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FROM FIRST ORDER PDE-SYSTEMS TO HARMONIC MAPS BETWEEN GENERALIZED LAGRANGE SPACES

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Abstract. Section 1 defines the geometrical notion of a harmonic map between two generalized Lagrange spaces. Section 2 analyzes the particular case of the harmonic maps between two Lagrange spaces of electrodynamics. Section 3 proves that the smooth solutions of certain important first order DE- or PDE-systems are harmonic maps between convenient generalized Lagrange spaces. Section 4 describes the common main geometrical properties of the generalized Lagrange structures which convert the solutions of the initial first order DE- or PDE-systems into harmonic maps.

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1. Introduction

Let $(M^m, g_{\alpha\beta})$ and (N^n, h_{ij}) be two generalized Lagrange spaces, where m, respectively n, is the dimension of the manifold M, respectively N. The manifold M, respectively N, has the coordinates $(a^\alpha)_{\alpha=\overline{1,m}}$, respectively $(x^i)_{i=\overline{1,n}}$. In these notations, the fundamental generalized Lagrange metrical tensors are locally expressed by:

1) $g_{\alpha\beta} = g_{\alpha\beta}(a, b)$, $\forall\ \alpha, \beta = \overline{1, m}$, where

$$(a, b) = (a^1, ...a^m, b^1, ..., b^m)$$

are the adapted coordinates on the tangent bundle TM;

2) $h_{ij} = h_{ij}(x, y)$, $\forall\ i, j = \overline{1, n}$, where

$$(x, y) = (x^1, ..., x^n, y^1, ..., y^n)$$

are the adapted coordinates on the tangent bundle TN.

Remark 1.1. *In this paper, the first m coordinates of the manifold $M \times N$ are indexed by $\alpha, \beta, \gamma, ...$ and the last n coordinates of the manifold $M \times N$ are indexed by $i, j, k, ...$.*

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Definition 1.2. *A tensor field P of type $(1,2)$ on the manifold $M\times N$, having all components null, except for $P^{\beta}_{\alpha i}(a,x)$ and $P^{j}_{\alpha i}(a,x)$, is called a* ***tensor of connection****.*

Remark 1.3. *An "a priori" fixed tensor of connection P allows us to construct two privileged directions b and y for the metrical tensors $g_{\alpha\beta}(a,b)$ and $h_{ij}(x,y)$, which will be used in a natural construction of the notion of a harmonic map between the generalized Lagrange spaces M and N.*

Now, let us fix an arbitrary tensor of connection P on $M\times N$ and let us assume that the manifold M is connected, compact, orientable and endowed with a Riemannian metric $\varphi_{\alpha\beta}(a)$.

Remark 1.4. *The above assumptions ensure the existence of a volume element and, implicitly, of a theory of integration on the manifold M.*

In our geometrical context, we can introduce the following notion:

Definition 1.5. *The functional*

$$E^P_{g\varphi h}: C^\infty(M,N)\to\mathbb{R}$$

defined by

$$E^P_{g\varphi h}(f)=\frac{1}{2}\int_M g^{\alpha\beta}(a,b(a,x^k,x^k_\gamma))h_{ij}(f(a),y(a,x^k,x^k_\gamma))x^i_\alpha x^j_\beta\sqrt{\varphi}da,$$

where the smooth map $f\in C^\infty(M,N)$ is locally expressed by

$$a=(a^1,...,a^m)\in M\xrightarrow{f}(x^1(a),...,x^n(a))\in N,$$

and

$$\begin{aligned}
&x^i_\alpha=\frac{\partial x^i}{\partial a^\alpha},\ \varphi=\det(\varphi_{\alpha\beta}),\\
&b(a,x^k,x^k_\gamma)=b^\gamma(a)\left.\frac{\partial}{\partial a^\gamma}\right|_a\overset{def}{=}\varphi^{\alpha\beta}(a)x^i_\alpha(a)P^\gamma_{\beta i}(a,f(a))\left.\frac{\partial}{\partial a^\gamma}\right|_a,\\
&y(a,x^k,x^k_\gamma)=y^k(a)\left.\frac{\partial}{\partial x^k}\right|_{f(a)}\overset{def}{=}\varphi^{\alpha\beta}(a)x^i_\alpha(a)P^k_{\beta i}(a,f(a))\left.\frac{\partial}{\partial x^k}\right|_{f(a)},
\end{aligned}$$

is called the $\begin{pmatrix} & P & \\ g & \varphi & h\end{pmatrix}$***-energy functional****.*

Definition 1.6. *A smooth map $f\in C^\infty(M,N)$ which is a critical point of the energy functional $E^P_{g\varphi h}$ is called a*

$$\begin{pmatrix} & P & \\ g & \varphi & h\end{pmatrix}\textit{-harmonic map}$$

or a ***harmonic map between the generalized Lagrange spaces***

$$(M^m, g_{\alpha\beta}(a,b)) \textbf{\textit{ and }} (N^n, h_{ij}(x,y))$$

with respect to the tensor of connection P ***and the Riemannian metric*** $\varphi_{\alpha\beta}(a)$.

