ and $\tau_\beta(t) = g(t)$. Let $g(T_\beta, T_\beta) = \epsilon = \pm 1$ and $g(N_\beta, N_\beta) = \eta = \pm 1$. By using (3), it follows that the centrode of β is given by

$$D_\beta(t) = -\epsilon\eta g(t)T_\beta(t) - \epsilon\eta c_0 B_\beta(t).$$

Let $\gamma(s) = D_\beta(f(s))$ be the reparametrization of D_β. Therefore,

$$\gamma(s) = -\epsilon\eta s T_\beta(f(s)) - \epsilon\eta c_0 B_\beta(f(s)).$$

We will prove that the curves α and γ are congruent. Differentiating the last equation with respect to s and using (1), we find

$$\gamma'(s) = -\epsilon\eta T_\beta(f(s)),$$

and thus $g(\gamma'(s), \gamma'(s)) = g(T_\beta(f(s)), T_\beta(f(s))) = \epsilon$. Consequently, $\gamma(s)$ is a unit speed non-null curve with

$$T_\gamma(s) = -\epsilon\eta T_\beta(f(s)).$$

Differentiating the previous equation two times with respect to s, and using (1), we find

$$|\kappa_\alpha(s)| = |\kappa_\gamma(s)|, \qquad |\tau_\alpha(s)| = |\tau_\gamma(s)|,$$

which means that α and γ are congruent curves. This proves the theorem. □

Remark 4.1. *Theorem 4.1 is also valid when the curve α has a nonconstant curvature and constant torsion $\tau_\alpha \neq 0$.*

It is shown in [2] that the Cartan frame of a null curve in E_1^4 can be defined to be parameter independent. Accordingly, we can define the Frenet frame of a null curve in E_1^3 in a similar way. Let $\beta(t)$ be a null curve in E_1^3, parameterized by an arbitrary parameter t. We define the tangent vector $T_\beta(t)$ and the principal normal vector $N_\beta(t)$ by

$$T_\beta(t) = \frac{\beta'(t)}{||\beta''(t)||}, \quad N_\beta(t) = T'_\beta(t). \tag{6}$$

Then the binormal vector $B_\beta(t)$ is the unique null vector field such that $g(T_\beta, B_\beta) = 1$, $g(N_\beta, B_\beta) = 0$. It is easy to see that the Frenet equations of $\beta(t)$ are given by

$$\begin{aligned} T'_\beta(t) &= \kappa_\beta(t)N_\beta(t), \\ N'_\beta(t) &= \tau_\beta(t)T_\beta(t) - \kappa_\beta(t)B_\beta(t), \\ B'_\beta(t) &= -\tau_\beta(t)N_\beta(t), \end{aligned} \tag{7}$$

where $\kappa_\beta(t) = 0$ if β is a straight line, or $\kappa_\beta(t) = 1$ in all other cases.

Theorem 4.2. *The centrode of a null rectifying curve $\beta(s)$, parameterized by arclength s, is in E_1^3 a null rectifying curve. Conversely, every null rectifying curve $\beta(s)$, parameterized by acrlength s, is in E_1^3 the centrode of some null curve $\gamma(t)$ with the torsion $\tau_\gamma(t) = ct$, where t is the arclength parameter and $c \in R_0$.*

Proof. Let $\beta(s)$ be a null rectifying curve in E_1^3, parameterized by arclength s and with $\kappa_\beta(s) = 1$. By theorem D, it follows that $\tau_\beta(s)/\kappa_\beta(s) = c_1 s + c_2$, where $c_1 \in R_0$, $c_2 \in R$, and hence $\tau_\beta(s) = c_1 s + c_2$. Consider the centrode of β given by

