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HILBERT SPACE VALUED GENERALIZED RANDOM PROCESSES – PART I[1]

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Abstract. Generalized random processes by various types of continuity are considered and classified as generalized random processes (GRPs) of type (I) and (II). Structure theorems for Hilbert space valued generalized random processes are obtained: Series expansion theorems for GRPs (I) considered as elements of the spaces $\mathcal{L}(\mathcal{A}, S(H)_{-1})$ are derived, and structure representation theorems for GRPs (II) on $\mathcal{K}\{M_p\}(H)$ on a set with arbitrary large probability are given.

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Introduction

Generalized random (stochastic) processes can be defined in various ways, depending on whether they are regarded as a family of random variables or as a family of trajectories, but also depending on the type of continuity implied on this family. It is well known that a classical stochastic process $X(t,\omega)$, $t \in T \subseteq \mathbb{R}^d$, $\omega \in \Omega$, can be regarded either as a family of random variables $X(t,\cdot)$, $t \in T$, as a family of trajectories $X(\cdot,\omega)$, $\omega \in \Omega$, or as a family of functions $X : T \times \Omega \to \mathbb{R}^n$ ($\mathbb{R}^n$ is the state space) such that for each fixed $t \in T$, $X(t,\cdot)$ is an $\mathbb{R}^n$–valued random variable and for each fixed $\omega \in \Omega$, $X(\cdot,\omega)$ is an $\mathbb{R}^n$–valued deterministic function (called a trajectory). For classical stochastic processes these three concepts are equivalent, but if we replace the space of trajectories with some space of deterministic generalized functions, or if we replace the space of random variables with some space of generalized random variables, then we get different types of generalized stochastic processes. The classification of generalized stochastic processes by various conditions of continuity leads to structural theorems such as integral representations and series expansions, which will be subject of this paper.

Let us give now a historical overview of various definitions of the concept of generalized random processes (GRPs). One possible definition of a GRP is the one used by J.B. Walsh (see [17]) as a measurable mapping $X : \Omega \to \mathcal{D}'(T)$.

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For each $\phi \in \mathcal{D}(T)$, the mapping $\Omega \to \mathbb{R}$, $\omega \mapsto \langle X(\omega), \phi\rangle$ is a random variable. This definition is motivated by the fact that trajectories of Brownian motion are nowhere differentiable and can be considered as elements of $\mathcal{D}'(T)$. K. Itô defined in [7] a GRP as a linear and continuous mapping from $L^2(\mathbb{R})$ to the space $L^2(\Omega)$ of random variables with finite second moments, while Inaba considered in [6] a GRP as a continuous mapping from a certain space of test functions to the space $L^2(\Omega)$. Also, Gel'fand and Vilenkin [2] have considered GRPs in this sense. Further on, we will refer to GRPs defined in this sense as to **GRPs of type (I)**. On the other hand, O. Hanš, M. Ullrich, L. Swartz and others (see [3], [8], [14], [16]) defined a GRP as a mapping $\xi : \Omega \times V \to \mathbb{C}$ such that for every $\varphi \in V$, $\xi(\cdot, \varphi)$ is a complex random variable, and for every $\omega \in \Omega$, $\xi(\omega, \cdot)$ is an element in V', where V denotes a topological vector space and V' its dual space. Further on, we will refer to GRPs defined in this sense as to **GRPs of type (II)**.

Note, if a GRP is of type (I), then we do not have continuity of sample paths for each fixed $\omega \in \Omega$, only continuity in distributional sense. If we assume for a GRP (I) the continuity for a.e. fixed $\omega \in \Omega$, then it is also of type (II). However in [17] Walsh proved that, if the underlying test space V is nuclear, then for a GRP (I) there exists a version which is also a GRP (II). This result need not hold true if V is not nuclear, e.g. if it is a Hilbert space, as it was shown in [12]. Vice versa, if a GRP is of type (II), this does not ensure its continuity as a mapping from the test space $\mathcal{K}\{M_p\}$ into $L^2(\Omega)$ or even $L^0(\Omega)$ equipped with convergence in probability. It was shown in [12] under which conditions a GRP of type (II) is also a GRP (I) up to a set of arbitrarily small measure.

T. Hida, Y. Kondratiev, B. Øksendal, H.-H. Kuo, and many others (refer to [4], [5]) have developed a very general concept of GRPs via chaos expansions. In [5], GRPs are defined as measurable mappings $T \to (S)_{-1}$, where $(S)_{-1}$ denotes the Kondratiev space, but one can consider also some other spaces of generalized random variables instead of it. Thus, they are pointwisely defined with respect to the parameter $t \in T$ and generalized with respect to $\omega \in \Omega$.

GRPs generalized with respect to both arguments were introduced in [13] and [11], where we in fact generalized and unified the concept of a GRP in Inaba's sense and the previous definition, and considered GRPs as linear continuous mappings from the Zemanian test space $\mathcal{A}$ into $(S)_{-1}$. There we gave structural properties of these GRPs by series expansions in spaces of Zemanian generalized functions and a simultaneous chaos expansion. Since these processes, as elements of $\mathcal{L}(\mathcal{A}, (S)_{-1})$, are very close to the concept of a GRP (I), we also call them GRPs of type (I).

The aim of this paper is to generalize the results obtained in [11] and [12] to Hilbert space valued GRPs. The results in [11] for GRPs (I) and in [12] for GRPs (II) hold for processes with a finite–dimensional state space. In this paper we derive similar results for GRPs with an infinite–dimensional state space H, where H is a separable Hilbert space.

The paper is organized in the following manner: In the introductory section (Section 1) we provide some basic terminology and background information from the theory of generalized functions, generalized random variables and generalized random processes. In Section 2 we derive chaos expansion theorems for GRPs (I) considered as elements of the spaces $\mathcal{L}(\mathcal{A}, S(H)_{-1})$ i.e. as linear continuous mappings defined on the space of Zemanian test fuctions $\mathcal{A}$ and taking values in the H–valued Kondratiev space $S(H)_{-1}$. We also prove that in the case when $\mathcal{A}$ is nuclear, a GRP (I) can also be regarded as an element of the space $\mathcal{L}(\mathcal{A}(H), S_{-1})$. Section 3 contains an integral representation theorem on sets with arbitrary large probability for GRPs (II) on $\mathcal{K}\{M_p\}(H)$ spaces.

With these theorems established, in Part II of the paper, we will derive the Wick product for GRPs (I) and use the series expansion machinery for solving a class of linear and a class of nonlinear evolution stochastic differential equations.

1. Basic concepts

In this introductory chapter we give a brief overview of some classes of deterministic (e.g. Schwartz, Zemanian) and stochastic (Hida, Kondratiev, etc.) generalized function spaces. Definitions of some basic concepts, their most important properties and relations are given, which are necessary to understand the methods used in the sequent parts of the paper. Most of the material presented here is familiar and therefore given without proofs, but with references for further reading.

We use the notation $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ for multi-indices, $D^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$ for the differential operator, and $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. The length of a multi-index α is defined as $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$.

1.1. Hermite functions

The *Hermite polynomial of order* n, $n \in \mathbb{N}_0$, is defined by

$$h_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}}),\ x \in \mathbb{R}.$$

It is well known that the family $\{\frac{1}{\sqrt{n!}} h_n : n \in \mathbb{N}_0\}$ constitutes an orthonormal basis of the space $L^2(\mathbb{R}, d\mu)$, where $d\mu = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ is the Gaussian measure. The *Hermite function of order* $n+1$, $n \in \mathbb{N}_0$, is defined as

$$\xi_{n+1}(x) = \frac{1}{\sqrt[4]{\pi}\sqrt{n!}} e^{-\frac{x^2}{2}} h_n(\sqrt{2}x), \quad x \in \mathbb{R}.$$

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}_0^d$ be a multi-index. Define $\xi_\alpha = \xi_{\alpha_1} \otimes \xi_{\alpha_2} \otimes \cdots \otimes \xi_{\alpha_d}$. The set of multi-indeces α can be ordered in an ascending sequence as it is described in [5]. Denote by $\alpha^{(j)}$ the jth multi-index in this ordering. Hence, the family of vectors ξ_α can also be enumerated into a countable family $\eta_j = \xi_{\alpha^{(j)}}$, $j \in \mathbb{N}$. The family of functions $\{\eta_j : j \in \mathbb{N}\}$ is an orthonormal basis of the space $L^2(\mathbb{R}^d)$.

1.2. Zemanian spaces

Let I be an open interval in $\mathbb{R}$, and let $\mathcal{R}$ be a formally self-adjoint linear differential operator of the form

$$\mathcal{R} = \theta_0 D^{n_1}\theta_1 \cdots D^{n_\nu}\theta_\nu = \bar{\theta}_\nu(-D)^{n_\nu}\cdots(-D)^{n_2}\bar{\theta}_1(-D)^{n_1}\bar{\theta}_0 \tag{1}$$

where $D = d/dx$, θ_k are smooth complex functions without zero-points in I, and n_k are integers $k = 1, 2, \ldots, \nu$. Suppose that there exist a sequence of real numbers $\langle\lambda_n\rangle_{n\in\mathbb{N}}$, $\lim_{n\to\infty}|\lambda_n| = \infty$, and a sequence of smooth functions $\langle\psi_n\rangle_{n\in\mathbb{N}}$ in $L^2(I)$ which are the eigenvalues and eigenfunctions, respectively, of the operator $\mathcal{R}$, i.e. $\mathcal{R}\psi_n = \lambda_n\psi_n$, $n \in \mathbb{N}$. We can enumerate them in an ascending order: $|\lambda_1| \le |\lambda_2| \le |\lambda_3| \le \cdots \to \infty$. This re-ordering is made just for technical reasons, but it is neither unique nor unavoidable. Suppose that $\{\psi_n : n \in \mathbb{N}\}$ forms a complete orthonormal system in $L^2(I)$ with respect to the usual inner product denoted by $(\cdot|\cdot)$. Each function $f \in L^2(I)$ can be represented as an infinite sum $f = \sum_{n=1}^{\infty}(f|\psi_n)\psi_n$ converging in $L^2(I)$. Define inductively: $\mathcal{R}^0 = \mathcal{I}$, $\mathcal{R}^{k+1} = \mathcal{R}(\mathcal{R}^k)$, $k \in \mathbb{N}$. Note, $\lambda_n = 0$ for some $n \in \mathbb{N}$ implies $\lambda_k = 0$ for every $k < n$. From now on, if $\lambda_n = 0$, we replace it with $\widetilde{\lambda_n} = 1$, else we put $\widetilde{\lambda_n} = \lambda_n$, $n \in \mathbb{N}$.