Example 1.7. *If* $g_{\alpha\beta}(a,b) = \varphi_{\alpha\beta}(a)$ *and* $h_{ij}(x,y) = h_{ij}(x)$ *are Riemannian metrics and the tensor of connection* P *is an arbitrary one, we recover the classical definition of a* ***harmonic map between two Riemannian manifolds*** *[1]. We remark that, in this particular case, the definition of the*

$$\begin{pmatrix} & P & \\ \varphi & \varphi & h \end{pmatrix}\textbf{\textit{-harmonic maps}}$$

is independent of the choice of the tensor of connection P.

Example 1.8. *If we consider that* $M = [a,b] \subset \mathbb{R}$, $\varphi_{11}(t) = g_{11}(t) = 1$, *where* $a^1 \overset{not}{=} t$, *and the tensor of connection is*

$$P = (P^1_{1i}(t,x), P^j_{1i}(t,x) = \delta^j_i),$$

then we obtain

$$C^\infty(M,N) \overset{not}{=} \Omega_{a,b}(N) = \{c : [a,b] \to N \mid c - C^\infty \text{ differentiable}\}.$$

Moreover, the energy functional E^P_{11h} *is*

$$E^P_{11h}(c) = \frac{1}{2}\int_a^b h_{ij}(c(t),\dot{c}(t))\frac{dx^i}{dt}\frac{dx^j}{dt}dt, \ \forall\ c \in \Omega_{a,b}(N),$$

where the smooth curve c is locally expressed by

$$t \in [a,b] \xrightarrow{c} (x^1(t), ..., x^n(t)) \in N,$$

and

$$\dot{c}(t) = \frac{dx^i}{dt} \cdot \left.\frac{\partial}{\partial x^i}\right|_{c(t)}.$$

In conclusion, the

$$\begin{pmatrix} & P & \\ 1 & 1 & h \end{pmatrix}\textbf{\textit{-harmonic curves}}$$

are exactly the ***geodesics*** *of the generalized Lagrange space* $(N, h_{ij}(x,y))$ *[2].*

Example 1.9. *If we take* $N = \mathbb{R}$, $h_{11}(x) = 1$, *where* $x^1 \overset{not}{=} x$, *and the tensor of connection is of the form*

$$P = (P^\beta_{\alpha 1}(a,x) = \delta^\beta_\alpha, P^1_{\beta 1}(a,x)),$$

then we obtain

$$C^{\infty}(M,N) \stackrel{not}{=} \mathcal{F}(M) = \{f : M \to \mathbb{R} \mid f - C^{\infty} \text{ differentiable}\}.$$

Moreover, the energy functional $E^P_{g\varphi 1}$ *becomes*

$$E^P_{g\varphi 1}(f) = \frac{1}{2}\int_M g^{\alpha\beta}(a, grad_{\varphi} f)x_{\alpha}x_{\beta}\sqrt{\varphi}da, \ \forall\, f \in \mathcal{F}(M),$$

where the smooth function f *is locally expressed by*

$$a = (a^1, ..., a^m) \in M \stackrel{f}{\to} x(a) \in \mathbb{R},$$

and

$$x_{\alpha} = \frac{\partial x}{\partial a^{\alpha}}, \ grad_{\varphi} f = \left[\varphi^{\alpha\beta}(a)x_{\alpha}(a)\right] \cdot \left.\frac{\partial}{\partial a^{\beta}}\right|_a .$$

2. Harmonic maps between two Lagrange spaces of electrodynamics

Let (M^m, L_M) and (N^n, L_N) be Lagrange spaces with the Lagrangians

$$L_M(a,b) = g_{\alpha\beta}(a)b^{\alpha}b^{\beta} + g_{\alpha\beta}(a)U^{\alpha}(a)b^{\beta} + F(a)$$

and

$$L_N(x,y) = h_{ij}(x)y^i y^j + h_{ij}(x)V^i(x)y^j + G(x),$$

where

- $g_{\alpha\beta}$ (resp. h_{ij}) is a Riemannian metric on the manifold M (resp. N) representing the gravitational potentials on M (resp. N);
- U^{α} (resp. V^i) is a vector field on M (resp. N) representing the electromagnetic potentials on M (resp. N);
- F (resp. G) is a smooth function on M (resp. N) representing the potential function on M (resp. N).