$$D_\beta(s) = \tau_\beta(s)T_\beta(s) + \kappa_\beta(s)B_\beta(s). \tag{8}$$

Differentiating the previous equation with respect to s and using (2), we find

$$D'_\beta(s) = c_1 T_\beta(s). \tag{9}$$

Consequently, $g(D'_\beta(s), D'_\beta(s)) = 0$, so D_β is a null curve. Differentiating (9) with respect to s yields $D''_\beta(s) = c_1 N_\beta(s)$. If $\{T_D, N_D, B_D\}$ is the Frenet frame of D_β, relation (6) implies $T_D(s) = D'_\beta(s)/||D''_\beta(s)||$, $N_D(s) = T'_D(s)$. Consequently, $T_D(s) = T_\beta(s)$, $N_D(s) = N_\beta(s)$, which implies $B_D(s) = B_\beta(s)$. Therefore, the position vector of the centrode D_β given by (8) lies in the plane $\{T_D, B_D\}$, which means that D_β is a rectifying curve.

Conversely, let $\beta(s)$ be a null rectifying curve in E_1^3, parameterized by the arclength s and with $\kappa_\beta(s) = 1$. Then Theorem D implies $\tau_\beta(s)/\kappa_\beta(s) = c_1 s + c_2$, whereby $c_1 \in R_0$, $c_2 \in R$, and thus $\tau_\beta(s) = c_1 s + c_2$. We may assume that $\tau_\beta(s) = c_1 s$. Next, we define the functions f and g by $f(s) = c_1 s$, $g(s) = f^{-1}(s)/c_1$. Up to the isometries of E^3_1, there exists null curve $\gamma(t)$ such that $\kappa_\gamma(t) = 1$, $\tau_\gamma(t) = g(t)$, where t is the arclength function of γ. By relation (4), the centrode of γ is given by

$$D_\gamma(t) = \tau_\gamma(t) T_\gamma(t) + \kappa_\gamma(t) B_\gamma(t).$$

Consequently, $D_\gamma(t) = g(t)T_\gamma(t) + B_\gamma(t)$. Let $\delta(s) = D_\gamma(f(s))$ be the reparametrization of D_γ. It follows that

$$\delta(s) = \frac{1}{c_1} s T_\gamma(f(s)) + B_\gamma(f(s)). \tag{10}$$

We will prove that the curves β and δ are congruent. Differentiating (10) with respect to s and using (7), we get

$$\delta'(s) = \frac{1}{c_1} T_\gamma(f(s)).$$

Differentiating the previous equation with respect to s and using (7), yields

$$\delta''(s) = N_\gamma(f(s)).$$

Therefore, $g(\delta''(s), \delta''(s)) = 1$. Let $\{T_\delta, N_\delta, B_\delta\}$ be the Frenet frame of δ. Then relation (6) implies $T_\delta(s) = \delta'(s)$, $N_\delta(s) = T'_\delta(s)$, and accordingly

$$T_\delta(s) = \frac{1}{c_1} T_\gamma(f(s)), \quad N_\delta(s) = N_\gamma(f(s)). \tag{11}$$

By taking the derivative of N_δ with respect to s and using (7), we obtain

$$N'_\delta(s) = s T_\gamma(f(s)) - c_1 B_\gamma(f(s)). \tag{12}$$

Next, relations (11) and (12) imply

$$N'_\delta(s) = c_1 s T_\delta(s) - c_1 B_\gamma(f(s)). \tag{13}$$

On the other hand, from (7) we have

$$N'_\delta(s) = \tau_\delta(s) T_\delta(s) - B_\delta(s), \tag{14}$$

so relations (13) and (14) yield

$$\tau_\delta(s) = c_1 s, \qquad B_\delta(s) = c_1 B_\gamma(f(s)).$$

Since $\kappa_\beta(s) = \kappa_\delta(s) = 1$ and $\tau_\beta(s) = \tau_\delta(s) = c_1 s$, it follows that β and γ are congruent curves, which proves the theorem. $\square$