Define:

$$\mathcal{A}_k = \{f = \sum_{n=1}^{\infty} a_n\psi_n : \sum_{n=1}^{\infty}|a_n|^2\widetilde{\lambda_n}^{2k} < \infty\}, \quad k \in \mathbb{Z}.$$

If $k \in \mathbb{N}_0$, then $\mathcal{A}_k \subseteq L^2(I)$. For each $k \in \mathbb{N}_0$, $\mathcal{A}_k$ is a Hilbert space when provided with the inner product $(f|g)_k = \sum_{n=1}^{\infty} a_n\bar{b}_n\widetilde{\lambda_n}^{2k}$, where $f = \sum_{n=1}^{\infty} a_n\psi_n$, $g = \sum_{n=1}^{\infty} b_n\psi_n \in \mathcal{A}_k$. Denote by $\|\cdot\|_k$ the norm induced by this inner product. The dual space $\mathcal{A}_k'$, equipped with the usual dual norm, is isomorphic with $\mathcal{A}_{-k}$. Thus, we have a sequence of linear continuous canonical inclusions

$$\cdots \subseteq \mathcal{A}_{k+1} \subseteq \mathcal{A}_k \subseteq \cdots \mathcal{A}_0 = L^2(I) \subseteq \mathcal{A}_{-1} \subseteq \mathcal{A}_{-2} \subseteq \cdots$$

The set

$$S = \{f \in L^2(I) : f = \sum_{n=1}^{m} a_n\psi_n,\ a_n \in \mathbb{C}, m \in \mathbb{N}\}, \tag{2}$$

i.e. the linear span of the set $\{\psi_n : n \in \mathbb{N}\}$, is dense in each $\mathcal{A}_k$, $k \in \mathbb{Z}$. Define:

$$\mathcal{A} = \bigcap_{k\in\mathbb{N}_0}\mathcal{A}_k = \left\{f \in L^2(I) : f = \sum_{n=1}^{\infty} a_n\psi_n,\ \forall k, \sum_{n=1}^{\infty}|a_n|^2\widetilde{\lambda_n}^{2k} < \infty\right\},$$

$$\mathcal{A}' = \bigcup_{k\in\mathbb{N}_0}\mathcal{A}_{-k} = \left\{f = \sum_{n=1}^{\infty} b_n\psi_n : \exists k, \sum_{n=1}^{\infty}|b_n|^2\widetilde{\lambda_n}^{-2k} < \infty\right\}.$$

The *Zemanian space of test functions* $\mathcal{A}$ is equipped with the projective topology, and its dual $\mathcal{A}'$, the *Zemanian space of generalized functions*, is equipped with the inductive topology which is equivalent to the strong dual topology. The action of a generalized function $f = \sum_{n=1}^{\infty} a_n\psi_n \in \mathcal{A}'$ onto a test function $\varphi = \sum_{n=1}^{\infty} b_n\psi_n$ is given by the dual pairing $\langle f, \varphi\rangle = \sum_{n=1}^{\infty} a_n b_n$. The orthonormal basis of $\mathcal{A}_k$, $k \in \mathbb{N}_0$, is the family of functions $\{\widetilde{\lambda_n}^{-k}\psi_n : n \in \mathbb{N}\}$. Note, $\mathcal{A}$ is nuclear if for some $c \in \mathbb{N}_0$ the condition $\sum_{n\in\mathbb{N}} \widetilde{\lambda_n}^{-2c} < \infty$ holds.

Example 1.1. In particular, for the choice $\mathcal{R} = -\frac{d^2}{dx^2} + x^2 + 1$, defined on a maximal domain in $L^2(\mathbb{R})$, $\mathcal{A}'$ is the space of tempered distributions $\mathcal{S}'(\mathbb{R})$.

Let $p \in \mathbb{N}$. Denote $\exp_p x = \underbrace{\exp(\exp(\cdots(\exp x))\cdots)}_{p}$ and define (see [10]) $\exp_p \mathcal{A}$ as the projective limit of the family

$$\exp_{p,k} \mathcal{A} = \{\varphi = \sum_{n=1}^{\infty} a_n\psi_n : \sum_{n=1}^{\infty} |a_n|^2(\exp_p \widetilde{\lambda_n})^{2k} < \infty\}, \quad k \in \mathbb{N}_0,$$

equipped with the norm $\|\varphi\|_{p,k} = \sum_{n=1}^{\infty} |a_n|^2(\exp_p \widetilde{\lambda_n})^{2k}$, $k \in \mathbb{N}_0$. Thus,

$$\exp_p \mathcal{A} = \bigcap_{k\in\mathbb{N}_0} \exp_{p,k} \mathcal{A}, \quad \exp_p \mathcal{A}' = \bigcup_{k\in\mathbb{N}_0} \exp_{p,-k} \mathcal{A}.$$

Clearly, S is dense in each $\exp_{p,k} \mathcal{A}$. The canonical inclusions $\exp_{p,k+1} \mathcal{A} \subseteq \exp_{p,k} \mathcal{A}$ are compact. Moreover, $\exp_p \mathcal{A}$ is nuclear if for some $c \in \mathbb{N}_0$ the series $\sum_{n=1}^{\infty}(\exp_p \widetilde{\lambda_n})^{-2c}$ converges. Define the pair of test and generalized function spaces $Exp\mathcal{A}$ and $Exp\mathcal{A}'$ as

$$Exp\mathcal{A} = projlim_{p\to\infty} \exp_p \mathcal{A}, \quad Exp\mathcal{A}' = indlim_{p\to\infty} \exp_p \mathcal{A}'.$$

The canonical inclusions $\exp_{p+1} \mathcal{A} \subseteq \exp_p \mathcal{A}$ are continuous and compact. The set S is dense in $\exp_p \mathcal{A}$ for each $p \in \mathbb{N}$. Hence, $Exp\mathcal{A}$ is dense in each $\exp_p \mathcal{A}$ as well as in $\mathcal{A}$.

1.2.1. Hilbert space valued Zemanian functions

Let H be a separable Hilbert space with orthonormal basis $\{e_i : i \in \mathbb{N}\}$. We will assume that $\mathcal{A}$ is nuclear, i.e. that there exists some $p \geq 0$, such that $\sum_{n=1}^{\infty} \widetilde{\lambda_n}^{-2p} < \infty$. This is necessary in order to have an isomorphism of $\mathcal{A}'(I; H)$ with the tensor product space $\mathcal{A}' \otimes H$ (refer to [15, Prop.50.7]). In general case, $\mathcal{A}' \otimes H$ would be isomorphic to a subspace of $\mathcal{A}'(I; H)$. Note that the notation $\otimes$ stands for the π–completition of the tensor product space, which is equivalent to the ϵ–completition in case $\mathcal{A}$ is nuclear.

Denote by $\mathcal{A}_k(I; H)$ the space of functions $f : I \to H$ of the form $f = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} a_{i,n}\psi_n e_i$ such that $\|f\|_{k;H} = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} |a_{i,n}|^2\widetilde{\lambda_n}^{2k} < \infty$. Let $\mathcal{A}(I; H) = projlim_{k\to\infty}\mathcal{A}_k(I; H)$. Clearly, $\mathcal{A}'_k(I; H)$ is isomorphic to $\mathcal{A}_{-k}(I; H)$ and we may define $\mathcal{A}'(I; H) = indlim_{k\to\infty}\mathcal{A}_{-k}(I; H)$.

A similar construction can be carried out also for the spaces $Exp\mathcal{A}$ and $Exp\mathcal{A}'$. We denote their Hilbert space valued versions by $Exp\mathcal{A}(I;H)$ and $Exp\mathcal{A}'(I;H)$.

1.3. Spaces of generalized random variables

The spaces of generalized random variables are stochastic analogues of deterministic generalized functions: they have no point value for $\omega \in \Omega$, only an average value with respect to a test random variable. For details refer to [4] or [5].

1.3.1. White Noise Space

Consider the Gel'fand triple $\mathcal{S}(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d) \subseteq \mathcal{S}'(\mathbb{R}^d)$, the Borel σ-algebra $\mathcal{B}$ generated by the weak topology on $\mathcal{S}'(\mathbb{R}^d)$ and the characteristic function $C(\phi) = \exp\{-\frac{1}{2}|\phi|^2_{L^2(\mathbb{R}^d)}\}$. According to the Bochner-Minlos theorem, there exists a unique measure μ on $(\mathcal{S}'(\mathbb{R}^d), \mathcal{B})$ such that for each $\phi \in \mathcal{S}(\mathbb{R}^d)$ the relation

$$\int_{\mathcal{S}'(\mathbb{R}^d)} e^{i\langle\omega,\phi\rangle} d\mu(\omega) = C(\phi)$$

holds. Here $\langle\omega,\phi\rangle$ is the dual pairing of $\mathcal{S}'(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$. The triplet $(\mathcal{S}'(\mathbb{R}^d), \mathcal{B}, \mu)$ is called *the white noise space* and μ is called the *white noise measure* or the *Gaussian measure* on $\mathcal{S}'(\mathbb{R}^d)$.

From now on we suppose that the basic probability space $(\Omega, \mathcal{F}, P)$ is the space $(\mathcal{S}'(\mathbb{R}^d), \mathcal{B}, \mu)$. Put $(L)^2 = L^2(\mathcal{S}'(\mathbb{R}^d), \mathcal{B}, \mu)$. It is a Hilbert space equipped with the inner product $(F|G)_{(L)^2} = E_\mu(F\overline{G})$.

In a multi-dimensional case, for a given $m \in \mathbb{N}$, $m > 1$, define $\mathcal{S}_m = \prod_{i=1}^m \mathcal{S}_i(\mathbb{R}^d)$, where $\mathcal{S}_i(\mathbb{R}^d)$ is a copy of $\mathcal{S}(\mathbb{R}^d)$, and let $\mathcal{S}'_m = \prod_{i=1}^m \mathcal{S}'_i(\mathbb{R}^d)$. Equip the space $\mathcal{S}'_m$ with the product Borel σ-algebra and with the product measure $\mu_m = \mu \times \cdots \times \mu$. The triple $(\mathcal{S}'_m, \mathcal{B}, \mu_m)$ is called the *m-dimensional d-parameter white noise space*. Put $(L)^{2,m} = L^2(\mathcal{S}'_m, \mu_m)$, and for $N \in \mathbb{N}$, $N > 1$, let $(L)^{2,m,N} = \bigoplus_{k=1}^N (L)^{2,m}$ be the direct sum of N identical copies of the m-dimensional d-parameter white noise space. Here the finite dimension N is the state space dimension. In Section 2, instead of the N–dimensional state space, we will also consider the infinite–dimensional case, when the state space is a separable Hilbert space H.

1.3.2. The Wiener-Itô chaos expansion

Denote by $\mathcal{I} = (\mathbb{N}_0^{\mathbb{N}})_c$ the set of sequences of integers which have only finitely many nonzero components. For a given $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathcal{I}$ define the *Fourier-Hermite polynomial* as $H_\alpha(\omega) = \prod_{i=1}^\infty h_{\alpha_i}(\langle\omega,\eta_i\rangle)$, $\omega \in \mathcal{S}'(\mathbb{R}^d)$. In the multi-dimensional valued case ($m > 1$) the orthonormal basis of the space $K = \bigoplus_{k=1}^m L^2(\mathbb{R}^d)$ is constituted of vectors of length m of the form $e^{(k)} = (0, \ldots, \eta_j, 0, \ldots)$, where η_j takes the ith place in the sequence, and i, j are integers such that $k = i + (j-1)m$, $i \in \{1, 2, \ldots, m\}$, $j \in \mathbb{N}$. In this case we define $H_\alpha^{(m)}(\omega) = \prod_{k=1}^\infty h_{\alpha_k}(\langle\omega, e^{(k)}\rangle)$, $\omega \in \mathcal{S}'_m$. The Fourier-Hermite polynomials

$H_\alpha^{(m)}$ form an orthogonal basis of $(L)^{2,m}$. It is also known that $\|H_\alpha^{(m)}\|_{(L)^{2,m}} = \sqrt{\alpha_1!\alpha_2!\cdots}$, $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathcal{I}$.