The fundamental metrical tensors of these Lagrangians are the Riemannian metrics

$$g_{\alpha\beta}(a) = \frac{1}{2}\frac{\partial^2 L_M}{\partial b^{\alpha}\partial b^{\beta}} \text{ and } h_{ij}(x) = \frac{1}{2}\frac{\partial^2 L_N}{\partial y^i \partial y^j}.$$

Taking now an arbitrary tensor of connection P on $M\times N$, the energy functional $E^P_{g\varphi h}$ becomes

$$E^P_{g\varphi h}(f) = \frac{1}{2}\int_M g^{\alpha\beta}(a)h_{ij}(f(a))x^i_{\alpha}x^j_{\beta}\sqrt{\varphi}da \stackrel{not}{=} E_{g\varphi h}(f), \ \forall\, f \in C^{\infty}(M,N).$$

Remark 2.1. *We remark that, in this particular case, the energy functional* $E_{g\varphi h}$ *is independent of the choice of the tensor of connection* P.

Obviously, the Euler-Lagrange equations of the energy functional $E_{g\varphi h}$ are the equations of the harmonic maps between the Lagrange spaces of electrodynamics (M^m, L_M) and (N^n, L_N) with respect to the Riemannian metric $\varphi_{\alpha\beta}(a)$, namely

$$g^{\alpha\beta}\left\{x^k_{\alpha\beta} - \left[G^\gamma_{\alpha\beta} + \frac{1}{2}\frac{\partial}{\partial a^\alpha}\left(\ln\frac{g}{\varphi}\right)\delta^\gamma_\beta\right]x^k_\gamma + H^k_{ij}x^i_\alpha x^j_\beta\right\} = 0,\ \forall\ k = \overline{1,n},$$

where

- $x^k_{\alpha\beta} = \dfrac{\partial^2 x^k}{\partial a^\alpha \partial a^\beta}$, $g = \det(g_{\alpha\beta})$, $\varphi = \det(\varphi_{\alpha\beta})$;
- $G^\gamma_{\alpha\beta}(a)$ are the Christoffel symbols of the Riemannian metric $g_{\alpha\beta}(a)$;
- $H^k_{ij}(x)$ are the Christoffel symbols of the Riemannian metric $h_{ij}(x)$.

Remark 2.2. *If we have $g_{\alpha\beta}(a) = \varphi_{\alpha\beta}(a)$, then we recover the classical equations of the harmonic maps between the Riemannian manifolds $(M^m, g_{\alpha\beta}(a))$ and $(N^n, h_{ij}(x))$ [1].*

Remark 2.3. *The coefficients*

$$\Delta^\gamma_{\alpha\beta} = G^\gamma_{\alpha\beta} + \frac{1}{2}\frac{\partial}{\partial a^\alpha}\left(\ln\frac{g}{\varphi}\right)\delta^\gamma_\beta$$

represent the components of a linear connection on the manifold M, which is produced by the Riemannian metrics $g_{\alpha\beta}(a)$ and $\varphi_{\alpha\beta}(a)$.

Using the last remark, we can introduce the following definition:

Definition 2.4. *A curve $c : I \subset \mathbb{R} \to M$ which is an autoparallel curve of the linear connection $\Delta^\gamma_{\alpha\beta}$ produced by the Riemannian metrics $g_{\alpha\beta}(a)$ and $\varphi_{\alpha\beta}(a)$ is called a (g,φ)-**geodesic** on the manifold M.*

Remark 2.5. *If the curve c is locally expressed by $c(t) = (a^\alpha(t))$, then the curve c is a (g,φ)-geodesic on the manifold M if and only if*

$$\frac{d^2 a^\gamma}{dt^2} = -\Delta^\gamma_{\alpha\beta}\frac{da^\alpha}{dt}\frac{da^\beta}{dt},\ \forall\ \gamma = \overline{1,m}. \tag{2.1}$$

Remark 2.6. *If we take $g_{\alpha\beta}(a) = \varphi_{\alpha\beta}(a)$, then we recover the classical definition of a geodesic on the Riemannian manifold $(M^m, g_{\alpha\beta}(a))$.*

Remark 2.7. *It is obvious that a (g,φ)-geodesic on the manifold M is a reparametrized geodesic of the Riemannian metric $g_{\alpha\beta}(a)$.*

