Novi Sad J. Math.
Vol. 37, No. 1, 2007, 107-114

COMPLEXITY OF MAL'CEV INTERPOLATION[1]

E. Aichinger [2], D. Mašulović[3] and R. Pöschel[4]

Abstract. A classical result in near-ring theory tells that the 4-interpolation property implies the n-interpolation property for all $n \in \mathbb{N}$. In the present note we are interested in the complexity of a special kind of interpolating terms which can be constructed from the 4-interpolations.

AMS Mathematics Subject Classification (2000): 08A40, 16Y30

Key words and phrases: near-ring, interpolation, local functional completeness

1. Introduction

Let A be a set with at least two elements, let $\mathbf{A} = (A, F)$ be an algebra and let $\operatorname{Pol}_k \mathbf{A}$ denote the set of k-ary polynomials of an algebra $\mathbf{A}$ and $\operatorname{Pol} \mathbf{A} = \bigcup_{k \geqslant 1} \operatorname{Pol}_k \mathbf{A}$. An *$n$-interpolation problem of dimension k* is a $2 \times n$ matrix $P = \left(\begin{smallmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_n \\ a_1 & a_2 & \dots & a_n \end{smallmatrix} \right)$ such that

- $a_1, \dots, a_n \in A$;
- $\mathbf{x}_1, \dots, \mathbf{x}_n \in A^k$ and $\mathbf{x}_i \neq \mathbf{x}_j$ whenever $i \neq j$.

A *solution to P* is a polynomial $f \in \operatorname{Pol}_k \mathbf{A}$ such that $f(\mathbf{x}_i) = a_i$ for all i. A set $F \subseteq A^{A^k}$ of k-ary operations has the *n-interpolation property* if for every n-interpolation problem $P = \left(\begin{smallmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_n \\ a_1 & a_2 & \dots & a_n \end{smallmatrix} \right)$ of dimension k there exists a solution $f \in F$ to P. An algebra $\mathbf{A}$ has the n-interpolation property if $\operatorname{Pol}_k \mathbf{A}$ has the n-interpolation property for all $k \geqslant 1$. An algebra $\mathbf{A}$ is *locally functionally complete* if it has the n-interpolation property for all $n \geqslant 2$. Our starting point in this paper is the following well-known theorem:

Theorem 1.1. (The 4-Interpolation Property for Near-rings [5]) *Let $(\Gamma, +, -, 0)$ be a group, and let F be a subnear-ring of $(\Gamma^\Gamma, +, -, 0, \circ)$ that has the 4-interpolation property. Then F has the n-interpolation property for all $n \geqslant 1$.*

---

[1]The work is partially supported by the Ministry for Science, Technologies and Development of the Republic of Serbia (Project No. 144017)

[2]Institute for Algebra, University of Linz, A-4040 Linz, Austria, e-mail: erhard@algebra.uni-linz.ac.at

[3]Department of Mathematics and Informatics, University of Novi Sad, Trg D. Obradovića 4, 21000 Novi Sad, Serbia, e-mail:masul@im.ns.ac.yu

[4]Institute for Algebra, Dresden University of Technology, D-01062 Dresden, Germany, e-mail: Reinhard.Poeschel@tu-dresden.de

The theorem came as a consequence of the Density Theorems for near-rings [2, 4]; a proof is stated in [1]. We remark that one does not have to assume that F contains the identity map. However, it turns out that this theorem is a consequence of the following (easy) fact:

Fact 1.2. *Let* $\mathbf{A}$ *be an algebra with a Mal'cev term* m*:*

$$m(x, y, y) = m(y, y, x) = x$$

and assume that every 4-interpolation problem in $\mathbf{A}$ *of dimension 1 has a solution in* $\mathbf{A}$*. Then* $\mathbf{A}$ *has the* n*-interpolation property for all* n*.*

Proof. We know from [3] that a Mal'cev algebra with at least two elements is locally functionally complete if and only if it is simple and nonaffine. And it is obvious that an algebra $\mathbf{A}$ where every 4-interpolation problem of dimension 1 has a solution has to be both simple and nonaffine. □

In this paper we are interested in procedures that build interpolating polynomials from "elementary 4-interpolations", that is, using Mal'cev operations to "add" constants and polynomials that solve the 4-inteprolation problems of dimension 1. We would like to know how complicated are the terms that implement the interpolation? In particular, how many 4-interpolations are necessary.

