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EXTENSION OF LIAPUNOV THEORY TO FIVE-POINT BOUNDARY VALUE PROBLEMS FOR THIRD ORDER DIFFERENTIAL EQUATIONS

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Abstract. This paper presents criteria for the existence and uniqueness of solutions to five-point boundary value problems associated with third order differential equations by using matching technique. 'Liapunov-like' functions are used as a tool to establish existence and uniqueness of solutions by matching two three-point boundary value problems.

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1. Introduction

In this paper we study the problem of the existence and uniqueness of solutions of five-point boundary value problems for the third order differential equation

$$y''' = f(x, y, y', y''). \tag{1.1}$$

Here f is assumed to be continuous on $[a, c] \times R^3$ and unique solutions to initial value problems associated with (1.1) exist and extend throughout $[a, c]$.

Several authors [1]-[8] used matching technique of solutions to obtain existence and uniqueness of solutions to three-point boundary value problems associated with $n^{th}(n \geq 3)$ order nonlinear differential equations.

The approach taken here is similar to that of Barr and Sherman [2] and Rao, Murty and Murty [7], and is based on the use of a solution matching technique and suitable 'Liapunov-like' function defined later. Recently, Henderson and Tisdel [5] obtained existence and uniqueness of the solutions of five-point boundary value problems with the following monotonicity assumption on f:
For all $w \in R$, $f(x, v_1, v_2, w) > f(x, u_1, u_2, w)$,

(i) when $x \in (a, b]$, $u_1 \geq v_1$ and $v_2 > u_2$, or

(ii) when $x \in [b, c)$, $u_1 \leq v_1$ and $v_2 > u_2$.

In this paper we replace this monotonicity condition by an appropriate 'Liapunov-like' function and establish existence and uniqueness of the solutions of five-point boundary value problems.

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Section 2 gives criteria under which solutions of (1.1) which satisfy boundary conditions at three points may be matched to obtain a unique solution of (1.1) satisfying boundary conditions at five points.

In section 3, with the help of a suitable 'Liapunov-like' function, we obtain at most one solution to the following three-point boundary value problems (1.1) satisfying

$$y(a) - y(x_1) = y_1,\ y(b) = y_2,\ y^{(i)}(b) = m\ (i = 1, 2) \tag{1.2_i}$$

and

$$y(b) = y_2,\ y^{(i)}(b) = m,\ y(x_2) - y(c) = y_3\ (i = 1, 2). \tag{1.3_i}$$

Further, using the hypothesis that the solutions exist for the problems (1.1) satisfying (1.2_i) and (1.3_i) a unique solution to the five-point boundary value problem (1.1) satisfying

$$y(a) - y(x_1) = y_1,\ y(b) = y_2,\ y(x_2) - y(c) = y_3 \tag{1.4}$$

is constructed.

2. Existence and uniqueness of solutions

In this section the following theorem illustrates how the solutions of two three-point boundary value problems are matched to obtain a unique solution to the five-point boundary value problem.

Theorem 2.1. *Let $y_1, y_2, y_3, b \in R$ with $a < x_1 < b < x_2 < c$ and suppose that*

(i) For each $m \in R$ there exist solutions of (1.1) satisfying (1.2_i) *or* (1.3_i)$(i = 1, 2)$.

(ii) For each $m \in R$ and each t there exists at most one solution of each of the following boundary value problems (1.1) satisfying

$$y(a) - y(x_1) = y_1,\ \ y(b) = y_2,\ \ y''(t) = m,\ where\ \ t \in (a, b]$$

and

$$y(b) = y_2,\ \ y''(t) = m,\ \ y(x_2) - y(c) = y_3,\ where\ \ t \in [b, c).$$

Then there exists a unique solution to the boundary value problem (1.1) satisfying (1.4).

Proof. Let $y_1(x, m)$ denote a solution of (1.1) satisfying (1.2_2) with the second derivative m at $x = b$. First we show that $y_1'(b, m)$ is an increasing function of m. If $m_2 > m_1$, by definition $y_1''(b, m_2) > y_1''(b, m_1)$. Let $w(x) = y_1(x, m_2) - y_1(x, m_1)$. We claim that $w''(x) > 0$ for all $x \in (a, b]$. Suppose to the contrary that there exists a point $p \in (a, b)$ such that $w''(p) \leq 0$. Since

$w''(x)$ is continuous, there exists a point $q \in [p, b)$ such that $w''(q) = 0$. This implies that $y_1''(q, m_2) = y_1''(q, m_1) = k$, which is a contradiction to our hypothesis (ii). Since $w'(x)$ is increasing on $(a, b]$ and $w(a) = w(x_1)$, there exists a point $r \in (a, x_1)$ such that $w'(r) = 0$ and $w'(x) > 0$ for all $x \in (r, b]$. In particular $w'(b) > 0$, and hence $y_1'(b, m)$ is strictly increasing function of m.