Theorem 2.8. *Let $f : (M^m, L_M) \to (N^n, L_N)$ be a smooth map which carries the (g, φ)-geodesics from M into h-geodesics on N. Then the smooth map f is a harmonic map between the Lagrange spaces of electrodynamics (M, L_M) and (N, L_N) with respect to the Riemannian metric $\varphi_{\alpha\beta}(a)$.*

Proof. Let $c : I \subset \mathbb{R} \to M$, locally expressed by $c(t) = (a^\alpha(t))$, be a (g, φ)-geodesic on the manifold M. Then, the relations (2.1) hold. Because the curve $\overline{c}(t) = f(c(t))$, locally expressed by $\overline{c}(t) = (x^i(a^\alpha(t)))$, is an h-geodesic on the manifold N, it follows that we have

$$\frac{d^2x^k}{dt^2} + H^k_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0,\ \forall\ k = \overline{1, n}. \tag{2.2}$$

But, it is obvious that we also have

$$\frac{dx^k}{dt} = x^k_\alpha \frac{da^\alpha}{dt} \text{ and } \frac{d^2x^k}{dt^2} = x^k_{\alpha\beta}\frac{da^\alpha}{dt}\frac{da^\beta}{dt} + x^k_\gamma \frac{d^2a^\gamma}{dt^2},\ \forall\ k = \overline{1, n},$$

Replacing these relations into (2.2), we obtain

$$\frac{d^2a^\gamma}{dt^2}x^k_\gamma + x^k_{\alpha\beta}\frac{da^\alpha}{dt}\frac{da^\beta}{dt} + H^k_{ij}x^i_\alpha x^j_\beta \frac{da^\alpha}{dt}\frac{da^\beta}{dt} = 0,\ \forall\ k = \overline{1, n}.$$

Now, using the relations (2.1), it follows that we have

$$\left(x^k_{\alpha\beta} - \Delta^\gamma_{\alpha\beta}x^k_\gamma + H^k_{ij}x^i_\alpha x^j_\beta\right)\frac{da^\alpha}{dt}\frac{da^\beta}{dt} = 0,\ \forall\ k = \overline{1, n}.$$

Obviously, because the (g, φ)-geodesic c on M is an arbitrary one, it follows what we were looking for. □

3. Geometrical interpretations of solutions of certain first order PDE-systems

The problem of finding a geometrical structure of Riemannian type on a manifold M such that the orbits of an arbitrary vector field X should be geodesics was intensively studied by Sasaki. The results were not satisfactory, but, in his study, Sasaki discovered the well known almost contact structures on a manifold of odd dimension [5]. After the introduction of the generalized Lagrange spaces by Miron and Anastasiei [2], the same problem was resumed by Udrişte [6], [8]. In his studies, he succeded to discover a Lagrange structure on M, depending on the vector field X and an associated $(1, 1)$-tensor field, such that the orbits of C^2 class should be geodesics. Moreover, he formulated some more general problems [6]:

1) Are there structures of Lagrange type such that the solutions of a first order PDE-systems to be *harmonic maps*?

2) What means a *harmonic map* in such Lagrange structures?

A partial answer to these questions is offered by the author of this paper in the work [4], using the notion of *harmonic map on a direction* between a Riemannian manifold and a generalized Lagrange manifold. The generalized notion of a *harmonic map between two generalized Lagrange manifolds*, introduced in [7], allows us to extend the results of the previous papers [4], [6], [8] and obtain beautiful geometrical interpretations for the solutions of certain first order PDE-systems in the sense of Udrişte questions. In this direction, note that for every smooth map $f \in C^\infty(M, N)$ we use the notation

$$\delta f = x^i_\alpha da^\alpha|_a \otimes \left.\frac{\partial}{\partial x^i}\right|_{f(a)} \in \Gamma(T^*M \otimes TN).$$

Now, let us consider that T is an arbitrary tensor of type $(1,1)$ on the manifold $M \times N$, having all components equal to zero, except for

$$(T^i_\alpha(a,x)),\ \forall\ i = \overline{1,n},\ \ \alpha = \overline{1,m}.$$

In this context, the tensor field T produces the first order PDE-system

$$\delta f = T \Leftrightarrow \frac{\partial x^i}{\partial a^\alpha} = T^i_\alpha(a, f(a)). \tag{3.1}$$