The Wiener-Itô expansion theorem (see [5]) states that each element $F \in (L)^{2,m,N}$ has a unique representation of the form

$$F(\omega) = \sum_{\alpha\in\mathcal{I}} c_\alpha H_\alpha^{(m)}(\omega), \quad \omega \in \mathcal{S}'_m, \quad c_\alpha \in \mathbb{R}^N,\ \alpha \in \mathcal{I},$$

such that $\|F\|^2_{(L)^{2,m,N}} = \sum_{\alpha\in\mathcal{I}} \alpha! c_\alpha^2$, where $c_\alpha^2 = (c_\alpha|c_\alpha)$ is the standard inner product in $\mathbb{R}^N$.

Example 1.2. Let $\varepsilon_j = (0, 0, \ldots, 1, 0, \ldots)$ be a sequence of zeros with the number 1 as the jth component. The one-dimensional d-parameter Brownian motion $B(t,\omega) = \langle \omega, \kappa_{[0,t]} \rangle$ has the expansion

$$B(t,\omega) = \sum_{j=1}^{\infty} \left(\int_0^t \eta_j(u) du \right) H_{\varepsilon_j}(\omega), \quad t \in \mathbb{R}^d,\ \omega \in \mathcal{S}'(\mathbb{R}^d).$$

1.3.3. The Kondratiev spaces

We will use the notation $\alpha^\beta = \alpha_1^{\beta_1}\alpha_2^{\beta_2}\cdots$ for the given multi-indices $\alpha, \beta \in \mathcal{I}$, and

$$(2\mathbb{N})^\gamma = \prod_{j=1}^{\infty} (2j)^{\gamma_j},$$

where $\gamma = (\gamma_1, \gamma_2, \ldots) \in \mathcal{I}$. Then, $\sum_{\alpha\in\mathcal{I}} (2\mathbb{N})^{-p\alpha} < \infty$ if and only if $p > 1$, and $\sum_{\alpha\in\mathcal{I}} e^{-p(2\mathbb{N})^\alpha} < \infty$ if and only if $p > 0$.

We will use the definition of the Kondratiev spaces given in [5], where the authors provide an equivalent construction of the original one introduced by Y. Kondratiev. The *space of the Kondratiev stochastic test functions (space of Kondratiev test random variables)* $(S)_\rho^{m,N}$ consists of those elements $f = \sum_{\alpha\in\mathcal{I}} c_\alpha H_\alpha^{(m)} \in (L)^{2,m,N}$, $c_\alpha \in \mathbb{R}^N$, $\alpha \in \mathcal{I}$, such that

$$\|f\|^2_{\rho,p} = \sum_{\alpha\in\mathcal{I}} c_\alpha^2 (\alpha!)^{1+\rho} (2\mathbb{N})^{p\alpha} < \infty, \quad \text{for all } p \in \mathbb{N}_0.$$

The *space of the Kondratiev stochastic generalized functions (space of Kondratiev generalized random variables)* $(S)_{-\rho}^{m,N}$ consists of formal expansions of the form $F = \sum_{\alpha\in\mathcal{I}} b_\alpha H_\alpha^{(m)}$, $b_\alpha \in \mathbb{R}^N$, $\alpha \in \mathcal{I}$, such that

$$\|F\|^2_{-\rho,-p} = \sum_{\alpha\in\mathcal{I}} b_\alpha^2 (\alpha!)^{1-\rho} (2\mathbb{N})^{-p\alpha} < \infty, \quad \text{for some } p \in \mathbb{N}_0.$$

The action of F onto a test function f is given by $\langle F, f \rangle = \sum_{\alpha\in\mathcal{I}} (b_\alpha|c_\alpha)\alpha!$ where $(\cdot|\cdot)$ is the standard inner product in $\mathbb{R}^N$.

The *generalized expectation* of F is defined as $E(F) = \langle F, 1 \rangle = b_0$.

The space $(S)_{-\rho}^{m,N}$ can also be constructed as the inductive limit of the family $(S_{-p})_{-\rho}^{m,N} = \{f = \sum_{\alpha\in\mathcal{I}} c_\alpha H_\alpha : \|f\|_{-\rho,-p} < \infty\}$, $p \in \mathbb{N}_0$.

In particular, for $\rho = 0$ the Kondratiev spaces are called the *Hida spaces* of test and generalized stochastic functions.

Example 1.3. One-dimensional d-parameter singular white noise is defined via the formal expansion

$$W(t,\omega) = \sum_{k=1}^{\infty} \eta_k(t) H_{\varepsilon_k}(\omega)$$

where $t \in \mathbb{R}^d$, and $\varepsilon_k,\ \eta_k$ are as in the previous example. Singular white noise belongs to the Hida space of generalized stochastic functions.

1.3.4. The Spaces $\exp(S)_{\rho}^{m,N}$ and $\exp(S)_{-\rho}^{m,N}$

In [13] and [11] the following space of generalized random variables was introduced: The space of stochastic test functions $exp(S)_{\rho}^{m,N}$ consists of those elements $f = \sum_{\alpha\in\mathcal{I}} c_\alpha H_\alpha^{(m)} \in (L)^{2,m,N}$, $c_\alpha \in \mathbb{R}^N$, $\alpha \in \mathcal{I}$, such that

$$\|f\|_{\rho,p,exp}^2 = \sum_{\alpha\in\mathcal{I}} c_\alpha^2 (\alpha!)^{1+\rho} e^{p(2\mathbb{N})^\alpha} < \infty, \quad \text{for all } p \in \mathbb{N}_0.$$

The space of stochastic generalized functions $exp(S)_{-\rho}^{m,N}$ consists of formal expansions of the form $F = \sum_{\alpha\in\mathcal{I}} b_\alpha H_\alpha^{(m)}$, $b_\alpha \in \mathbb{R}^N$, $\alpha \in \mathcal{I}$, such that

$$\|F\|_{-\rho,-p,exp}^2 = \sum_{\alpha\in\mathcal{I}} b_\alpha^2 (\alpha!)^{1-\rho} e^{-p(2\mathbb{N})^\alpha} < \infty, \quad \text{for some } p \in \mathbb{N}_0.$$

Note, for each $\rho \in [0,1]$, $exp(S)_{\rho}^{m,N}$ is nuclear and $exp(S)_{-\rho}^{m,N} \subseteq exp(S)_{-1}^{m,N}$ (the canonical inclusion $\exp(S)_{1}^{m,N} \subseteq exp(S)_{\rho}^{m,N}$ is compact). Moreover, the following relationship to the Kondratiev spaces holds:

$$exp(S)_{\rho}^{m,N} \subseteq (S)_{\rho}^{m,N} \subseteq (L)^{2,m,N} \subseteq (S)_{-\rho}^{m,N} \subseteq exp(S)_{-\rho}^{m,N}.$$

The canonical inclusion $exp(S)_{\rho}^{m,N} \subseteq (S)_{\rho}^{m,N}$ is compact. From the construction it follows that $exp(S)_{\rho}^{m,N}$ is dense in $(L)^{2,m,N}$, i.e. $exp(S)_{\rho}^{m,N} \subseteq (L)^{2,m,N} \subseteq exp(S)_{-\rho}^{m,N}$ is a Gel'fand triple.

1.3.5. Hilbert space valued generalized random variables

Let H be a separable Hilbert space with orthonormal basis $\{e_i : i \in \mathbb{N}\}$. As already suggested, we will treat H as the state space, i.e. we replace the N–dimensional case with an infinitedimensional state space (see [9] and the references therein). While in [9] the case $m = 1$ is considered, we keep our white noise space dimension to be $m \geq 1$.

Recall that the basic probability space is $(\mathcal{S}'_m, \mathcal{B}, \mu_m)$. Denote by $L^{2,m}(\Omega; H)$ the space of functions on Ω with values in H which are square integrable with respect to μ_m. The family of functions $\{H^{(m)}_\alpha e_i :\ i \in \mathbb{N}, \alpha \in \mathcal{I}\}$ is an orthogonal basis of the Hilbert space $L^{2,m}(\Omega; H)$. Each element of $L^{2,m}(\Omega; H)$ can be represented in either of the following forms:

$$f(\omega) = \sum_{i=1}^{\infty} a_i(\omega) e_i, \qquad a_i = \langle f, e_i\rangle_H \in (L)^{2,m}, \qquad \sum_{i=1}^{\infty} \|a_i\|^2_{(L)^{2,m}} < \infty,$$

$$f(\omega) = \sum_{i=1}^{\infty}\sum_{\alpha\in\mathcal{I}} a_{i,\alpha} H^{(m)}_\alpha(\omega) e_i,\ a_{i,\alpha} = \langle f, H^{(m)}_\alpha e_i\rangle \in \mathbb{R},\ \sum_{i=1}^{\infty}\sum_{\alpha\in\mathcal{I}} \alpha! |a_{i,\alpha}|^2 < \infty,$$

$$f(\omega) = \sum_{\alpha\in\mathcal{I}} a_\alpha H^{(m)}_\alpha(\omega),\ \ a_\alpha = \langle f, H^{(m)}_\alpha\rangle_{(L)^{2,m}} \in H,\ \sum_{\alpha\in\mathcal{I}} \alpha! \|a_\alpha\|^2_H < \infty.$$

Now, one can build up spaces of H–valued generalized random variables (H–valued Kondratiev spaces and others) over $L^{2,m}(\Omega; H)$ following the same ideas as in the $\mathbb{R}^N$–valued case. Let $\rho \in [0,1]$.