It is a well-known fact (see [6]) that if $\mathbf{A}$ is a discriminator algebra then the discriminator and constants suffice to solve any interpolation problem. Namely, every discriminator algebra has the switching term

$$\text{if}(x, y, u, v) = \begin{cases} u, & x = y \\ v, & x \neq y \end{cases}$$

so an interpolation problem such as e.g. $\left(\begin{smallmatrix} p_1 & p_2 & p_3 & p_4 \\ a_1 & a_2 & a_3 & a_4 \end{smallmatrix}\right)$ can be straightforwardly solved by

$$f(x) = \text{if}(x, c_{p_1}, c_{a_1}, \text{if}(x, c_{p_2}, c_{a_2}, \text{if}(x, c_{p_3}, c_{a_3}, \text{if}(x, c_{p_4}, c_{a_4}, c_{a_4}))))$$

where c_a is a constant symbol with the obvious interpretation. This is why we are interested in those situations where one *has* to use the 4-interpolations.

2. Mal'cev interpolation algebras

For a set A let $\mathcal{L}_A = \{\mu\} \cup \Phi_A \cup C_A$ be the *language of Mal'cev interpolation on* A, where μ is a ternary operation symbol, Φ_A is a set of unary operation symbols indexed by 4-interpolation problems over A of dimension 1:

$$\Phi_A = \left\{\varphi_{\left(\begin{smallmatrix} x_1 & x_2 & x_3 & x_4 \\ a_1 & a_2 & a_3 & a_4 \end{smallmatrix}\right)} : \left(\begin{smallmatrix} x_1 & x_2 & x_3 & x_4 \\ a_1 & a_2 & a_3 & a_4 \end{smallmatrix}\right) \text{ is a 4-interpolation problem of dimension 1}\right\}$$

and $C_A = \{c_a : a \in A\}$ is a set of constant symbols indexed by elements of A.

We would like to consider interpretations of $\mathcal{L}_A$ in various Mal'cev algebras whose algebraic type need not contain $\mathcal{L}_A$. Let $\mathcal{F}$ be an algebraic type and $\mathbf{A} = \langle A, F\rangle$ an $\mathcal{F}$-algebra. The *interpretation of* $\mathcal{L}_A$ *in* $\mathbf{A}$ is a pair $(\mathbf{A}, \sigma)$ where $\sigma : \mathcal{L}_A \to \text{Pol}(\mathbf{A})$ is a mapping such that

- $\mathrm{ar}(\sigma(\mu)) = 3$ and $\sigma(\mu)$ is a Mal'cev term operation of $\mathbf{A}$;
- $\mathrm{ar}(\sigma(\varphi)) = 1$ for all $\varphi \in \Phi_A$, and $\sigma(\varphi_{\left(\begin{smallmatrix} x_1 & x_2 & x_3 & x_4 \\ a_1 & a_2 & a_3 & a_4 \end{smallmatrix}\right)})(x_i) = a_i$, for $i = 1, 2, 3, 4$; and
- $\mathrm{ar}(\sigma(c_a)) = 0$ for all $c_a \in C_a$, and $\sigma(c_a) = a$.

For any set of variables X, σ uniquely extends to a map $\mathrm{Term}_{\mathcal{L}_A}(X) \to \mathrm{Pol}(\mathbf{A})$ which we also denote by σ. For a term $t \in \mathrm{Term}_{\mathcal{L}_A}(X)$, instead of $\sigma(t)$ we write t^σ.

Definition 2.1. An algebra $\mathbf{A}$ is a *Mal'cev interpolation algebra* if there exists an interpretation of $\mathcal{L}_A$ in $\mathbf{A}$. The *set of Mal'cev interpolating polynomials* over A is the least set of $\mathcal{L}_A$-terms such that

- c and $\varphi(x)$ are Mal'cev interpolating polynomials for all $c \in C_A$ and $\varphi \in \Phi_A$;
- if t is a Mal'cev interpolating polynomial, then so is $\varphi(t)$ for every $\varphi \in \Phi_A$;
- if t_1, t_2, t_3 are Mal'cev interpolating polynomials, then so is $\mu(t_1, t_2, t_3)$.

A *Mal'cev interpolating polynomial for an n-interpolation problem P w.r.t. the interpretation* $(\mathbf{A}, \sigma)$ is a Mal'cev interpolating polynomial t such that t^σ solves P.