Remark 3.1. *If $(M, \varphi_{\alpha\beta}(a))$ and $(N, \psi_{ij}(x))$ are Riemannian manifolds, then we can construct a natural scalar product on $\Gamma(T^*M \otimes TN)$ by*

$$\langle T, S\rangle = \varphi^{\alpha\beta}(a)\psi_{ij}(x)T^i_\alpha(a,x)S^j_\beta(a,x),$$

where

$$T = T^i_\alpha(a,x)\ da^\alpha\big|_a \otimes \left.\frac{\partial}{\partial x^i}\right|_x \ \text{ and } S = S^j_\beta(a,x)\ da^\beta\big|_a \otimes \left.\frac{\partial}{\partial x^j}\right|_x.$$

Under the previous assumptions, we can prove the following result:

Theorem 3.2. *If (M, φ) and (N, ψ) are Riemannian manifolds and the map $f \in C^\infty(M,N)$ is a solution of the first order PDE-system (3.1), then f is a solution of the variational problem associated to the functional*

$$\mathcal{L}_T : C^\infty(M,N)\backslash\{f \mid \exists\ a \in M \ \text{ such that } \langle \delta f, T\rangle (a) = 0\} \to \mathbb{R}_+$$

defined by

$$\mathcal{L}_T(f) = \frac{1}{2}\int_M \frac{||\delta f||^2||T||^2}{\langle \delta f, T\rangle^2}\sqrt{\varphi}da = \frac{1}{2}\int_M \frac{||T||^2}{\langle \delta f, T\rangle^2}\varphi^{\alpha\beta}\psi_{ij}x^i_\alpha x^j_\beta\sqrt{\varphi}da.$$

Proof. The Cauchy inequality for the scalar product $\langle\ ,\ \rangle$ holds good. It follows that the following inequality is true:

$$\langle T, S\rangle^2 \le ||T||^2||S||^2,\ \forall\ T, S \in \Gamma(T^*M \otimes TN),$$

with equality if and only if there exists $\mathcal{K} \in \mathcal{F}(M \times N)$ such that $T = \mathcal{K}S$. Consequently, for every smooth map $f \in C^\infty(M, N)$ we have

$$\mathcal{L}_T(f) = \frac{1}{2}\int_M \frac{||\delta f||^2||T||^2}{\langle \delta f, T\rangle^2}\sqrt{\varphi}da \geq \frac{1}{2}\int_M \sqrt{\varphi}da.$$

Obviously, if the map f is a solution of the first order PDE-system (3.1), then we obtain

$$\mathcal{L}_T(f) = \frac{1}{2}\int_M \sqrt{\varphi}da,$$

that is the map f is a global minimum point for the functional $\mathcal{L}_T$. □

Remark 3.3. *The global minimum points of the functional $\mathcal{L}_T$ are solutions of the first order PDE-system $\delta f = \mathcal{K}T$, where $\mathcal{K} \in \mathcal{F}(M \times N)$, not necessarily with $\mathcal{K} = 1$.*

Remark 3.4. *In some particular cases of the first order PDE-system (3.1) the functional $\mathcal{L}_T$ becomes exactly a functional of the type*

$$\begin{pmatrix} & P & \\ g & \varphi & h \end{pmatrix}\textbf{-}\textit{\textbf{energy}}.$$

This fact means that the solutions of these particular first order PDE-systems become harmonic maps between two convenient generalized Lagrange spaces.

Fundamental examples.

1. Orbits

For $M = ([a, b], \varphi_{11}(t) = 1)$ and $T = \xi \in \Gamma(TN)$, the first order PDE-system (3.1) becomes the DE-system of orbits for ξ, namely

$$\frac{dx^i}{dt} = \xi^i(c(t)), \tag{3.2}$$

where the curve $c : [a, b] \to N$ is locally expressed by $c(t) = (x^i(t))$. Moreover, the functional $\mathcal{L}_\xi$ is

$$\mathcal{L}_\xi(c) = \frac{1}{2}\int_a^b \frac{||\xi||^2_\psi}{\left[\xi^b(\dot{c}(t))\right]^2}\psi_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}dt,$$

where

$$\xi^b(\dot{c}(t)) = \psi_{kl}\xi^l\frac{dx^k}{dt}.$$

Hence it folows that the functional $\mathcal{L}_\xi$ is a

$$\begin{pmatrix} & P & \\ 1 & 1 & h \end{pmatrix}\textbf{-energy}$$

(see **Example 1.8**). The generalized Lagrange metric on the manifold N is

$$h_{ij} : TN\backslash\{(x,y) \mid \xi^b(y) = 0\} \to \mathbb{R},$$

where

$$\xi^b(y) = \psi_{kl}\xi^l y^k$$

and

$$h_{ij}(x,y) = \frac{||\xi||^2_\psi}{\left[\xi^b(y)\right]^2}\psi_{ij}(x) = \psi_{ij}(x)\exp\left[2\ln\frac{||\xi||_\psi}{|\xi^b(y)|}\right].$$