Define $S(H)^m_\rho$ as the space of functions $f \in L^{2,m}(\Omega; H)$, $f(\omega) = \sum_{i=1}^\infty \sum_{\alpha\in\mathcal{I}} a_{i,\alpha} H^{(m)}_\alpha(\omega) e_i$, $a_{i,\alpha} \in \mathbb{R}$, such that for all $p \in \mathbb{N}_0$,

$$\|f\|^2_{\rho,p;H} = \sum_{i=1}^{\infty}\sum_{\alpha\in\mathcal{I}} \alpha!^{1+\rho} |a_{i,\alpha}|^2 (2\mathbb{N})^{p\alpha} = \sum_{\alpha\in\mathcal{I}}\sum_{i=1}^{\infty} \alpha!^{1+\rho} |a_{i,\alpha}|^2 (2\mathbb{N})^{p\alpha} < \infty.$$

Define $S(H)^m_{-\rho}$ as the space of formal expansions $F(\omega) = \sum_{i=1}^\infty \sum_{\alpha\in\mathcal{I}} b_{i,\alpha} H^{(m)}_\alpha(\omega) e_i$, $b_{i,\alpha} \in \mathbb{R}$, such that for some $q \in \mathbb{N}_0$,

$$\|F\|^2_{-\rho,-q;H} = \sum_{i=1}^{\infty}\sum_{\alpha\in\mathcal{I}} \alpha!^{1-\rho} |b_{i,\alpha}|^2 (2\mathbb{N})^{-q\alpha} = \sum_{\alpha\in\mathcal{I}}\sum_{i=1}^{\infty} \alpha!^{1-\rho} |b_{i,\alpha}|^2 (2\mathbb{N})^{-q\alpha} < \infty.$$

Note, we can also write

$$f(\omega) = \sum_{\alpha\in\mathcal{I}}\sum_{i=1}^{\infty} a_{i,\alpha} H^{(m)}_\alpha(\omega) e_i = \sum_{\alpha\in\mathcal{I}} a_\alpha H^{(m)}_\alpha(\omega) = \sum_{i=1}^{\infty} a_i(\omega) e_i,$$

where $a_\alpha = \sum_{i=1}^\infty a_{i,\alpha} e_i \in H$ and $a_i = \sum_{\alpha\in\mathcal{I}} a_{i,\alpha} H^{(m)}_\alpha(\omega) \in (S)^m_\rho$. Also,

$$\|f\|^2_{\rho,p;H} = \sum_{\alpha\in\mathcal{I}} \alpha!^{1+\rho} \|a_\alpha\|^2_H (2\mathbb{N})^{p\alpha} = \sum_{i=1}^{\infty} \|a_i\|^2_{\rho,p}.$$

The same holds also for

$$F(\omega) = \sum_{\alpha\in\mathcal{I}}\sum_{i=1}^{\infty} b_{i,\alpha} H^{(m)}_\alpha(\omega) e_i = \sum_{\alpha\in\mathcal{I}} b_\alpha H^{(m)}_\alpha(\omega) = \sum_{i=1}^{\infty} b_i(\omega) e_i,$$

where $b_\alpha = \sum_{i=1}^{\infty} b_{i,\alpha} e_i \in H$ and $b_i = \sum_{\alpha\in\mathcal{I}} b_{i,\alpha} H_\alpha^{(m)}(\omega) \in (S)_{-\rho}^m$. Also,

$$\|F\|_{-\rho,-q;H}^2 = \sum_{\alpha\in\mathcal{I}} \alpha!^{1-\rho} \|b_\alpha\|_H^2 (2\mathbb{N})^{-q\alpha} = \sum_{i=1}^{\infty} \|b_i\|_{-\rho,-q}^2.$$

The action of F onto f is given by

$$\langle F, f\rangle = \sum_{\alpha\in\mathcal{I}} \alpha! \langle b_\alpha, a_\alpha\rangle_H.$$

Similarly as for the finite-dimensional case we have

$$S(H)_1^{(m)} \subseteq S(H)_\rho^{(m)} \subseteq S(H)_0^{(m)} \subseteq L^{2,m}(\Omega; H) \subseteq S(H)_{-0}^{(m)} \subseteq S(H)_{-\rho}^{(m)} \subseteq S(H)_{-1}^{(m)}.$$

The same construction can be carried out for the exponential growth rate spaces. Let $expS(H)_\rho^m$ be the space of functions $f \in L^{2,m}(\Omega; H)$, $f(\omega) = \sum_{i=1}^{\infty} \sum_{\alpha\in\mathcal{I}} a_{i,\alpha} H_\alpha^{(m)}(\omega) e_i$, $a_{i,\alpha} \in \mathbb{R}$, such that for all $p \in \mathbb{N}_0$,

$$\|f\|_{\rho,p,exp;H}^2 = \sum_{i=1}^{\infty} \sum_{\alpha\in\mathcal{I}} \alpha!^{1+\rho} |a_{i,\alpha}|^2 e^{p(2\mathbb{N})^\alpha} = \sum_{\alpha\in\mathcal{I}} \sum_{i=1}^{\infty} \alpha!^{1+\rho} |a_{i,\alpha}|^2 e^{p(2\mathbb{N})^\alpha} < \infty.$$

The corresponding space of stochastic generalized functions $expS(H)_{-\rho}^m$ consists of formal expansions of the form $F(\omega) = \sum_{i=1}^{\infty} \sum_{\alpha\in\mathcal{I}} b_{i,\alpha} H_\alpha^{(m)}(\omega) e_i$, $b_{i,\alpha} \in \mathbb{R}$, such that for some $q \in \mathbb{N}_0$,

$$\|F\|_{-\rho,-q,exp;H}^2 = \sum_{i=1}^{\infty} \sum_{\alpha\in\mathcal{I}} \alpha!^{1-\rho} |b_{i,\alpha}|^2 e^{-q(2\mathbb{N})^\alpha} = \sum_{\alpha\in\mathcal{I}} \sum_{i=1}^{\infty} \alpha!^{1-\rho} |b_{i,\alpha}|^2 e^{-q(2\mathbb{N})^\alpha} < \infty.$$

Both $S(H)_\rho^m$ and $expS(H)_\rho^m$ are countably Hilbert spaces and

$$expS(H)_\rho^m \subseteq S(H)_\rho^m \subseteq L^{2,m}(\Omega; H) \subseteq S(H)_{-\rho}^m \subseteq expS(H)_{-\rho}^m.$$

In general, $S(H)_\rho^m$ and $expS(H)_\rho^m$ are not nuclear spaces. They would be nuclear e.g. if H were finite–dimensional (see [15, Prop.50.1]).

Note, since $(S)_\rho^m$ and $exp(S)_\rho^m$ are nuclear spaces, by [15, Prop.50.7] we have again (as for the Schwartz spaces in the deterministic case) an isomorphism with the tensor product spaces:

$$S(H)_{-\rho}^m \cong (S)_{-\rho}^m \otimes H, \qquad expS(H)_{-\rho}^m \cong exp(S)_{-\rho}^m \otimes H.$$

Remark. For technical simplicity, in the sequel we consider only the case $m = 1$, but all results can be carried over to the general case.

Examples of elements of $S(H)_0$ are H–valued Brownian motion and singular white noise (see [9]).

1.4. Expansion theorems for GRPs (I)

We give now an overview of the results obtained in [13] and [11]. Elements of the spaces $\mathcal{L}(\mathcal{A},(S)_{-1})$ and $\mathcal{L}(\mathcal{A}, exp(S)_{-1})$, but also of $\mathcal{L}(Exp\mathcal{A},(S)_{-1})$ and $\mathcal{L}(Exp\mathcal{A}, exp(S)_{-1})$, are GRPs (I). As already mentioned, these processes are generalized both by the time-parameter t and by the random parameter ω. Further we will also denote by $[\cdot,\cdot]$ the action of an element from $\mathcal{L}(\mathcal{A},(S)_{-1})$ or $\mathcal{L}(\mathcal{A}, exp(S)_{-1})$ onto an element from $\mathcal{A}$, and with $\langle\cdot,\cdot\rangle$ the classical dual pairing in $\mathcal{A}'$ and $\mathcal{A}$.

Consider, for example, GRPs (I) as elements of the space $\mathcal{A}^* = \mathcal{L}(\mathcal{A},(S)_{-1})$. Elements of $\mathcal{A}^*_k = \mathcal{L}(\mathcal{A}_k,(S)_{-1})$ are called GRPs (I) of $\mathcal{R}$-order k. We have a chain of continuous canonical inclusions

$$(L^2(I))^* = \mathcal{A}^*_0 \subseteq \mathcal{A}^*_1 \subseteq \cdots \subseteq \mathcal{A}^*_k \subseteq \mathcal{A}^* = \bigcup_{k\in\mathbb{N}_0} \mathcal{A}^*_k.$$

For technical reasons we assume that the set of multi-indices $\mathcal{I}$ is ordered in a lexicographic order and denote by α^j, $j\in\mathbb{N}$, the jth element in this ordering.

Definition 1.1. *Let $f_j \in \mathcal{A}'$, $j = 1,2,\ldots,m$ and let $\theta_{\alpha^j} \in (S)_{-1}$, $j = 1,2,\ldots,m$. Then $\sum_{j=1}^m f_j\otimes\theta_{\alpha^j}$ is a GRP (I), i.e. an element of $\mathcal{A}^*$ defined by*

$$\left[\sum_{j=1}^m f_j\otimes\theta_{\alpha^j},\varphi\right] = \sum_{j=1}^m \langle f_j,\varphi\rangle\theta_{\alpha^j}, \qquad \varphi\in\mathcal{A}. \tag{3}$$

Theorem 1.1. *Let $k\in\mathbb{N}_0$. The following conditions are equivalent:*

(i) $\Phi\in\mathcal{A}^*_k$.

(ii) Φ *can be represented in the form*

$$\Phi = \sum_{j=1}^\infty f_j\otimes H_{\alpha^j}, \qquad f_j\in\mathcal{A}_{-k}, \quad j\in\mathbb{N}, \tag{4}$$

and there exists $k_0\in\mathbb{N}_0$ such that for each bounded set $B\subseteq\mathcal{A}_k$

$$\sup_{\varphi\in B}\sum_{j=1}^\infty |\langle f_j,\varphi\rangle|^2\,(2\mathbb{N})^{-k_0\alpha^j} < \infty. \tag{5}$$

(iii) Φ *can be represented in the form (4) and there exists $k_1\in\mathbb{N}_0$ such that*

$$\sum_{j=1}^\infty \|f_j\|^2_{-k}(2\mathbb{N})^{-k_1\alpha^j} < \infty. \tag{6}$$

Since $\mathcal{A}^*$ is constructed as the inductive limit of the family $\mathcal{A}^*_k$, $k\in\mathbb{N}_0$, we obtain the following expansion theorem for a GRP (I).

Theorem 1.2. *$\Phi \in \mathcal{A}^*$ if and only if there exist k, $k_0 \in \mathbb{N}_0$, such that series expansion (4) and condition (5) hold.*

It has been shown in [11] that these expansion theorems for GRPs (I) are consistent with the expansion theorems of Pettis–integrable generalized stochastic processes defined pointwisely as in [5].

Analogously to the previous theorems concerning GRPs (I) with values in $(S)_{-1}$ one can consider GRPs (I) taking values in other spaces of generalized stochastic functions. Consider, for example, the space $exp(S)_{-1}$, which will provide a larger class of GRPs (I). Let ${}^{exp}\mathcal{A}^* = \mathcal{L}(\mathcal{A},\, exp(S)_{-1})$ and ${}^{exp}\mathcal{A}^*_k = \mathcal{L}(\mathcal{A}_k,\, exp(S)_{-1})$ be GRPs (I) and GRPs (I) of $\mathcal{R}$-order k, respectively. Further, let all the other terms be defined analogously as for $\mathcal{A}^*$; i.e. we replace $(S)_{-1}$ with $exp(S)_{-1}$ in Definition 1.1 and else where necessary.

Theorem 1.3. *Let $k \in \mathbb{N}_0$. The following conditions are equivalent:*

(i) $\Phi \in {}^{exp}\mathcal{A}^*_k$.