Let $y_2(x, m)$ denote a solution of (1.1) satisfying (1.3_2) with the second derivative m at $x = b$. In a similar way it can be shown that $y_2'(b, m)$ is a strictly decreasing function of m.

Now it is claimed that $y_1'(b, m)$ has no jump discontinuities as a function of m. Suppose to the contrary that $y_1'(b, m)$ has a jump discontinuity at $m = m_1$ and if $y_1'(b, m_1^-) = u$, $y_1'(b, m_1) = v$, $y_1'(b, m_1^+) = w$, then $u < v < w$. Choose $k \neq v$ and consider a solution $Y(x)$ of (1.1) satisfying $y(a) - y(x_1) = y_1$, $y(b) = y_2$, $y'(b) = k$. Since $Y''(b)$ exists, $Y''(b) = p$. For $p = m_1$, $y_1(x, p) \equiv Y(x)$ which leads to a contradiction. Similarly, $y_2'(b, m)$ has no jump discontinuities.

Now we show that $y_1'(b, .) : R \overset{onto}{\to} R$.

Let $z_0 \in R$, the boundary value problem (1.1) satisfying $y(a) - y(x_1) = y_1$, $y(b) = y_2$, $y'(b) = z_0$ has a solution ϕ. Let $\phi''(b) = q$, then $y_1(x, q) \equiv \phi(x)$ implies $y_1'(b, q) = \phi'(b) = z_0$. Similarly, $y_2'(b, .)$ also maps from R onto R.

Thus both $y_1'(b, m)$ and $y_2'(b, m)$ are continuous strictly monotone functions of m, whose ranges are the set of real numbers. Denote $Y'(b, m) = y_1'(b, m) - y_2'(b, m)$

$$Y'(b, m) \to \infty \quad as \quad m \to +\infty$$

$$Y'(b, m) \to -\infty \quad as \quad m \to -\infty$$

. Thus there exists an $m_0 \in R$ such that $y_1'(b, m_0) = y_2'(b, m_0)$. By definition of $y_1(x, m_0)$ and $y_2(x, m_0)$ we have $y_1(b, m_0) = y_2(b, m_0)$ and $y_1''(b, m_0) = y_2''(b, m_0)$. Thus

$$y(x) = \begin{cases} y_1(x, m_0), & a \leq x \leq b \\ y_2(x, m_0), & b \leq x \leq c \end{cases}$$

is a solution of (1.1) satisfying (1.4).

To establish uniqueness, suppose that $y_1(x)$ and $y_2(x)$ are two distinct solutions of (1.1) satisfying (1.4). Let $w(x) = y_1(x) - y_2(x)$, then $w(a) - w(x_1) = 0$, $w(b) = 0$, $w(x_2) - w(c) = 0$. Thus there exists a point $p_1 \in (a, x_1)$ and a point $p_2 \in (x_2, c)$ such that $w'(p_1) = w'(p_2) = 0$. This implies that there exists a point $p_3 \in (p_1, p_2)$ such that $w''(p_3) = 0$, i.e $y_1''(p_3) = y_2''(p_3)$ which is again a contradiction to hypothesis (ii). □

The matching of solutions in the above theorem in hypothesis (i) was accomplished by depending on hypothesis (ii), which is related to uniqueness of the solutions of four-point boundary value problems. Hence it is preferable to match solutions of three-point boundary value problems without the help of a hypothesis involving four-point boundary value problems. This was achieved in the next section with the use of 'Liapunov-like' function.

3. Liapunov function - Existence and uniqueness of solutions

In this section, we define the Liapunov function and establish lemmas which are useful for proving our main theorem regarding the existence and uniqueness of five-point boundary value problems.

Suppose y_1 and y_2 be two solutions of (1.1) satisfying (1.2_i) or (1.3_i) $(i = 1, 2)$, write $y = y_1 - y_2$ then

$$y''' = F(x, y, y', y'') = f(x, y + y_2, y' + y_2', y'' + y_2'') - f(x, y_2, y_2', y_2'') \tag{3.1}$$

and $F(x, 0, 0, 0) = 0$. The boundary conditions (1.2_i), (1.3_i) respectively become

$$y(a) - y(x_1) = 0,\ y(b) = 0,\ y^{(i)}(b) = 0\ (i = 1, 2) \tag{3.2_i}$$

and

$$y(b) = 0,\ y^{(i)}(b) = 0,\ y(x_2) - y(c) = 0\ (i = 1, 2). \tag{3.3_i}$$

Hence $y(x) \equiv 0$ is a solution of (3.1) satisfying (3.2_i) or $(3.3_i)(i = 1, 2)$. Thus we have proved the following.