Remark 3.5. *This DE-system was studied in other way by Udrişte in the papers [6], [8].*

2. Pfaffian systems

For $N = (\mathbb{R}, \psi_{11}(x) = 1)$ and $T = A \in \Gamma(T^*M)$, the first order PDE-system (3.1) becomes the Pfaffian system

$$\frac{\partial x}{\partial a^\alpha} = A_\alpha(a), \tag{3.3}$$

where the function $f : M \to \mathbb{R}$ is locally expressed by $f(a) = x(a)$. Moreover, the functional $\mathcal{L}_A$ becomes

$$\mathcal{L}_A(f) = \frac{1}{2}\int_M \frac{||A||^2_\varphi}{[A(grad_\varphi f)]^2}\varphi^{\alpha\beta}x_\alpha x_\beta\sqrt{\varphi}da.$$

Hence it follows that the functional $\mathcal{L}_A$ is a

$$\begin{pmatrix} & P & \\ g & \varphi & 1 \end{pmatrix}\textbf{-energy}$$

(see **Example 1.9**). The generalized Lagrange metric on the manifold M is

$$g_{\alpha\beta} : TM\backslash\{(a,b) \mid A(b) = 0\} \to \mathbb{R},$$

where

$$A(b) = A_\alpha b^\alpha$$

and

$$g_{\alpha\beta}(a,b) = \frac{[A(b)]^2}{||A||^2_\varphi}\varphi_{\alpha\beta}(a) = \varphi_{\alpha\beta}(a)\exp\left[2\ln\frac{|A(b)|}{||A||_\varphi}\right].$$

Remark 3.6. *The preceding cases appear also in the paper [7] and, from another point of view, in the paper [4]. The following case is the main novelty of this paper.*

3. A general case

If we have

$$T^i_\alpha(a,x) = \sum_{r=1}^{t} \xi^i_r(x) A^r_\alpha(a),$$

where $\{\xi_r\}_{r=\overline{1,t}} \subset \Gamma(TN)$ are vector fields on N and $\{A^r\}_{r=\overline{1,t}} \subset \Gamma(T^*M)$ are 1-forms on M, then the first order PDE-system (3.1) reduces to

$$\frac{\partial x^i}{\partial a^\alpha} = \sum_{r=1}^{t} \xi^i_r(f(a)) A^r_\alpha(a). \tag{3.4}$$

Remark 3.7. *Without loss of generality of the problem, we can suppose that $\{\xi_r\}_{r=\overline{1,t}} \subset \Gamma(TN)$ (resp. $\{A^r\}_{r=\overline{1,t}} \subset \Gamma(T^*M)$) are linearly independent. Under these assumptions, we have $t \leq \min\{m,n\}$, where $m = \dim M$ and $n = \dim N$.*

3.1. Let us assume that $\{\xi_r\}_{r=\overline{1,t}} \subset \Gamma(TN)$ is an orthonormal system of vector fields with respect to the Riemannian metric $\psi_{ij}(x)$ on N and let $B \in \Gamma(T^*M)$ be an arbitrary unit 1-form on M. Under these assumptions, by a simple calculation, we obtain

$$||T||^2 = \varphi^{\alpha\beta}\psi_{ij}\xi^i_r A^r_\alpha \xi^j_s A^s_\beta = \sum_{r,s=1}^{t} \langle \xi_r, \xi_s \rangle_\psi \langle A^r, A^s \rangle_\varphi = \sum_{r=1}^{t} ||A^r||^2_\varphi,$$

$$\langle \delta f, T \rangle = \varphi^{\alpha\beta}\psi_{ij} x^i_\alpha \xi^j_r A^r_\beta B^\mu B_\mu.$$

Defining the tensor of connection by

$$P^\gamma_{i\beta}(a,x) = \psi_{ij}(x)\xi^j_r(x) A^r_\beta(a) B^\gamma(a)$$

and putting

$$b^\gamma = \varphi^{\alpha\beta} x^i_\alpha P^\gamma_{i\beta},$$

the functional $\mathcal{L}_T$ takes the form

$$\begin{aligned} \mathcal{L}_T(f) &= \frac{1}{2}\int_M \frac{\sum_{r=1}^{t} ||A^r||^2_\varphi}{[B(b)]^2} \varphi^{\alpha\beta}\psi_{ij} x^i_\alpha x^j_\beta \sqrt{\varphi}\, da = \\ &= \frac{1}{2}\int_M g^{\alpha\beta}(a,b)\psi_{ij}(f(a)) x^i_\alpha x^j_\beta \sqrt{\varphi}\, da. \end{aligned}$$