(ii) Φ *can be represented in the form*

$$\Phi = \sum_{j=1}^{\infty} f_j \otimes H_{\alpha^j}, \qquad f_j \in \mathcal{A}_{-k}, \quad j \in \mathbb{N}, \tag{7}$$

and there exists $k_0 \in \mathbb{N}_0$, such that for each bounded set $B \subseteq \mathcal{A}_k$

$$\sup_{\varphi \in B} \sum_{j=1}^{\infty} |\langle f_j, \varphi \rangle|^2 e^{-k_0 (2\mathbb{N})^{\alpha^j}} < \infty. \tag{8}$$

(iii) Φ *can be represented in the form (7) and there exists $k_1 \geq 0$, such that*

$$\sum_{j=1}^{\infty} \|f_j\|^2_{-k} e^{-k_1 (2\mathbb{N})^{\alpha^j}} < \infty.$$

For examples of GRPs (I) refer to [11].

1.5. $\mathcal{K}\{M_p\}$ spaces

Now we give a brief overview of some basic notions from the theory of $\mathcal{K}\{M_p\}$ spaces, which are constructed similarly as the tempered distributions, but are more general. For further details refer to [1].

Let $(M_p)_{p \in \mathbb{N}_0}$ be a sequence of continuous functions on $\mathbb{R}$ such that the following conditions are satisfied:

$$1 \leq M_p(x) \leq M_{p'}(x),\ x \in \mathbb{R},\ \ p < p'. \tag{9}$$

(P) For every $p \in \mathbb{N}_0$ there is $p' \in \mathbb{N}_0$ such that

$$\lim_{|x| \to \infty} M_p(x) M_{p'}^{-1}(x) = 0.$$

(N) For every $p \in \mathbb{N}_0$ there is $p' \in \mathbb{N}_0$ such that $M_p M_{p'}^{-1} \in L^1(\mathbb{R})$.

$\mathcal{K}\{M_p\}$ is defined as a space of smooth functions $\varphi \in C^\infty(\mathbb{R})$ endowed with the family of norms

$$\| \varphi \|_p = \sup\{M_p(x)|\varphi^{(i)}(x)| : \ x \in \mathbb{R}, \ i \le p\} \ , \ p \in \mathbb{N}_0.$$

For the properties of $\mathcal{K}\{M_p\}$ and its strong dual $\mathcal{K}'\{M_p\}$ we refer the reader to [1]. In this paper we shall consider a subclass of such spaces. Namely, as in [2, p.82], we shall assume that $\{M_p, \ p \in \mathbb{N}_0\}$ are smooth functions such that

(I) for every $k, p \in \mathbb{N}_0$ there exist $p' \in \mathbb{N}_0$ and $C > 0$ such that

$$|M_p^{(k)}(x)| \le C M_{p'}(x), \quad x \in \mathbb{R}.$$

With the quoted conditions on M_p, $p \in \mathbb{N}_0$, the sequence of norms $\| \cdot \|_p$, $p \in \mathbb{N}_0$, is equivalent to the sequence of norms

$$\| \varphi \|_{p,2} = \sup\{(\int_{\mathbb{R}} |M_p(x)\varphi^{(i)}(x)|^2 dx)^{1/2} : \ \ i \le p\}, \ p \in \mathbb{N}_0.$$

For example, if we choose $M_p(x) = (1 + |x|^2)^{\frac{p}{2}}$, we obtain the space of rapidly decreasing functions $\mathcal{S}(\mathbb{R})$ as $\mathcal{K}\{M_p\}$.

Further on we will also assume that the weight functions M_p satisfy condition

(T) for every $p \in \mathbb{N}_0$ there exist $\tilde{p} \in \mathbb{N}_0$ such that

$$M_p(x - u)M_p(u) \le M_{\tilde{p}}(x), \quad 0 \le u \le |x|, \ x \in \mathbb{R}.$$

Note that the functions $M_p(x) = (1 + |x|^2)^{\frac{p}{2}}$, and the functions defined to be $M_p(x) = e^{p|x|^r}$, for $|x| > x_0 > 0$ and smooth around zero, $r \in [1, \infty)$, satisfy condition (T).

1.5.1. H–valued $\mathcal{K}\{M_p\}$ spaces

Conditions (P) and (N) imply that $\mathcal{K}\{M_p\}$ is a nuclear space. Thus, if H is a Hilbert space with orthonormal basis $\{e_i : \ i \in \mathbb{N}\}$, we can consider the H–valued $\mathcal{K}\{M_p\}$ spaces, denoted by $\mathcal{K}\{M_p\}(H)$, as the tensor product $\mathcal{K}\{M_p\}(H) \cong \mathcal{K}\{M_p\} \otimes H$. Thus, a function ϕ belongs to $\mathcal{K}\{M_p\}(H)$ if and only if it is of the form $\phi = \sum_{i=1}^\infty \phi_i e_i$, $\phi_i \in \mathcal{K}\{M_p\}$, $i \in \mathbb{N}$, and $\|\phi\|_{p,2;H}^2 = \sum_{i=1}^\infty \|\phi_i\|_p^2 < \infty$ for all $p \in \mathbb{N}_0$.

1.6. Structure theorems for GRPs (II)

We give now an overview of the results obtained in [12]. Let V be a topological vector space, V' its dual space, $(\Omega, \mathcal{F}, P)$ be a probability space and $Z^p = L^p(\Omega)$, $p \ge 1$, be the space of random variables X such that

$\int_\Omega |X|^p dP < \infty$. By $L^r(\mathbb{R}^n)$, $r \geq 1$, we denote the space of r–integrable functions with respect to the Lebesgue measure m.

Definition 1.2. *A GRP (II) is a mapping $\xi : \Omega \times V \to \mathbb{C}$ such that for every $\varphi \in V$, $\xi(\cdot, \varphi)$ is a complex random variable and for every $\omega \in \Omega$, $\xi(\omega, \cdot)$ is an element in V'.*

As in [2], we will suppose that the expectation of ξ, denoted by $E(\xi(\omega, \varphi)) = m(\varphi)$, $\varphi \in V$, exists and belongs to V'. Due to this fact we shall suppose that $E(\xi(\cdot, \varphi)) = 0$ for every $\varphi \in V$.

The proof of the following results can be found in [12].

Theorem 1.4. *Let $G = \prod_{i=1}^n (\alpha_i, \beta_i) \subset \mathbb{R}^n$, $-\infty \leq \alpha_i < \beta_i \leq \infty$, $i = 1, 2, ..., n$, and let ξ be a GRP on $\Omega \times L^r(G)$, $r > 1$.*

a) There exists a function $f : \Omega \times G \to \mathbb{C}$ such that

(i) for every $x \in G$, $f(\cdot, x)$ is measurable and for every $\omega \in \Omega$, $f(\omega, \cdot) \in L^p(G)$, $p = r/(r-1)$.

(ii)

$$\xi(\omega, \varphi) = \int_G f(\omega, t)\varphi(t)dt, \quad \omega \in \Omega, \quad \varphi \in L^r(G).$$

b) Let $G = \mathbb{R}^n$. If there exists $A \in \mathcal{F}$ such that $P(A) = 0$ and

$$|\xi(\omega, \varphi)| \leq C(\omega)||\varphi||_r\ ,\ \omega \in \Omega \backslash A,$$

then the correlation operator $C_\xi(\cdot, \cdot)$ has a representation

$$C_\xi(\varphi, \psi) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \varphi(t) E(f(\omega, t)\overline{f(\omega, s)})\overline{\psi(s)} dt ds, \quad \varphi, \psi \in L^r(G).$$

Theorem 1.5. *a) Let ξ be a GRP on $\mathcal{K}\{M_p\}$. Then for every $\varepsilon > 0$ there exist $d \in \mathbb{N}_0$, $M \in \mathcal{F}$ satisfying $P(M) \geq 1 - \varepsilon$, and functions $f_\alpha : \Omega \times \mathbb{R} \to \mathbb{C}$, $\alpha = 0, 1, ..., d$, such that $f_\alpha(\cdot, t)$ is measurable for every $t \in \mathbb{R}$, $f_\alpha(\omega, \cdot)$ is in $L^2(\mathbb{R})$ for every $\omega \in M$, $\alpha = 0, 1, ..., d$ and*

$$\xi(\omega, \varphi) = \sum_{\alpha=0}^{d} \int_{\mathbb{R}} f_\alpha(\omega, t) M_d(t) \varphi^{(\alpha)}(t) dt, \quad \omega \in M, \quad \varphi \in \mathcal{K}\{M_p\}, \tag{10}$$

$$\sum_{\alpha=0}^{d} \parallel f_\alpha(\omega, \cdot) \parallel_{L^2} \leq d, \quad \omega \in M. \tag{11}$$

In particular, if there exist $C(\omega) > 0$, $\omega \in \Omega$, and $d \in \mathbb{N}$ such that

$$|\xi(\omega, \varphi)| \leq C(\omega) \parallel \varphi \parallel_{d,2},\ \omega \in \Omega,\ \varphi \in \mathcal{K}\{M_p\}, \tag{12}$$

then representation (10) is valid on the whole Ω.

b) Moreover, if ξ is also a continuous mapping from $\mathcal{K}\{M_p\}$ to Z^2, then for almost every $t, s \in \mathbb{R}$ there exist $E(f_\alpha(\cdot,t)\overline{f_\beta(\cdot,s)})$, $\alpha \le d$, $\beta \le d$ and the correlation operator $C_\xi(\varphi,\psi)$, $\varphi,\psi \in \mathcal{K}\{M_p\}$ has the representation

$$C_\xi(\varphi,\psi) =$$

$$= \sum_{\alpha=0}^{d}\sum_{\beta=0}^{d}\int_{\mathbb{R}}\int_{\mathbb{R}} E(f_\alpha(\cdot,t)\overline{f_\beta(\cdot,s)})M_d(t)\overline{M_d(s)}\varphi^{(\alpha)}(t)\overline{\psi^{(\beta)}(s)}dtds.$$

c) If ξ is a GRP on $\mathcal{K}\{M_p\}$ such that (12) holds and $\omega \mapsto C(\omega)$ is in Z^2, then $\xi : \mathcal{K}\{M_p\} \to Z^2$ is continuous and (10) holds for every $\omega \in \Omega$. Condition $C(\cdot) \in Z^2$ is sufficient but not necessary for the continuity of $\xi : \mathcal{K}\{M_p\} \to Z^2$.