It is easy to see that every finite Mal'cev interpolation algebra $\mathbf{A}$ is functionally complete. Moreover, if $n > |A|^k$ there are no n-interpolation problems over A of dimension k. Therefore, in the sequel we consider only infinite Mal'cev interpolation algebras.

Let $t \in \mathrm{Term}_{\mathcal{F}}(X)$ be an $\mathcal{F}$-term and let $\mathcal{H} \subseteq \mathcal{F}$. The *$\mathcal{H}$-complexity of t* is defined recursively by

- $|x|_{\mathcal{H}} = 0$ for all $x \in X$,
- $|f(t_1, \ldots, t_k)|_{\mathcal{H}} = |t_1|_{\mathcal{H}} + \ldots + |t_k|_{\mathcal{H}} + \begin{cases} 0, & f \notin \mathcal{H}, \\ 1, & f \in \mathcal{H}. \end{cases}$

Let $\mathbf{A}$ be a Mal'cev interpolation algebra and let $(\mathbf{A}, \sigma)$ be an interpretation of $\mathcal{L}_A$ in $\mathbf{A}$. Let $P = \left(\begin{smallmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \ldots & \mathbf{x}_n \\ a_1 & a_2 & \ldots & a_n \end{smallmatrix}\right)$ be an n-interpolation problem over A of dimension k. By

$$\mathrm{Cmp}_{(\mathbf{A},\sigma)} P = \min\{|t|_{\mathcal{L}_A} : t \text{ is a Mal'cev interpolating polynomial which solves } P \text{ w.r.t. } (\mathbf{A}, \sigma)\}$$

we denote the minimal complexity of a Mal'cev interpolating polynomial that solves P w.r.t. $(\mathbf{A}, \sigma)$. If $\mathrm{Cmp}_{(\mathbf{A},\sigma)} P = |t|_{\mathcal{L}_A}$, we say that t is a *minimal polynomial for P w.r.t.* $(\mathbf{A}, \sigma)$.

The *complexity of an n-interpolation problem P of dimension k in a Mal'cev interpolation algebra* $\mathbf{A}$ is then given by

$$\mathrm{Cmp}_{\mathbf{A}} P = \sup\{\mathrm{Cmp}_{(\mathbf{A},\sigma)} P : \sigma \text{ is an interpretation of } \mathcal{L}_A \text{ in } \mathbf{A}\}.$$

Note that $\mathrm{Cmp}_{\mathbf{A}} P$ is an integer or $\aleph_0$. The *complexity of Mal'cev term interpolation in a Mal'cev interpolation algebra* $\mathbf{A}$ is given by

$$\mathrm{Cmp}_{\mathbf{A}}(n,k) = \sup\{\mathrm{Cmp}_{\mathbf{A}} P : P \text{ is an } n\text{-interpolation problem over } A \text{ of dimension } k\}.$$

Since every subset of $\{0,1,2,\dots\} \cup \{\aleph_0\}$ has a supremum in in the same set, we see that $\mathrm{Cmp}_{\mathbf{A}}(n,k)$ is an integer or $\aleph_0$.

We measure the *complexity of Mal'cev term interpolation* by the following two functions:

$$\mathrm{Cmp}_*(n,k) = \min\{\mathrm{Cmp}_{\mathbf{A}}(n,k) : \mathbf{A} \text{ is an infinite Mal'cev interpolation algebra}\}$$
$$\mathrm{Cmp}^*(n,k) = \sup\{\mathrm{Cmp}_{\mathbf{A}}(n,k) : \mathbf{A} \text{ is an infinite Mal'cev interpolation algebra}\}.$$

Clearly, $\mathrm{Cmp}_*(n) \geqslant 1$ for all $n \in \mathbb{N}$. This is not a useful lower bound, but the proof of Theorem 3.1 makes use of this simple fact.

3. A lower and an upper bound on the complexity of Mal'cev interpolation

In this section we provide a lower and an upper bound on the complexity of Mal'cev interpolation. We show that there exist positive constants γ_1 and γ_2 such that $\mathrm{Cmp}_*(n,k) \geqslant \gamma_1 \cdot n$ and $\mathrm{Cmp}^*(n,k) \leqslant \gamma_2 \cdot n^2 \log n$.