5. The rectifying curves as extremal curves in $\mathbf{E}_1^3$

If α is a non-null curve given by $\alpha(t) = \rho(t)y(t)$, where $\rho(t) \neq 0$ is an arbitrary function and $y(t)$ is a curve lying in $S_1^2(1)$ (or in $H_0^2(1)$), then y is called the *pseudospherical projection* of α (or the *projection* of α in $H_0^2(1)$). Let κ_α and $\vartheta_\alpha = ||\alpha'||$ be the curvature and the speed of α, and let κ_g be the geodesic curvature of y. We consider the function F_α defined by $F_\alpha = \vartheta_\alpha^4 \kappa_\alpha^2/\rho^2$. In the following theorems, we show that in the set of non-null curves with the same projection y, only the rectifying curve α has a property that the corresponding function F_α takes its extremal value, equal to κ_g^2. Therefore, the non-null rectifying curves are the *extremal curves*. Depending on the causal character of curves α and y, the extremal value κ_g^2 can be the maximum or the minimum value of the function F_α.

Theorem 5.1. *If $y(t)$ is a unit speed spacelike curve in $S_1^2(1)$ with the geodesic curvature κ_g, $\rho > 0$ arbitrary function and $\alpha(t) = \rho(t)y(t)$ the curve with the spacelike rectifying plane in E_1^3, then*

$$F_\alpha \leq \kappa_g^2,$$

where the equality sign holds if and only if α is a rectifying curve.

Proof. Assume that the curve α is given by $\alpha(t) = \rho(t)y(t)$, where $y(t)$ is a unit speed spacelike curve in $S_1^2(1)$ and $\rho(t) > 0$. Also assume that α has the spacelike rectifying plane. Since

$$\alpha'(t) = \rho'(t)y(t) + \rho(t)y'(t),$$

it follows that

$$g(y \times y', \alpha') = 0.$$

Thus $y \times y'$ lies in the timelike plane spanned by $\{N_\alpha, B_\alpha\}$ and has the equation

$$y \times y' = \cosh(\gamma)N_\alpha + \sinh(\gamma)B_\alpha, \tag{15}$$

where $\gamma = \gamma(t)$ is an arbitrary function. Next, we may decompose the vector field $y''(t)$ with respect to the orthonormal frame $\{y, y', y \times y'\}$ by

$$y'' = g(y'', y)g(y, y)y + g(y'', y')g(y', y')y' + g(y'', y \times y')g(y \times y', y \times y')y \times y'.$$

By definition, the geodesic curvature κ_g of y is given by

$$\kappa_g(t) = g(y''(t), y(t) \times y'(t)),$$

and hence

$$y''(t) = -y(t) - \kappa_g(t)y(t) \times y'(t). \tag{16}$$

Differentiating (15) with respect to t, using (16) and Frenet equations for arbitrary speed curves, we obtain

$$-\kappa_g y' = \vartheta_\alpha \kappa_\alpha \cosh(\gamma)T_\alpha + (\gamma' + \tau_\alpha \vartheta_\alpha)(\sinh(\gamma)N_\alpha + \cosh(\gamma)B_\alpha), \tag{17}$$

where $\vartheta_\alpha = ||\alpha'||$ is the speed of α. Since $y' \times (y \times y') = -y$, it follows that

$$\kappa_g y' \times (y \times y') = -\kappa_g y. \tag{18}$$

Substituting (15) and (17) in (18), we get

$$\kappa_g y = (\gamma' + \tau_\alpha \vartheta_\alpha) T_\alpha - \vartheta_\alpha \kappa_\alpha \cosh(\gamma)(\sinh(\gamma) N_\alpha + \cosh(\gamma) B_\alpha). \tag{19}$$