2. Hilbert space valued GRPs (I)

Now we expand on the results of [11] and consider Hilbert space valued GRPs. Let H be a separable Hilbert space with orthonormal basis $\{e_i : i \in \mathbb{N}\}$. We replace the Kondratiev space $(S)_{-1}$ with the H–valued Kondratiev space $S(H)_{-1}$ and define H–valued GRPs (I) as linear continuous mappings from the Zemanian test space $\mathcal{A}$ into $S(H)_{-1}$, i.e. as elements of

$$\mathcal{A}(H)^* = \mathcal{L}(\mathcal{A}, S(H)_{-1}).$$

Elements of $\mathcal{A}(H)^*_k = \mathcal{L}(\mathcal{A}_k, S(H)_{-1})$ are called H–valued GRPs (I) of $\mathcal{R}$-order k. Thus, $L \in \mathcal{A}(H)^*_k$ if and only if there exists $k_0 \in \mathbb{N}$ such that $L \in \mathcal{L}(\mathcal{A}_k, S(H)_{-1,-k_0})$. Note that $\mathcal{L}(\mathcal{A}_k, S(H)_{-1,-k_0})$ is a Banach space with the usual dual norm

$$\|L\|^*_{-k;H} = \sup\{\|[L,g]\|_{-1,-k_0;H} : g \in \mathcal{A}_k, \|g\|_k \le 1\}.$$

Clearly, $\mathcal{A}(H)'_k \subseteq \mathcal{A}(H)^*_k$, and $\|f\|^*_{-k;H} = \|f\|_{-k;H}$ if $f \in \mathcal{A}(H)_{-k}$. We have a chain of continuous canonical inclusions

$$L^2(I;H) = \mathcal{A}(H)^*_0 \subseteq \mathcal{A}(H)^*_1 \subseteq \cdots \subseteq \mathcal{A}(H)^*_k \subseteq \mathcal{A}(H)^* = \bigcup_{k\in\mathbb{N}_0} \mathcal{A}(H)^*_k.$$

Definition 2.1. *Let $f_j \in \mathcal{A}'$, $j = 1,2,\ldots,m$ and let $\theta_{\alpha^j} \in S(H)_{-1}$, $j = 1,2,\ldots,m$. Then $\sum_{j=1}^m f_j \otimes \theta_{\alpha^j}$ is an element of $\mathcal{A}(H)^*$ defined by*

$$\left[\sum_{j=1}^m f_j \otimes \theta_{\alpha^j}, \varphi\right] = \sum_{j=1}^m \langle f_j,\varphi\rangle\theta_{\alpha^j}, \qquad \varphi \in \mathcal{A}. \tag{13}$$

Recall, each $\theta_{\alpha^j} \in S(H)_{-1}$ can be represented as

$$\theta_{\alpha^j}(\omega) = \sum_{i=1}^\infty\sum_{k=1}^\infty \theta_{jik}H_{\alpha^k}(\omega)e_i, \quad \theta_{jik} \in \mathbb{R}. \tag{14}$$

Thus, (13) can be written in an equivalent form

$$\sum_{j=1}^{m}\langle f_j,\varphi\rangle\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\theta_{jik}H_{\alpha^k}(\omega)e_i=\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\langle F_{ik},\varphi\rangle H_{\alpha^k}(\omega)e_i$$
$$=\left[\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}F_{ik}\otimes H_{\alpha^k}(\omega)e_i,\varphi\right],$$

where $F_{ik}=\sum_{j=1}^{m}f_j\theta_{jik}\in\mathcal{A}'$.

Lemma 2.1. *Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}'$ and $\langle\theta_{\alpha^j}\rangle_{j\in\mathbb{N}}$ be a sequence in $S(H)_{-1}$. If there exists $k_0\in\mathbb{N}_0$ such that for any bounded set $B\in\mathcal{A}$,*

$$\sup_{\varphi\in B}\sum_{j=1}^{\infty}|\langle f_j,\varphi\rangle|\cdot\|\theta_{\alpha^j}\|_{-1,-k_0;H}<\infty, \tag{15}$$

then $\sum_{j=1}^{\infty}f_j\otimes\theta_{\alpha^j}$ defined by

$$\sum_{j=1}^{\infty}f_j\otimes\theta_{\alpha^j}=\lim_{m\to\infty}\sum_{j=1}^{m}f_j\otimes\theta_{\alpha^j}$$

is an element of $\mathcal{A}(H)^*$.

Proof. Denote $\Upsilon=\sum_{j=1}^{\infty}f_j\otimes\theta_{\alpha^j}$ and $\Upsilon_m=\sum_{j=1}^{m}f_j\otimes\theta_{\alpha^j}$, $m\in\mathbb{N}$. Clearly, $\Upsilon_m\in\mathcal{A}^*$. By (15), $\Upsilon_m\in\mathcal{L}(\mathcal{A},S(H)_{-1,-k_0})$.

The sequence of partial sums Υ_m, $m\in\mathbb{N}$, is a Cauchy sequence in $\mathcal{A}(H)^*$ because, for given $\varepsilon>0$ and $m>n$,

$$\|[\Upsilon_m,\varphi]-[\Upsilon_n,\varphi]\|_{-1,-k_0;H}=\sum_{j=n+1}^{m}|\langle f_j,\varphi\rangle|\cdot\|\theta_{\alpha^j}\|_{-1,-k_0;H}<\varepsilon,$$

if n,m are large enough.

Since

$$\|\Upsilon_m\|^*=\sup\{\|[\Upsilon,\varphi]\|_{-1,-k_0;H}:\varphi\in\mathcal{A},\|\varphi\|\le 1\}$$
$$\le\sup\{\sum_{j=1}^{m}|\langle f_j,\varphi\rangle|\cdot\|\theta_{\alpha^j}\|_{-1,-k_0;H}:\varphi\in\mathcal{A},\|\varphi\|\le 1\},$$

it follows that

$$\sup_{m\in\mathbb{N}}\|\Upsilon_m\|^*\le\sup_{m\in\mathbb{N}}\left(\sup\{\sum_{j=1}^{m}|\langle f_j,\varphi\rangle|\cdot\|\theta_{\alpha^j}\|_{-1,-k_0;H}:\varphi\in\mathcal{A},\|\varphi\|\le 1\}\right)$$
$$\le\sup\{\sum_{j=1}^{\infty}|\langle f_j,\varphi\rangle|\cdot\|\theta_{\alpha^j}\|_{-1,-k_0;H}:\varphi\in\mathcal{A},\|\varphi\|\le 1\}<\infty,$$

by condition (15). According to the Banach-Steinhaus theorem,

$$\Upsilon = lim_{m\to\infty}\Upsilon_m$$

also belongs to $\mathcal{L}(\mathcal{A}, S(H)_{-1,-k_0})$. □

Theorem 2.1. *Let $k \in \mathbb{N}_0$. The following conditions are equivalent:*

(i) $\Phi \in \mathcal{A}(H)^*_k$.

(ii) Φ *can be represented in the form*

$$\Phi = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f_{ij} \otimes H_{\alpha^j}\, e_i, \qquad f_{ij} \in \mathcal{A}_{-k}, \quad i,j \in \mathbb{N}, \tag{16}$$

and there exists $k_0 \in \mathbb{N}_0$ such that for each bounded set $B \subseteq \mathcal{A}_k$

$$\sup_{\varphi\in B}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\langle f_{ij}, \varphi\rangle|^2 (2\mathbb{N})^{-k_0\alpha^j} < \infty. \tag{17}$$

Proof. Let $\Phi \in \mathcal{A}^*_k = \mathcal{L}(\mathcal{A}_k, S(H)_{-1})$. There exists $k_0 \in \mathbb{N}_0$, such that $\Phi \in \mathcal{L}(\mathcal{A}_k, S(H)_{-1;-k_0})$. The mapping $f_{ij} : \mathcal{A}_k \to \mathbb{R}$ given by

$$\varphi \mapsto ([\Phi, \varphi]|H_{\alpha^j} e_i)_{-1,-k_0;H}$$

is linear and continuous for each $H_{\alpha^j} e_i$, i.e. $f_{ij} \in \mathcal{A}'_k = \mathcal{A}_{-k}$ for each $i, j \in \mathbb{N}$. Thus,

$$\langle f_{ij}, \varphi\rangle = ([\Phi, \varphi]|H_{\alpha^j} e_i)_{-1,-k_0;H}\,, \quad \varphi \in \mathcal{A}_k, \quad j \in \mathbb{N}.$$

Also, $[\Phi, \varphi] \in S(H)_{-1;-k_0}$ has the expansion

$$[\Phi, \varphi] = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} ([\Phi, \varphi]|H_{\alpha^j} e_i)_{-1,-k_0;H}\, H_{\alpha^j} e_i. \tag{18}$$

The series on the right-hand side of (18) converges if and only if

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \left|([\Phi, \varphi]|H_{\alpha^j} e_i)_{-1,-k_0;H}\right|^2 (2\mathbb{N})^{-k_0\alpha^j} = \sum_{j=1}^{\infty} |\langle f_j, \varphi\rangle|^2 (2\mathbb{N})^{-k_0\alpha^j} < \infty,$$

which yields (17). Now, by Definition 2.1, (18) is equal to

$$[\Phi, \varphi] = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\langle f_{ij}, \varphi\rangle H_{\alpha^j} e_i = \left[\sum_{j=1}^{\infty} f_{ij} \otimes H_{\alpha^j} e_i, \varphi\right],$$

and this implies (16).

Conversely, let $\Phi = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f_{ij} \otimes H_{\alpha^j} e_i$, where $f_{ij} \in \mathcal{A}_{-k}$, $i, j \in \mathbb{N}$, and let (17) hold for any bounded set $B \subseteq \mathcal{A}_k$.

Since $[\Phi, \varphi] \in S(H)_{-1;-k_0}$, it has the expansion

$$\begin{aligned}
[\Phi, \varphi] &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}([\Phi, \varphi]|H_{\alpha^j} e_i)_{-1,-k_0;H} H_{\alpha^j} e_i \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}([\sum_{l=1}^{\infty}\sum_{k=1}^{\infty} f_{lk} \otimes H_{\alpha^k} e_l, \varphi]|H_{\alpha^j} e_i)_{-1,-k_0;H} H_{\alpha^j} e_i \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\langle f_{lk}, \varphi\rangle H_{\alpha^k} e_l|H_{\alpha^j} e_i)_{-1,-k_0;H} H_{\alpha^j} e_i \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\langle f_{ij}, \varphi\rangle (2\mathbb{N})^{-k_0\alpha^j} H_{\alpha^j} e_i,\ \varphi \in \mathcal{A}_k,
\end{aligned}$$

where in the last step the orthogonality of the basis $H_{\alpha^j} e_i$ was used.