Theorem 3.1. *There exists a positive constant* γ_1 *such that* $\mathrm{Cmp}_*(n,k) \geqslant \gamma_1 \cdot n$ *for all positive integers* n *and* k.

Proof. If $n \in \{1,2\}$ the claim is obviously true since $\mathrm{Cmp}_{\mathbf{A}}(n,k) \geqslant 1$ for all infinite Mal'cev interpolation algebras $\mathbf{A}$. So, let $n \geqslant 3$ and let k be arbitrary. In order to show that $\mathrm{Cmp}_*(n,k) \geqslant \gamma_1 \cdot n$ for some $\gamma_1 > 0$ it suffices to show that for every infinite Mal'cev interpolation algebra $\mathbf{A}$ there exists an interpretation σ and an n-interpolation problem P of dimension k such that $\mathrm{Cmp}_{(\mathbf{A},\sigma)} P \geqslant \gamma_1 \cdot n$.

We shall say that vectors $\mathbf{x} = (x_1,\dots,x_k)$ and $\mathbf{x}' = (x'_1,\dots,x'_k)$ are disjoint if $\{x_1,\dots,x_k\} \cap \{x'_1,\dots,x'_k\} = \varnothing$. Also, we shall say that a vector $\mathbf{x} = (x_1,\dots,x_k)$ is disjoint from a set S if $\{x_1,\dots,x_k\} \cap S = \varnothing$.

Take any n mutually disjoint vectors $\mathbf{x}_1, \dots, \mathbf{x}_n \in A^k$ and any n distinct $a_1, \dots, a_n \in A$, and consider the n-interpolation problem $P = \left(\begin{smallmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_n \\ a_1 & a_2 & \dots & a_n \end{smallmatrix}\right)$ of dimension k. Furthermore, let $\mathbf{x}_i = (x_{i1},\dots,x_{ik})$ for $i \in \{1,\dots,n\}$ and let $d \in A$ be arbitrary. Since $\mathbf{A}$ is a Mal'cev interpolation algebra, every interpolation problem has a solution in $\mathbf{A}$. For any 4-interpolation problem $\left(\begin{smallmatrix} y_1 & y_2 & y_3 & y_4 \\ b_1 & b_2 & b_3 & b_4 \end{smallmatrix}\right)$ of dimension 1 consider the following interpolation problem:

$$Q_{\left(\begin{smallmatrix} y_1 & y_2 & y_3 & y_4 \\ b_1 & b_2 & b_3 & b_4 \end{smallmatrix}\right)} = \begin{pmatrix} y_1 & y_2 & y_3 & y_4 & x_{i_1} & \dots & x_{i_m} \\ b_1 & b_2 & b_3 & b_4 & d & \dots & d \end{pmatrix}$$

where $\{x_{i_1},\dots,x_{i_m}\} = \{x_{11},\dots,x_{1k},\dots,x_{n1},\dots,x_{nk}\} \setminus \{y_1,y_2,y_3,y_4\}$.

Let $f_{\left(\begin{smallmatrix} y_1 & y_2 & y_3 & y_4 \\ b_1 & b_2 & b_3 & b_4 \end{smallmatrix}\right)} \in \operatorname{Pol}(\mathbf{A})$ be a solution to $Q_{\left(\begin{smallmatrix} y_1 & y_2 & y_3 & y_4 \\ b_1 & b_2 & b_3 & b_4 \end{smallmatrix}\right)}$ and define the interpretation σ of $\mathcal{L}_A$ in $\mathbf{A}$ as follows:

- $\sigma(\mu)$ is any Mal'cev term operation of $\mathbf{A}$;
- $\sigma(c_a) = a$ for all $a \in A$; and
- $\sigma(\varphi_{\left(\begin{smallmatrix} y_1 & y_2 & y_3 & y_4 \\ b_1 & b_2 & b_3 & b_4 \end{smallmatrix}\right)}) = f_{\left(\begin{smallmatrix} y_1 & y_2 & y_3 & y_4 \\ b_1 & b_2 & b_3 & b_4 \end{smallmatrix}\right)}$.

By the construction of σ, if $z, z' \notin \{y_1, y_2, y_3, y_4\}$ then $\varphi^\sigma(z) = \varphi^\sigma(z')$ for every $\varphi \in \Phi_A$.