Consequently, the functional $\mathcal{L}_T$ is a

$$\begin{pmatrix} & P & \\ g & \varphi & \psi \end{pmatrix}\textbf{-energy}.$$

The generalized Lagrange metric on the manifold M is

$$g_{\alpha\beta} : TM \backslash \{(a,b) \mid B(b) = 0\} \to \mathbb{R},$$

where

$$B(b) = B_\alpha b^\alpha$$

and

$$g_{\alpha\beta}(a,b) = \frac{[B(b)]^2}{\sum_{r=1}^t ||A^r||^2_\varphi}\varphi_{\alpha\beta}(a) = \varphi_{\alpha\beta}(a)\exp\left[2\ln\frac{|B(b)|}{\sqrt{\sum_{r=1}^t ||A^r||^2_\varphi}}\right].$$

3.2. As above, let us assume that $\{A^r\}_{r=\overline{1,t}} \subset \Gamma(T^*M)$ is an orthonormal system of 1-forms with respect to the metric $\varphi^{\alpha\beta}(a)$ on M and let $X \in \Gamma(TN)$ be an arbitrary unit vector field on N. By analogy to **3. 1.** we have

$$||T||^2 = \sum_{r=1}^t ||\xi_r||^2_\psi \text{ and } \langle \delta f, T\rangle = \varphi^{\alpha\beta}\psi_{ij}x^i_\alpha \xi^j_r A^r_\beta X^k X_k.$$

Taking the tensor of connection

$$P^k_{i\beta}(a,x) = \psi_{ij}(x)\xi^j_r(x)A^r_\beta(a)X^k(x)$$

and putting

$$y^k = \varphi^{\alpha\beta}x^i_\alpha P^k_{i\beta},$$

the functional $\mathcal{L}_T$ takes the form

$$\begin{aligned}\mathcal{L}_T(f) &= \frac{1}{2}\int_M \frac{\sum_{r=1}^t ||\xi_r||^2_\psi}{[X^b(y)]^2}\varphi^{\alpha\beta}\psi_{ij}x^i_\alpha x^j_\beta\sqrt{\varphi}da = \\ &= \frac{1}{2}\int_M \varphi^{\alpha\beta}(a)h_{ij}(f(a),y)x^i_\alpha x^j_\beta\sqrt{\varphi}da.\end{aligned}$$

Obviously, the functional $\mathcal{L}_T$ is a

$$\begin{pmatrix} & P & \\ \varphi & \varphi & h\end{pmatrix}\textbf{-energy}.$$

The generalized Lagrange metric tensor on the manifold N is

$$h_{ij} : TN\backslash\{(x,y) \mid X^b(y) = 0\} \to \mathbb{R},$$

where

$$X^b(y) = \psi_{kl}X^l y^k$$

and

$$h_{ij}(x,y) = \frac{\sum_{r=1}^t ||\xi_r||^2_\psi}{[X^b(y)]^2}\psi_{ij}(x) = \psi_{ij}(x)\exp\left[2\ln\frac{\sqrt{\sum_{r=1}^t ||\xi_r||^2_\psi}}{|X^b(y)|}\right].$$

Remark 3.8. *We assumed above the* ***a priori*** *existence of a Riemannian metric φ (resp. ψ) on the manifold M (resp. N) such that the system of covectors $\{A^r\}_{r=\overline{1,t}}$ (resp. of vectors $\{\xi_r\}_{r=\overline{1,t}}$) is orthonormal. This fact is always possible. In conclusion, our assumptions upon the orthonormality of the above systems of covectors or vectors do not restrict the generality of the problem.*

4. Generalized Lagrange geometry associated to certain first order PDE-systems

We remark that in the studies of all preceding cases of first order PDE-systems we proved that the smooth solutions of the initial PDE-system become harmonic maps between two generalized Lagrange spaces, in the sense defined in this paper. It is important to note that the generalized Lagrange structures that convert these solutions into harmonic maps are of the type

$$GL^n = (M^n, e^{2\sigma(x,y)}\gamma_{ij}(x)),$$

where $\sigma : TM\backslash\{\text{Hyperplane}\} \to \mathbb{R}$ is a smooth function. We emphasize that in these spaces, using the ideas exposed in [2] and [3], we can construct a generalized Lagrange geometry and a field theory. Obviously, these geometrical Lagrange theories can be regarded as natural ones, associated to the studied first order PDE-system. In this direction, we assume that the generalized Lagrange space GL^n satisfies the following axiom:

Axiom: The space GL^n is endowed with the nonlinear connection

$$N^i_j(x,y) = \Gamma^i_{jk}(x)y^k,$$

where $\Gamma^i_{jk}(x)$ are the Christoffel symbols of the Riemannian metric $\gamma_{ij}(x)$.