The sequence of partial sums $\Phi_m = \sum_{i=1}^{m}\sum_{j=1}^{m} f_{ij} \otimes H_{\alpha^j} e_i$, $m \in \mathbb{N}$, is a Cauchy sequence in $\mathcal{L}(\mathcal{A}_k, S(H)_{-1;-k_0})$ because, for given $\varepsilon > 0$,

$$\|[\Phi_m, \varphi] - [\Phi_n, \varphi]\|^2_{-1,-k_0;H} = \sum_{i=n+1}^{m}\sum_{j=n+1}^{m} |\langle f_{ij}, \varphi\rangle|^2 (2\mathbb{N})^{-k_0\alpha^j} < \varepsilon,$$

if we choose n, m large enough. This yields that $\langle \Phi_m\rangle_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{A}(H)^*_k$. Also, (17) implies that

$$\sup_{m\in\mathbb{N}} \|\Phi_m\|^{*2}_{-k;H} \leq \sup\{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\langle f_{ij}, \varphi\rangle|^2 (2\mathbb{N})^{-k_0\alpha^j} : \varphi \in \mathcal{A}_k, ||\varphi||_k \leq 1\} < \infty.$$

Thus, due to the Banach-Steinhaus theorem, $\Phi_0 = \lim_{m\to\infty} \Phi_m \in \mathcal{A}(H)^*_k$. So it has to be of the form

$$\Phi_0 = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \tilde{f}_{ij} \otimes H_{\alpha^j} e_i.$$

It remains to show that $\Phi_0 = \Phi$. Since

$$\begin{aligned}
\langle \tilde{f}_{ij}, \varphi\rangle - \langle f_{ij}, \varphi\rangle &= ([\Phi_0, \varphi]|H_{\alpha^j} e_i)_{-1,-k_0;H} - ([\Phi, \varphi]|H_{\alpha^j} e_i)_{-1,-k_0;H} \\
&= ([\lim_{m\to\infty} \Phi_m, \varphi]|H_{\alpha^j} e_i)_{-1,-k_0;H} - ([\Phi, \varphi]|H_{\alpha^j} e_i)_{-1,-k_0;H} \\
&= \lim_{m\to\infty} ([\Phi_m - \Phi, \varphi]|H_{\alpha^j} e_i)_{-1,-k_0;H} \\
&= \lim_{m\to\infty} ([\sum_{i=m+1}\sum_{j=m+1}^{\infty} f_{ij} \otimes H_{\alpha^j} e_i, \varphi]|H_{\alpha^j} e_i)_{-1,-k_0;H} \\
&= \lim_{m\to\infty} \sum_{i=m+1}\sum_{j=m+1}^{\infty} |\langle f_{ij}, \varphi\rangle|^2 (2\mathbb{N})^{-k_0\alpha^j} = 0
\end{aligned}$$

for any $\varphi \in \mathcal{A}_k$, it implies that $\tilde{f}_{ij} = f_{ij}$, $i, j \in \mathbb{N}$. □

Corollary 2.1. *If Φ can be represented in the form (16) and there exists $k_1 \in \mathbb{N}_0$ such that*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \|f_{ij}\|_{-k}^2 (2\mathbb{N})^{-k_1\alpha^j} < \infty, \tag{19}$$

then $\Phi \in \mathcal{A}(H)_k^*$.

Proof. According to the Cauchy-Schwartz inequality, it is obvious that (19) implies (17). □

Remark: Note, that in the finite dimensional valued case we had an equivalence between (5) and (6) in Theorem 1.1. But now (17) does not necessarily imply (19). This implication would be true only if $S(H)_{-1}$ were a nuclear space (see [2, Theorem 1, page 67]), which it is not.

Since $\mathcal{A}(H)^*$ is constructed as the inductive limit of the family $\mathcal{A}(H)_k^*$, $k \in \mathbb{N}_0$, we obtain the following expansion theorem for an H–valued GRP (I).

Theorem 2.2. *$\Phi \in \mathcal{A}(H)^*$ if and only if there exist $k, k_0 \in \mathbb{N}_0$ such that series expansion (16) and condition (17) hold.*

Let U be a H–valued GRP in the sense of [5], given by the expansion

$$U(t,\omega) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}(t) H_{\alpha^j}(\omega) e_i, \qquad t \in \mathbb{R}, \quad \omega \in \mathcal{S}'(\mathbb{R}),$$

such that $a_{ij}(t) \in L^1_{loc}(\mathbb{R})$, $i, j \in \mathbb{N}$. Then there is a H–valued GRP (I), denoted by $\tilde{U}$ associated with U, such that

$$[\tilde{U}, \varphi](\omega) = \int_{\mathbb{R}} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}(t) H_{\alpha^j}(\omega) e_i \varphi(t) dt = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \langle \tilde{a}_{ij}, \varphi \rangle H_{\alpha^j}(\omega) e_i,$$

$\omega \in \mathcal{S}'(\mathbb{R})$, where $\tilde{a}_{ij} \in \mathcal{S}'(\mathbb{R})$ is the generalized function associated with the function $a_{ij}(t) \in L^1_{loc}(\mathbb{R})$, $i, j \in \mathbb{N}$. Thus, $\tilde{U}$ has the expansion

$$\tilde{U} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \tilde{a}_{ij} \otimes H_{\alpha^j}.$$

The expansion theorems for ${}^{exp}\mathcal{A}(H)^* = \mathcal{L}(\mathcal{A}, exp S(H)_{-1})$ can also be stated as in the case of a one–dimensional state space:

Theorem 2.3. *Let $k \in \mathbb{N}_0$. The following conditions are equivalent:*

(i) $\Phi \in {}^{exp}\mathcal{A}(H)_k^*$.

(ii) Φ *can be represented in the form*

$$\Phi = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f_{ij} \otimes H_{\alpha^j}\ e_i, \qquad f_{ij} \in \mathcal{A}_{-k}, \quad i,j \in \mathbb{N}, \tag{20}$$

and there exists $k_0 \in \mathbb{N}_0$ *such that for each bounded set* $B \subseteq \mathcal{A}_k$

$$\sup_{\varphi\in B} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\langle f_{ij}, \varphi\rangle|^2 e^{-k_0(2\mathbb{N})^{\alpha^j}} < \infty. \tag{21}$$

Corollary 2.2. *If* Φ *can be represented in the form (20) and there exists* $k_1 \in \mathbb{N}_0$ *such that*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \|f_{ij}\|_{-k}^2 e^{-k_1(2\mathbb{N})^{\alpha^j}} < \infty,$$

then $\Phi \in {}^{exp}\mathcal{A}(H)_k^*$.

Theorem 2.4. $\Phi \in {}^{exp}\mathcal{A}(H)^*$ *if and only if there exist* k, $k_0 \in \mathbb{N}_0$ *such that the series expansion (20) and condition (21) hold.*

2.1. GRPs (I) on nuclear spaces

Recall, since $(S)_{-1}$ is a nuclear space, we have $S(H)_{-1} \cong (S)_{-1} \otimes H$. Assume now that $\mathcal{A}$ is also a nuclear space (this is not a strict restriction since in most cases it is one). Then, by Proposition 50.7. in [15], we have $\mathcal{L}(\mathcal{A}, S(H)_{-1}) \cong \mathcal{A}' \otimes S(H)_{-1}$. Combining this with the previous remark, we can now consider GRPs (I) as elements of $\mathcal{A}' \otimes (S)_{-1} \otimes H$, or, if we regroup the spaces, also as elements of $\mathcal{A}' \otimes H \otimes (S)_{-1}$, which is again by nuclearity of $\mathcal{A}$ isomorphic to $\mathcal{A}'(I;H) \otimes (S)_{-1}$. In other words, it is equivalent whether we consider the state space H as the codomain of the generalized random variables or as the codomain of the deterministic generalized functions representing the trajectories of the process.

Similarly as we did for GRPs (I), we have a representation for elements of $\mathcal{A}'(I;H)$. A function g belongs to $\mathcal{A}_{-k}(I;H)$ if and only if it is of the form $\sum_{i=1}^{\infty} g_i \otimes e_i$, $g_i \in \mathcal{A}_{-k}$ and $sup_{\varphi\in B} \sum_{i=1}^{\infty} |\langle g_i, \varphi\rangle|^2 < \infty$ holds for each bounded set $B \subseteq \mathcal{A}_k$. The sum $\sum_{i=1}^{\infty} g_i \otimes e_i$ is defined by the action $\langle \sum_{i=1}^{\infty} g_i \otimes e_i, \varphi\rangle = \sum_{i=1}^{\infty}\langle g_i, \varphi\rangle e_i$, $\varphi \in \mathcal{A}$, provided the latter sum converges in H.

Thus, if Φ is a GRP (I) given by the expansion $\Phi = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f_{ij} \otimes H_{\alpha^j} e_i$, $f_{ij} \in \mathcal{A}_{-k}$, we can rewrite its action in the following manner:

$$\left[\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f_{ij} \otimes H_{\alpha^j} e_i, \varphi\right] = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\langle f_{ij}, \varphi\rangle H_{\alpha^j} e_i = \sum_{j=1}^{\infty}\langle\sum_{i=1}^{\infty} f_{ij} \otimes e_i, \varphi\rangle H_{\alpha^j}.$$

Also, from $sup_{\varphi\in B} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\langle f_{ij}, \varphi\rangle|^2 (2\mathbb{N})^{-p\alpha^j} < \infty$, $B \subseteq \mathcal{A}_k$, we get $sup_{\varphi\in B} \sum_{j=1}^{\infty} \|\langle g_j, \varphi\rangle\|_H^2 (2\mathbb{N})^{-p\alpha^j} < \infty$, where $g_j = \sum_{i=1}^{\infty} f_{ij} \otimes e_i \in \mathcal{A}_{-k}(I;H)$ and $\|g_j\|_{-k;H}^2 = \sum_{i=1}^{\infty} |f_{ij}|^2 \tilde{\lambda}_i^{-k}$. In view of these facts we can now reformulate our representation theorem for GRPs (I):

Theorem 2.5. *Let $k \in \mathbb{N}_0$. The following conditions are equivalent:*

(i) $\Phi \in \mathcal{A}(H)_k^*$.

(ii) Φ *can be represented in the form*

$$\Phi = \sum_{j=1}^{\infty} f_j \otimes H_{\alpha^j}, \qquad f_j \in \mathcal{A}_{-k}(I;H), \quad j \in \mathbb{N}, \tag{22}$$

and there exists $k_0 \in \mathbb{N}_0$ *such that for each bounded set* $B \subseteq \mathcal{A}_k$

$$\sup_{\varphi \in B} \sum_{j=1}^{\infty} \|\langle f_j, \varphi \rangle\|_H^2 \, (2\mathbb{N})^{-k_0 \alpha^j} < \infty. \tag{23}$$

Corollary 2.3. *If* Φ *can be represented in the form (22) and there exists* $k_1 \in \mathbb{N}_0$ *such that*

$$\sum_{j=1}^{\infty} \|f_j\|_{-k;H}^2 (2\mathbb{N})^{-k_1 \alpha^j} < \infty, \tag{24}$$

then $\Phi \in \mathcal{A}(H)_k^*$.

3. Hilbert space valued GRPs (II)

Recall, H is a separable Hilbert space over $\mathbb{C}$ with orthonormal basis $\{e_n : n \in \mathbb{N}\}$. While for GRPs (I) we had $\mathcal{L}(\mathcal{A}(H);(S)_{-1}) \cong \mathcal{L}(\mathcal{A};S(H)_{-1})$, i.e. it was equivalent whether H was the codomain of the x–variable function space or the ω–variable function space, for GRPs (II) we have a different situation already by the definition of a GRP (II).

Now we state the definition of a Hilbert space valued GRP (II) and the corresponding structure theorems for them. We will restrict our attention to GRPs on $\mathcal{K}\{M_p\}$ spaces.

Definition 3.1. *An* H*-valued GRP (II) is a mapping* $\xi : \Omega \times \mathcal{K}\{M_p\}(H) \to \mathbb{C}$ *such that:*

(i) for every $\varphi \in \mathcal{K}\{M_p\}(H)$, $\xi(\cdot, \varphi)$ *is a complex random variable,*

(ii) for every $\omega \in \Omega$, $\xi(\omega, \cdot)$ *is an element in* $\mathcal{K}'\{M_p\}(H)$.