Now, let u be a minimal polynomial that solves P, so that $\operatorname{Cmp}_{(\mathbf{A},\sigma)} P = |u|_{\mathcal{L}_A}$, let $\varphi_{\left(\begin{smallmatrix} y_{11} & y_{12} & y_{13} & y_{14} \\ b_{11} & b_{12} & b_{13} & b_{14} \end{smallmatrix}\right)}$, $\varphi_{\left(\begin{smallmatrix} y_{21} & y_{22} & y_{23} & y_{24} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{smallmatrix}\right)}$, $\dots$, $\varphi_{\left(\begin{smallmatrix} y_{m1} & y_{m2} & y_{m3} & y_{m4} \\ b_{m1} & b_{m2} & b_{m3} & b_{m4} \end{smallmatrix}\right)}$ be the list of all symbols from Φ_A that occur in u and let

$$Y = \{y_{11}, y_{12}, y_{13}, y_{14}, \dots, y_{m1}, y_{m2}, y_{m3}, y_{m4}\}.$$

If there exist distinct indices i and j such that both $\mathbf{x}_i$ and $\mathbf{x}_j$ are disjoint from Y then a simple induction shows that $u^\sigma(\mathbf{x}_i) = u^\sigma(\mathbf{x}_j)$, which is impossible due to the choice of P and u. Therefore, at most one of the $\mathbf{x}_i$'s can be disjoint from Y, or, in other words, at least $n-1$ of the vectors $\mathbf{x}_1, \dots, \mathbf{x}_n$ are not disjoint from Y. Since all the $\mathbf{x}_i$'s are mutually disjoint, it follows that $|Y| \geqslant n-1$, whence $4m \geqslant |Y| \geqslant n-1$. This shows that $\operatorname{Cmp}_{(\mathbf{A},\sigma)} P = |u|_{\mathcal{L}_A} \geqslant |u|_{\Phi_A} = m \geqslant \frac{n-1}{4}$. □

Theorem 3.2. *There exists a positive constant* γ_2 *such that* $\operatorname{Cmp}^*(n,k) \leqslant \gamma_2 \cdot n^2 \log n$ *for all positive integers* n *and* k.

Proof. Clearly, it suffices to show that $\operatorname{Cmp}_{\mathbf{A}}(n,k) \leqslant \gamma_2 \cdot n^2 \log n$ for every Mal'cev interpolation algebra $\mathbf{A}$. So, take any Mal'cev interpolation algebra $\mathbf{A} = (A, F)$ and an arbitrary interpretation σ of $\mathcal{L}_A$. Let $P = \left(\begin{smallmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_n \\ a_1 & a_2 & \dots & a_n \end{smallmatrix}\right)$ be an arbitrary n-interpolation problem of dimension $m \geqslant 1$.

Let $p, q, r \in A$ be three arbitrary distinct elements of A and let us first construct a sequence of polynomials $g_2, \dots, g_n$ such that

$$g_k \text{ solves } \begin{pmatrix} \mathbf{x}_1 & \dots & \mathbf{x}_{k-1} & \mathbf{x}_k \\ p & \dots & p & q \end{pmatrix}.$$

We can always find a polynomial which solves a 2-interpolation problem $\left(\begin{smallmatrix} \mathbf{x}_l & \mathbf{x}_k \\ a & b \end{smallmatrix}\right)$. Namely, since $\mathbf{x}_k \neq \mathbf{x}_l$, there exists a coordinate j such that $x_{k,j} \neq x_{l,j}$. Then the function we are looking for is $\varphi^\sigma_{\left(\begin{smallmatrix} x_{l,j} & x_{k,j} & u & v \\ a & b & u' & v' \end{smallmatrix}\right)}$ for some u, u', v, $v' \in A$.

If $\mu^\sigma(p,q,r) = r$, for each k we can inductively construct terms $g'_{k/2}$ and $g''_{k/2}$ so that

$$g'_{k/2} \text{ solves } \begin{pmatrix} \mathbf{x}_1 & \dots & \mathbf{x}_{\lfloor \frac{k-1}{2} \rfloor} & \mathbf{x}_k \\ q & \dots & q & p \end{pmatrix}$$