Next, taking the scalar product of (19) with T_α, we find

$$\kappa_g g(y, T_\alpha) = \gamma' + \tau_\alpha \vartheta_\alpha. \tag{20}$$

Moreover, since $T_\alpha = (\rho'/\vartheta_\alpha) y + (\rho/\vartheta_\alpha) y'$, it follows that

$$g(y, T_\alpha) = \frac{\rho'}{\vartheta_\alpha}. \tag{21}$$

Then (20) and (21) imply

$$\gamma' + \tau_\alpha \vartheta_\alpha = \frac{\kappa_g \rho'}{\vartheta_\alpha}. \tag{22}$$

Substituting (22) in (17), we obtain

$$-\vartheta_\alpha \kappa_g y' = \vartheta_\alpha^2 \kappa_\alpha \cosh(\gamma) T_\alpha + \kappa_g \rho' (\sinh(\gamma) N_\alpha + \cosh(\gamma) B_\alpha).$$

Next, using the equation $g(y'(t), y'(t)) = 1$ we get

$$\kappa_g^2 = \frac{\vartheta_\alpha^4 \kappa_\alpha^2 \cosh^2(\gamma)}{\rho^2}.$$

Consequently, the last equation implies the inequality

$$\kappa_g^2 \geq \frac{\vartheta_\alpha^4 \kappa_\alpha^2}{\rho^2} = F_\alpha. \tag{23}$$

It is easy to see that in (23) the equality sign holds if and only if $\cosh(\gamma) = 1$. Moreover, relation (15) implies $\cosh(\gamma) = 1$ if and only if $y \times y' = N_\alpha$. On the other hand, differentiating the equation $\alpha(t) = \rho(t) y(t)$ with respect to t, we get

$$\alpha'' = \rho'' y + 2\rho' y' + \rho y''.$$

Next, using relation (16) we obtain

$$(\alpha' \times \alpha'') \times \alpha' = (2\rho'^2 - \rho\rho'' + \rho^2)(\rho y - \rho' y') + \rho(\rho^2 - \rho'^2) \kappa_g y \times y'. \tag{24}$$

Since $\alpha' \times \alpha''$ is in the direction of B_α, it follows that $(\alpha' \times \alpha'') \times \alpha'$ is in the direction of $N_\alpha = y \times y'$. Accordingly, relation (24) implies the differential equation

$$2\rho'^2 - \rho\rho'' + \rho^2 = 0.$$

The solution of the previous equation is given by $\rho(t) = a/\cos(t)$, $a \in R_0^+$. Therefore, by statement (i) of Theorem A it follows that α is a rectifying curve if and only if in (23) the equality sign holds. This completes the proof of the theorem.

In a similar way, we obtain the following two theorems. □

Theorem 5.2. *If $y(t)$ is a unit speed timelike (spacelike) curve in $S_1^2(1)$ ($H_0^2(1)$) with the geodesic curvature κ_g, $\rho > 0$ arbitrary function and $\alpha(t) = \rho(t)y(t)$ the timelike curve in E_1^3, then*

$$F_\alpha \geq \kappa_g^2,$$

where the equality sign holds if and only if α is a rectifying curve.

Theorem 5.3. *If $y(t)$ is a unit speed timelike (spacelike) curve in $S_1^2(1)$ ($H_0^2(1)$) with the geodesic curvature κ_g, $\rho > 0$ arbitrary function and $\alpha(t) = \rho(t)y(t)$ the spacelike curve with the timelike rectifying plane in E_1^3, then*

$$F_\alpha \leq \kappa_g^2,$$

where the equality sign holds if and only if α is a rectifying curve.

Next we consider the null curve $\beta(t)$ in E_1^3 and its corresponding function $F_\beta = \vartheta_\beta^4 \kappa_\beta^2/\rho^2$, where $\vartheta_\beta(t) = \sqrt{||\beta''(t)||}$, $\kappa_\beta(t) = 1$, and $\rho > 0$, is an arbitrary differentiable function. In the following theorem we show that in the set of null curves with the same projection y, lying in $S_1^2(1)$ or in $H_0^2(1)$, all curves are rectifying. Consequently, the null rectifying curves are not the extremal curves.