Lemma 3.1. *The problem (1.1) satisfying* (1.2_i) *or* $(1.3_i)(i = 1, 2)$ *has a unique solution if and only if* $y(x) \equiv 0$ *is the only solution of (3.1) satisfying* (3.2_i) *or* $(3.3_i)(i = 1, 2)$.

Definition 3.1. *A Liapunov function* $V(x, y, y', y'')$ *is a continuous locally Lipschitzian real valued function with respect to* (y, y', y''). *Corresponding to* $V(x, y, y', y'')$ *we define*

$$V_f'(x, y, y', y'') = \lim_{h \to 0^+} \inf \frac{1}{h}[V(x + h, y + hy', y' + hy'', y'' + hf) - V(x, y, y', y'')]$$

$$V'(x, y, y', y'') = \lim_{h \to 0^+} \inf \frac{1}{h}[V(x+h, y(x+h), y'(x+h), y''(x+h)) - V(x, y, y', y'')]$$

where f *is a function defined and continuous on a domain* $M = [a, c] \times N$, *where* $[a, c]$ *is an interval on the real line and* $N \subset R^3$. *Choose* $M = M_1 \cup M_2$, *where* $M_1 = [a, b] \times N$ *and* $M_2 = [b, c] \times N$.

Lemma 3.2. *If* $V(x, y, y', y'')$ *is a Liapunov function and* $y(x)$ *is a solution of (1.1) then* $V'(x, y, y', y'') = V_f'(x, y, y', y'')$ *and* $V(x, y, y', y'')$ *is nonincreasing(nondecreasing) if and only if* $V_f'(x, y, y', y'') \leq 0$ $(V_f'(x, y, y', y'') \geq 0)$.

Proof. Analogous to the proof of Yoshizawa [p.4 of [9]]. □

Lemma 3.3. *For F defined in (3.1), if there exists a Liapunov function $V(x,y,y',y'')$ defined on M_1 such that*
(i) $V(x,y,y',y'')=0$ if $y=0$
(ii) $V(x,y,y',y'')>0$ if $y\neq 0$
(iii) $V'_F(x,y,y',y'')\geq 0$ in the interior of M_1.
Then for each $m\in R$, there exists at most one solution to the three-point boundary value problems (1.1) satisfying $(1.2_i)(i=1,2)$.

Proof. The proof of the problem (1.1) satisfying (1.2_2) will be given. A similar proof holds for the other boundary value problem. Suppose $y_1(x)$ and $y_2(x)$ are two distinct solutions of (1.1) satisfying (1.2_2). Write $w(x)=y_1(x)-y_2(x)$, then

$$w'''=F(x,w,w',w''),\ w(a)-w(x_1)=0,\ w(b)=0,\ w''(b)=0, \tag{3.4}$$

where $F(x,0,0,0)=0$. From Lemma 3.1 it suffices to show that $w(x)\equiv 0$ is the only solution of (3.4). Suppose $\phi(x)$ is a nontrivial solution of (3.4), then there exists a $\eta\in(a,b)$ such that $\phi(\eta)\neq 0$. Hence

$$V(\eta,\phi(\eta),\phi'(\eta),\phi''(\eta))>0. \tag{3.5}$$

Since $V'_F(x,y,y',y'')\geq 0$ in the interior of M_1 and from Lemma 3.2 it follows that $V(x,y,y',y'')$ is nondecreasing along the solution $\phi(x)$. Thus $\eta<b$ implies

$$V(\eta,\phi(\eta),\phi'(\eta),\phi''(\eta))\leq V(b,\phi(b),\phi'(b),\phi''(b))=0. \tag{3.6}$$

Hence (3.5) and (3.6) contradict each other and hence $y_1(x)\equiv y_2(x)$. □

Lemma 3.4. *For F defined in (3.1), if there exists a Liapunov function $V(x,y,y',y'')$ defined on M_2 such that*
(i) $V(x,y,y',y'')=0$ if $y=0$
(ii) $V(x,y,y',y'')>0$ if $y\neq 0$
(iii) $V'_F(x,y,y',y'')\leq 0$ in the interior of M_2.
Then for each $m\in R$, there exists at most one solution to the three-point boundary value problems (1.1) satisfying (1.3_i) $(i=1,2)$.

Proof. Analogous to the proof of Lemma 3.3. □

Theorem 3.1. *Let $y_1,y_2,y_3,b\in R$ with $a<x_1<b<x_2<c$. Suppose that*
(i) For each $m\in R$ there exist solutions of (1.1) satisfying (1.2_i) or (1.3_i) $(i=1,2)$.
(ii) $V(x,y,y',y'')$ is a Liapunov function as in Lemmas 3.3 and 3.4.
Then there exists at most one solution to the boundary value problem (1.1) satisfying (1.4).