	$\mathbf{x}_1$	$\dots$	$\mathbf{x}_{\lfloor\frac{k-1}{2}\rfloor}$	$\mathbf{x}_{\lfloor\frac{k-1}{2}\rfloor+1}$	$\dots$	$\mathbf{x}_{k-1}$	$\mathbf{x}_k$
h'	q	$\dots$	q	p or q	$\dots$	p or q	p
h''	q or r	$\dots$	q or r	q	$\dots$	q	r
c_r	r	$\dots$	r	r	$\dots$	r	r
$\mu(h', h'', c_r)$	q or r	$\dots$	q or r	r	$\dots$	r	p
$\varphi_{\left(\begin{smallmatrix} p & q & r & u \\ q & p & p & p \end{smallmatrix}\right)}(\mu(h', h'', c_r))$	p	$\dots$	p	p	$\dots$	p	q

Figure 1: Case 1. $\mu^\sigma(p, q, r) = r$

and

$$g''_{k/2} \text{ solves } \begin{pmatrix} \mathbf{x}_{\lfloor\frac{k-1}{2}\rfloor+1} & \dots & \mathbf{x}_{k-1} & \mathbf{x}_k \\ q & \dots & q & r \end{pmatrix}.$$

In order to take care of values $(g'_k)^\sigma(\mathbf{x}_i)$ for $\lfloor\frac{k-1}{2}\rfloor + 1 \leqslant i \leqslant k-1$ and $(g''_k)^\sigma(\mathbf{x}_j)$ for $1 \leqslant j \leqslant \lfloor\frac{k-1}{2}\rfloor$, note that there exists a sequence $\varphi'_1, \dots, \varphi'_s \in \Phi_A$ of the form $\varphi_{\left(\begin{smallmatrix} p & q & u & v \\ p & q & p & p \end{smallmatrix}\right)}$ and a sequence $\varphi''_1, \dots, \varphi''_t \in \Phi_A$ of the form $\varphi_{\left(\begin{smallmatrix} q & r & u & v \\ q & r & q & q \end{smallmatrix}\right)}$ such that $h' = \varphi'_1(\dots(\varphi'_s(g'_{k/2}))\dots)$ solves

$$\begin{pmatrix} \mathbf{x}_1 & \dots & \mathbf{x}_{\lfloor\frac{k-1}{2}\rfloor} & \mathbf{x}_{\lfloor\frac{k-1}{2}\rfloor+1} & \dots & \mathbf{x}_{k-1} & \mathbf{x}_k \\ q & \dots & q & p \text{ or } q & \dots & p \text{ or } q & p \end{pmatrix}$$

and $h'' = \varphi''_1(\dots(\varphi''_t(g''_{k/2}))\dots)$ solves

$$\begin{pmatrix} \mathbf{x}_1 & \dots & \mathbf{x}_{\lfloor\frac{k-1}{2}\rfloor} & \mathbf{x}_{\lfloor\frac{k-1}{2}\rfloor+1} & \dots & \mathbf{x}_{k-1} & \mathbf{x}_k \\ q \text{ or } r & \dots & q \text{ or } r & q & \dots & q & r \end{pmatrix}.$$

Then $\mu(h', h'', c_r)$ solves

$$\begin{pmatrix} \mathbf{x}_1 & \dots & \mathbf{x}_{\lfloor\frac{k-1}{2}\rfloor} & \mathbf{x}_{\lfloor\frac{k-1}{2}\rfloor+1} & \dots & \mathbf{x}_{k-1} & \mathbf{x}_k \\ q \text{ or } r & \dots & q \text{ or } r & r & \dots & r & p \end{pmatrix}$$

and finally, $\varphi_{\left(\begin{smallmatrix} p & q & r & u \\ q & p & p & p \end{smallmatrix}\right)}(\mu(h', h'', c_r))$ solves

$$\begin{pmatrix} \mathbf{x}_1 & \dots & \mathbf{x}_{\lfloor\frac{k-1}{2}\rfloor} & \mathbf{x}_{\lfloor\frac{k-1}{2}\rfloor+1} & \dots & \mathbf{x}_{k-1} & \mathbf{x}_k \\ p & \dots & p & p & \dots & p & q \end{pmatrix}.$$

This procedure can be schematically represented as in Fig. 1.

In case $\mu^\sigma(p, q, r) \in \{p, q\}$ the procedure is the same, and the corresponding scheme is given in Fig. 2, while in case $\mu^\sigma(p, q, r) = s \notin \{p, q, r\}$ we again follow the same procedure according to the scheme given in Fig. 3.

Due to the divide and conquer approach, one easily deduces that the complexity of the terms g_k is $O(k \log k)$.