Under the preceding axiom, the generalized Lagrange space GL^n verifies a constructive-axiomatic formulation of General Relativity due to Ehlers, Pirani and Schild [2]. Moreover, the space GL^n represents a convenient relativistic geometrical model because it has the same conformal and projective properties as the Riemannian space $R^n = (M, \gamma_{ij}(x))$.

We recall that in the generalized Lagrangian theory of electromagnetism, the electromagnetic tensors F_{ij} and f_{ij} of the generalized Lagrange space GL^n are

$$F_{ij} = \left(g_{ip}\frac{\delta\sigma}{\delta x^j} - g_{jp}\frac{\delta\sigma}{\delta x^i}\right)y^p,\ f_{ij} = \left(g_{ip}\frac{\partial\sigma}{\partial y^j} - g_{jp}\frac{\partial\sigma}{\partial y^i}\right)y^p.$$

In the sequel, developping the formalism presented in [2], [3] and denoting by r^i_{jkl} the curvature tensor field of the Riemannian metric $\gamma_{ij}(x)$, the following Maxwell equations of the electromagnetic tensors F_{ij} and f_{ij} hold:

$$\begin{cases} F_{ij|k} + F_{jk|i} + F_{ki|j} = -\sum_{(i,j,k)} g_{ip}r^h_{qjk}\dfrac{\partial\sigma}{\partial y^h}y^p y^q, \\ F_{ij}|_k + F_{jk}|_i + F_{ki}|_j = -(f_{ij|k} + f_{jk|i} + f_{ki|j}), \\ f_{ij}|_k + f_{jk}|_i + f_{ki}|_j = 0, \end{cases}$$

where $_{|i}$ (resp. $|_a$) represents the $h-$ (resp. $v-$) covariant derivative induced by the Cartan canonical metrical d-connection $C\Gamma(N)$ of the generalized Lagrange space GL^n.

In the construction of the gravitational field equations of the generalized Lagrange space GL^n we use the notations:

$$\begin{cases} r_{ij} = r^k_{ijk},\ r = \gamma^{ij} r_{ij},\ \dfrac{\delta}{\delta x^i} = \dfrac{\partial}{\partial x^i} - N^j_i \dfrac{\partial}{\partial y^j}, \\ \sigma^H = \gamma^{kl} \dfrac{\delta\sigma}{\delta x^k}\dfrac{\delta\sigma}{\delta x^l},\ \sigma^V = \gamma^{ab}\dfrac{\partial\sigma}{\partial y^a}\dfrac{\partial\sigma}{\partial y^b},\ \overline{\sigma} = \gamma^{ij}\sigma_{ij},\ \dot{\sigma} = \gamma^{ab}\dot{\sigma}_{ab}, \end{cases}$$

where

$$\sigma_{ij} = \frac{\delta\sigma}{\delta x^i}|_j + \frac{\delta\sigma}{\delta x^i}\frac{\delta\sigma}{\delta x^j} - \frac{1}{2}\gamma_{ij}\sigma^H,\ \dot{\sigma}_{ab} = \left.\frac{\partial\sigma}{\partial y^a}\right|_b + \frac{\partial\sigma}{\partial y^a}\frac{\partial\sigma}{\partial y^b} - \frac{1}{2}\gamma_{ab}\sigma^V.$$

In this context, the Einstein equations of the generalized Lagrange space GL^n take the form

$$\begin{cases} r_{ij} - \dfrac{1}{2} r\gamma_{ij} + t_{ij} = \mathcal{K} T^H_{ij} \\ (2-n)(\dot{\sigma}_{ab} - \dot{\sigma}\gamma_{ab}) = \mathcal{K} T^V_{ab}, \end{cases}$$

where T^H_{ij} and T^V_{ab} are the $h-$ and the $v-$components of the energy momentum tensor field, $\mathcal{K}$ is the gravific constant and

$$t_{ij} = (n-2)(\gamma_{ij}\overline{\sigma} - \sigma_{ij}) + \gamma_{ij} r_{st} y^s \gamma^{tp}\frac{\partial\sigma}{\partial y^p} + \frac{\partial\sigma}{\partial y^i} r^a_{tja} y^t - \gamma_{js}\gamma^{ap}\frac{\partial\sigma}{\partial y^p} r^s_{ita} y^t.$$

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