Theorem 5.4. *If $y(t)$ is a unit speed timelike (spacelike) curve in $S_1^2(1)$ ($H_0^2(1)$) with the geodesic curvature κ_g, $\rho > 0$ arbitrary function and $\beta(t) = \rho(t)y(t)$ the null curve in E_1^3, then:*

(i) β is a rectifying curve;

(ii) $F_\beta = \kappa_g^2$;

(iii) $\kappa_g(t) = ce^{3t}$, $c \in R_0^+$.

Proof. Let $y(t)$ be a unit speed timelike curve in $S_1^2(1)$ and $\rho(t) > 0$ an arbitrary function. If β is a null curve given by $\beta(t) = \rho(t)y(t)$, then $\beta'(t) = \rho'(t)y(t) + \rho(t)y'(t)$, and hence $g(\beta'(t), \beta'(t)) = \rho'^2(t) - \rho^2(t) = 0$. It follows that $\rho(t) = e^t$, so Theorem B implies that β is a rectifying curve. This proves statement (i).

By statement (i), β is a rectifying curve with the equation $\beta(t) = e^t y(t)$. Differentiating the previous equation two times with respect to t, we find

$$\beta''(t) = e^t(y(t) + 2y'(t) + y''(t)). \tag{25}$$

By assumption of the theorem, vector fields y, y', and $y \times y'$ form an orthonormal frame along y. Consequently, we may decompose the vector field y'' by

$$y'' = g(y'', y)g(y, y)y + g(y'', y')g(y', y')y' + g(y'', y \times y')g(y \times y', y \times y')y \times y'. \tag{26}$$

By definition, the geodesic curvature κ_g of y is given by

$$\kappa_g(t) = g(y(t) \times y'(t), y''(t)). \tag{27}$$

Further relations (26) and (27) imply $y''(t) = y(t) + \kappa_g(t) y(t) \times y'(t)$, and therefore

$$g(y'', y'') = 1 + \kappa_g^2. \tag{28}$$

Using (25) and (28) we get

$$g(\beta''(t), \beta''(t)) = e^{2t} \kappa_g^2(t). \tag{29}$$

Therefore, since $\vartheta_\beta(t) = \sqrt{||\beta''(t)||}$, $\kappa_\beta(t) = 1$ and using (29), we obtain

$$F_\beta = \vartheta_\beta^4 \kappa_\beta^2 / \rho^2 = \kappa_g^2.$$

This proves statement (ii).

By statement (i), β is a rectifying curve, so by definition the position vector of β has the equation

$$\beta(s) = \lambda(s) T_\beta(s) + \mu(s) B_\beta(s), \tag{30}$$

for some differentiable functions λ and μ in the arclength function s. Differentiating (30) with respect to s and using (2), we obtain

$$\lambda(s) = s + a_1, \quad a_1 \in R, \quad \mu(s) = a_2, \quad a_2 \in R_0. \tag{31}$$

Further relations (30) and (31) imply

$$g(\beta(s), \beta(s)) = c_1 s + c_2, \quad c_1 \in R_0, \quad c_2 \in R.$$

On the other hand, since $\beta(t) = e^t y(t)$, we easily get

$$g(\beta(t), \beta(t)) = e^{2t},$$

where $t = t(s)$. It follows that

$$e^{2t} = c_1 s + c_2. \tag{32}$$

Moreover, since $s(t) = \int_0^t \sqrt{||\beta''(u)||}\, du$, we find $(ds/dt)^4 = g(\beta''(t), \beta''(t))$, so relation (29) implies

$$s'(t) = \sqrt{e^t \kappa_g(t)}. \tag{33}$$

On the other hand, relation (32) implies

$$s'(t) = 2e^{2t}/c_1. \tag{34}$$

Finally, from (33) and (34) we obtain $\kappa_g(t) = ce^{3t}$, $c \in R_0^+$, which proves statement (iii). □

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