	$\mathbf{x}_1$	$\dots$	$\mathbf{x}_{\lfloor \frac{k-1}{2} \rfloor}$	$\mathbf{x}_{\lfloor \frac{k-1}{2} \rfloor + 1}$	$\dots$	$\mathbf{x}_{k-1}$	$\mathbf{x}_k$
h'	p	$\dots$	p	p or q	$\dots$	p or q	q
c_q	q	$\dots$	q	q	$\dots$	q	q
h''	q or r	$\dots$	q or r	q	$\dots$	q	r
$\mu(h', c_q, h'')$	p or q	$\dots$	p or q	p or q	$\dots$	p or q	r
$\varphi_{\left(\begin{smallmatrix} p & q & r & u \\ p & p & q & p \end{smallmatrix}\right)}(\mu(h', c_q, h''))$	p	$\dots$	p	p	$\dots$	p	q

Figure 2: Case 2. $\mu^\sigma(p, q, r) \in \{p, q\}$

	$\mathbf{x}_1$	$\dots$	$\mathbf{x}_{\lfloor \frac{k-1}{2} \rfloor}$	$\mathbf{x}_{\lfloor \frac{k-1}{2} \rfloor + 1}$	$\dots$	$\mathbf{x}_{k-1}$	$\mathbf{x}_k$
h'	p or q	$\dots$	p or q	q	$\dots$	q	p
c_q	q	$\dots$	q	q	$\dots$	q	q
h''	q	$\dots$	q	q or r	$\dots$	q or r	r
$\mu(h', c_q, h'')$	p or q	$\dots$	p or q	q or r	$\dots$	q or r	s
$\varphi_{\left(\begin{smallmatrix} p & q & r & s \\ p & p & p & q \end{smallmatrix}\right)}(\mu(h', c_q, h''))$	p	$\dots$	p	p	$\dots$	p	q

Figure 3: Case 3. $\mu^\sigma(p, q, r) = s \notin \{p, q, r\}$

Using the terms $g_2, \dots, g_n$ we now construct a term f such that $f^\sigma(\mathbf{x}_i) = a_i$, $i \in \{1, \dots, n\}$. We shall inductively construct a sequence of terms f_2, f_3, $\dots$, $f_n = f$ such that

$$f_j \text{ solves } \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_j \\ a_1 & a_2 & \dots & a_j \end{pmatrix}.$$

The construction of f_2 is straightforward. Let us now show how to construct f_{k+1} starting from f_k. Let $z := f_k^\sigma(\mathbf{x}_{k+1})$ so that

$$f_k \text{ solves } \begin{pmatrix} \mathbf{x}_1 & \dots & \mathbf{x}_{k-1} & \mathbf{x}_k & \mathbf{x}_{k+1} \\ a_1 & \dots & a_{k-1} & a_k & z \end{pmatrix}.$$

If $z = a_{k+1}$ then $f_{k+1} = f_k$. Otherwise, $f_{k+1} = \mu(f_k, c_z, \varphi_{\left(\begin{smallmatrix} p & q & u & v \\ z & a_{k+1} & u & v \end{smallmatrix}\right)}(g_{k+1}))$ has the required properties. Since the complexity of g_k is $O(k \log k)$, it easily follows that the complexity of f is $O(n^2 \log n)$. This concludes the proof. □

References

[1] Aichinger, E., Local Interpolation Near-rings as a Frame-work for the Density Theorems. Contributions to General Algebra, Vol. 9, Verlag Hölder-Pichler-Tempsky, Wien - Verlag B.G. Teubner, Stuttgart, 1995, 27-36.

[2] Betsch, G., Primitive near-rings. Math. Z. 130 (1973), 351-361.

[3] Gumm, H. P., Algebras in congruence permutable varieties: Geometrical properties of affine algebras. Algebra Universalis 9 (1979), 8-34.

[4] Pilz, G., Primitive Near-rings and Interpolation Theory. Institutsbericht 1, Hochschule Linz, Institut für Mathematik, 4040 Linz, Austria, April 1974.

[5] Pilz, G., Near-rings, 2nd Ed. Amsterdam, New York, Oxford: North-Holland Publishing Company 1983.

[6] Werner, H., Eine Characterisierung funktional vollständiger Algebren. Arch. Math. (Basel) 21 (1970), 381-385

Received by the editors November 13, 2006