Proof. From Lemmas 3.3 and 3.4 the solutions of (1.1) satisfying (1.2_i) or (1.3_i)($i = 1, 2$) are unique. Let $y_1(x, m)$ denote the solution of (1.1) satisfying (1.2_2). If $m_2 > m_1$, then $y_1''(b, m_2) > y_1''(b, m_1)$. Let $w(x) = y_1(x, m_2) - y_1(x, m_1)$, since $w''(x)$ is continuous and $w''(b) > 0$, either

(1) $w''(x) > 0$ for all $x \in [a, b]$

(2) $w''(a) = 0$ and $w''(x) > 0$ for all $x \in (a, b]$

or

(3) there exists a point $q \in (a, b)$ such that $w''(q) = 0$ and $w''(x) > 0$ for all $x \in (q, b]$ holds.

First we show that in all the above three cases there exists a point $p \in (a, b)$ such that $w'(x) > 0$ for all $x \in (p, b]$. Suppose that (1) or (2) holds. Since $w(a) = w(x_1)$ there exists a point $p \in (a, x_1)$ such that $w'(p) = 0$. This, together with (1) or (2) implies $w'(x) > 0$ for all $x \in (p, b]$.

Suppose (3) holds. We claim that there exists a point $p \in [q, b)$ such that $w'(p) > 0$, thus from (3), $w'(x) > 0$ for all $x \in (p, b]$. Suppose to the contrary that $w'(x) < 0$ for all $x \in [q, b)$. Since $w(b) = 0$ implies $w(q) > 0$. Hence

$$V(q, w(q), w'(q), w''(q)) > 0. \tag{3.7}$$

Since $V_F'(x, y, y', y'') \geq 0$ in the interior of M_1 and from Lemma 3.2 it follows that $V(x, y, y', y'')$ is nondecreasing along $w(x)$. Since $q < b$ implies

$$V(q, w(q), w'(q), w''(q)) \leq V(b, w(b), w'(b), w''(b)) = 0. \tag{3.8}$$

Thus (3.7) and (3.8) contradict each other. In particular in all three cases $w'(b) > 0$, therefore $y_1'(b, m)$ is a strictly increasing function of m.

Let $y_2(x, m)$ denote a solution of (1.1) satisfying (1.3_2). A proof similar to the above shows that $y_2'(b, m)$ is a strictly decreasing function of m.

The remaining proof of the existence of solutions of five-point boundary value problem (1.1) satisfying (1.4) is obtained by matching the solutions of two three-point boundary value problems (1.1) satisfying (1.2_2) and (1.1) satisfying (1.3_2) and follows as in the proof of Theorem 2.1.

To establish uniqueness, suppose that $y_1(x)$ and $y_2(x)$ are two distinct solutions of (1.1) satisfying (1.4). Write $y(x) = y_1(x) - y_2(x)$, then

$$y'''(x) = F(x, y, y', y'') = 0, y(a) - y(x_1) = 0, y(b) = 0, y(x_2) - y(c) = 0, \tag{3.9}$$

where $F(x, 0, 0, 0) = 0$. Suppose $y_0(x)$ is a non-trivial solution of (3.9). There exists a point $p \in [a, c]$ such that $y_0(p) \neq 0$. Since $y_0(b) = 0$, we have either $p \in [a, b)$ or $p \in (b, c]$ and $y_0(p) \neq 0$. For $p \in [a, b)$ we have

$$V(p, y_0(p), y_0'(p), y_0''(p)) > 0. \tag{3.10}$$

Since $V_F'(x, y, y', y'') \geq 0$ in the interior of M_1 and from Lemma 3.2 it follows that $V(x, y, y', y'')$ is nondecreasing along $y_0(x)$. Since $p < b$ implies

$$V(p, y_0(p), y_0'(p), y_0''(p)) \leq V(b, y_0(b), y_0'(b), y_0''(b)) = 0, \tag{3.11}$$

which is again a contradiction. Similarly, the other case follows. Hence the uniqueness holds. □

Remark 3.1. It may be noted that even when the monotonicity condition is not satisfied by f, a Liapunov function satisfying all the requirements as in Theorem 3.1 may exist to ensure the existence and uniqueness of solutions to five-point boundary value problems as seen from the following example:

$$y''' + y' = 0 \tag{3.12}$$

$$y(0) - y(\frac{\pi}{4}) = \frac{-1}{\sqrt{2}}, \;\; y(\frac{\pi}{2}) = 1, \;\; y(\frac{3\pi}{4}) - y(\pi) = \frac{1}{\sqrt{2}}.$$

Here $V(x, y, y', y'') = y^2$ is a Liapunov function for (3.12) on $M = [0, \pi] \times N = M_1 \cup M_2$, where $M_1 = [0, \frac{\pi}{2}] \times N$, $M_2 = [\frac{\pi}{2}, \pi] \times N$ and $N \subset R^3$.

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