For $r > 1$ denote $L^r(\mathbb{R}^n; H) = L^r(\mathbb{R}^n) \otimes H$ and recall that its dual is $L^p(\mathbb{R}^n; H)$, $p = r/(r-1)$. The dual pairing of $f \in L^p(\mathbb{R}^n; H)$, $\varphi \in L^r(\mathbb{R}^n; H)$ can be written as $\int_{\mathbb{R}^n} \langle f(t), \varphi(t) \rangle_H dt$. It can easily be checked that the following H–valued version of Theorem 1.4 holds.

Theorem 3.1. *Let* $G = \prod_{i=1}^n (\alpha_i, \beta_i) \subset \mathbb{R}^n$, $-\infty \leq \alpha_i < \beta_i \leq \infty$, $i = 1, 2, ..., n$, *and let* ξ *be a GRP on* $\Omega \times L^r(G; H)$, $r > 1$. *There exists a function* $f : \Omega \times G \to H$ *such that* *(i) for every* $x \in G$*,* $f(\cdot, x)$ *is measurable and for every* $\omega \in \Omega$*,* $f(\omega, \cdot) \in L^p(G; H)$*,* $p = r/(r-1)$*.*

(ii)

$$\xi(\omega, \varphi) = \int_G \langle f(\omega, t), \varphi(t)\rangle_H dt, \quad \omega \in \Omega, \quad \varphi \in L^r(G; H).$$

The following H–valued analogue of Theorem 1.5 holds:

Theorem 3.2. *a) Let* ξ *be an* H*–valued GRP (II). Then for every* $\varepsilon > 0$ *there exist* $d \in \mathbb{N}_0$*,* $M \in \mathcal{F}$ *satisfying* $P(M) \geq 1 - \varepsilon$*, and functions* $f_\alpha : \Omega \times \mathbb{R} \to H$*,* $\alpha = 0, 1, ..., d$*, such that* $f_\alpha(\cdot, t)$ *is measurable for every* $t \in \mathbb{R}$*,* $f_\alpha(\omega, \cdot)$ *is in* $L^2(\mathbb{R}; H)$ *for every* $\omega \in M$*,* $\alpha = 0, 1, ..., d$ *and*

(25)
$$\xi(\omega, \varphi) = \sum_{\alpha=0}^{d} \int_{\mathbb{R}} \langle f_\alpha(\omega, t), M_d(t)\varphi^{(\alpha)}(t)\rangle_H dt, \quad \omega \in M, \quad \varphi \in \mathcal{K}\{M_p\}(H),$$

(26)
$$\sum_{\alpha=0}^{d} \| f_\alpha(\omega, \cdot) \|_{L^2(\mathbb{R};H)} \leq d, \quad \omega \in M.$$

In particular, if there exist $C(\omega) > 0$*,* $\omega \in \Omega$*, and* $d \in \mathbb{N}$ *such that*

(27)
$$|\xi(\omega, \varphi)| \leq C(\omega) \| \varphi \|_{d,2;H}, \ \omega \in \Omega, \ \varphi \in \mathcal{K}\{M_p\}(H),$$

then representation (25) is valid on the whole Ω*.*

b) Moreover, if ξ *is also a continuous mapping from* $\mathcal{K}\{M_p\}(H)$ *to* Z^2*, then for almost every* $t, s \in \mathbb{R}$ *there exist* $E(\langle f_\alpha(\cdot, t), \overline{f_\beta(\cdot, s)}\rangle_H)$*,* $\alpha \leq d$*,* $\beta \leq d$ *and the correlation operator* $C_\xi(\varphi, \psi)$*,* $\varphi, \psi \in \mathcal{K}\{M_p\}$ *has the representation*

$$C_\xi(\varphi, \psi) =$$

$$= \sum_{\alpha=0}^{d} \sum_{\beta=0}^{d} \int_{\mathbb{R}} \int_{\mathbb{R}} E(\langle f_\alpha(\cdot, t), \overline{f_\beta(\cdot, s)}\rangle_H) M_d(t)\overline{M_d(s)} \langle \varphi^{(\alpha)}(t), \overline{\psi^{(\beta)}(s)}\rangle_H dtds.$$

c) If ξ *is a GRP on* $\mathcal{K}\{M_p\}(H)$ *such that (27) holds and* $\omega \mapsto C(\omega)$ *is in* Z^2*, then* $\xi : \mathcal{K}\{M_p\}(H) \to Z^2$ *is continuous and (25) holds for every* $\omega \in \Omega$*. Condition* $C(\cdot) \in Z^2$ *is sufficient but not necessary for the continuity of* $\xi : \mathcal{K}\{M_p\}(H) \to Z^2$*.*

Proof. a) Since for every $\omega \in \Omega$, $\xi(\omega, \cdot)$ is in $\mathcal{K}'\{M_p\}(H)$, it follows that for every $\omega \in \Omega$ there exist $C(\omega) > 0$ and $p(\omega) \in \mathbb{N}$ such that

$$|\xi(\omega, \varphi)| \leq C(\omega) \| \varphi \|_{p(\omega),2;H}, \ \varphi \in \mathcal{K}\{M_p\}(H).$$

We can assume that $p(\omega) \geq C(\omega)$. For every $\varphi \in \mathcal{K}\{M_p\}(H)$ and $N \in \mathbb{N}$, put

$$A_N(\varphi) = \{\omega \in \Omega : \ |\xi(\omega, \varphi)| < N \parallel \varphi \parallel_{N,2;H}\}, \ \ A_N = \bigcap_{\varphi \in \mathcal{K}\{M_p\}(H)} A_N(\varphi).$$

Since $\mathcal{K}\{M_p\}(H)$ is separable, it contains a countable dense subset D and $A_N = \bigcap_{\varphi \in D} A_N(\varphi) \in \mathcal{F}$. Thus, from

$$\Omega = \bigcup_{N=1}^{\infty} A_N \quad \text{and} \quad A_N \subset A_{N+1}, \ \ N \in \mathbb{N},$$

it follows that for the given $\varepsilon > 0$ there exists an integer d such that $P(A_d) \geq 1 - \varepsilon$. Denote $M = A_d$. It follows

$$|\xi(\omega, \varphi)| \leq d \parallel \varphi \parallel_{d,2;H}, \ \ \omega \in M, \ \varphi \in \mathcal{K}\{M_p\}(H).$$

We extend ξ on the whole Ω by

$$\xi_1(\omega, \varphi) = \begin{cases} \xi(\omega, \varphi), & \omega \in M \\ 0, & \omega \notin M \end{cases}, \ \varphi \in \mathcal{K}\{M_p\}(H). \tag{28}$$

Further, put $R = \{\varphi \in \mathcal{K}\{M_p\}(H) : \ \parallel \varphi \parallel_{d,2;H} \leq 1\}$ and

$$S(\omega) = \sup_{\varphi \in R} |\xi_1(\omega, \varphi)| = \sup_{\varphi \in D \bigcap R} |\xi_1(\omega, \varphi)|, \ \omega \in \Omega.$$

It follows that S is measurable on Ω, $S(\omega) \leq d, \omega \in \Omega$. Thus,

$$|\xi_1(\omega, \varphi)| \leq S(\omega) \parallel \varphi \parallel_{d,2;H}, \ \ \varphi \in \mathcal{K}\{M_p\}(H), \ \omega \in \Omega. \tag{29}$$

Inequality (29) holds also for the space $H^d_M(\mathbb{R}; H) \subset H^d(\mathbb{R}; H)$, where $H^d(\mathbb{R}; H) \cong H^d(\mathbb{R}) \otimes H$ is the H–valued Sobolev space, and $H^d_M = \{\varphi \in H^d(\mathbb{R}; H) : M_d \varphi^{(\alpha)} \in L^2(\mathbb{R}; H), \ \alpha = 0, 1, \ldots, d\}$, equipped with the topology induced by the norm $||\varphi||_{d,L^2;H} = \sum_{\alpha=0}^{d} ||M_d \varphi^{(\alpha)}||_{L^2(\mathbb{R};H)}$.

We need the following consequence of (29):

$$\text{if } (\varphi_\nu)_{\nu \in \mathbb{N}} \text{ is a sequence in } \mathcal{K}\{M_p\}(H) \ \text{ and } \ \varphi_\nu \to 0 \text{ in } \ H^d_M, \quad \text{then } \ \xi_1(\omega, \varphi_\nu) \to 0, \nu \to \infty. \tag{30}$$

Let $\Gamma_d = \prod_{i=0}^{d} L^2(\mathbb{R}; H)$ and endow it with the scalar product $((\varphi_\alpha), (\psi_\alpha)) = \sum_{\alpha=0}^{d} \int_{\mathbb{R}} \langle \varphi_\alpha, \overline{\psi_\alpha} \rangle_H dt$, $(\varphi_\alpha), (\psi_\alpha) \in \Gamma_d$. Clearly, Γ_d is a Hilbert space. Define a mapping $\theta : \mathcal{K}\{M_p\}(H) \to \Gamma_d$ by $\theta(\varphi) = (M_d \varphi, M_d \varphi', \ldots, M_d \varphi^{(d)})$, $\varphi \in \mathcal{K}\{M_p\}(H)$, which is injective, and denote $\Delta = \theta(\mathcal{K}\{M_p\}(H))$. Note that

$$\overline{\Delta} = \theta(H^d_M). \tag{31}$$

Define a mapping $\Omega \times \Gamma_d \to \mathbb{C}$, for every $\omega \in \Omega$, by

$$F(\omega, \psi) = \begin{cases} \xi_1(\omega, \theta^{-1}(\psi)), & \psi \in \Delta \\ \lim_{\nu \to \infty} \xi_1(\omega, \theta^{-1}(\psi_\nu)), & \psi \in \bar{\Delta}, \quad \psi_\nu \in \Delta, \quad \psi_\nu \overset{L^2(\mathbb{R};H)}{\longrightarrow} \psi, \\ 0, & \psi \in \bar{\Delta}^{\perp}. \end{cases}$$

The existence of the limit follows from (30) and (31). Thus,

$$F(\omega,\tilde{\psi}) = F(\omega,\psi),\ \tilde{\psi}\in\Gamma_d,\ \ \tilde{\psi}=\psi+\psi^{\perp},\ \psi\in\bar{\Delta},\psi^{\perp}\in\bar{\Delta}^{\perp}.$$

Clearly, $F(\cdot,\tilde{\psi})$ is measurable for any $\tilde{\psi}\in\Gamma_d$. Let $\varphi\in\mathcal{K}\{M_p\}(H)$, $\omega\in\Omega$. We have

$$|F(\omega,\theta(\varphi))| \leq S(\omega)\parallel\varphi\parallel_{d,2;H} = S(\omega)\parallel\theta(\varphi)\parallel_{\Gamma_d}.$$

So, for every $\omega\in\Omega$, $F(\omega,\cdot)$ is a continuous linear functional on Γ_d and it is of the form

$$F(\omega,\cdot)=\sum_{\alpha=0}^{d} F_\alpha(\omega,\cdot),\ \omega\in\Omega.$$

Here $F_\alpha(\omega,\cdot)$, $\in\Omega$, $\alpha\leq d$, are continuous linear functionals on the subspaces $\Gamma_{d,\alpha}\subset\Gamma_d$, $\alpha\leq d$, where

$$\Gamma_{d,\alpha}=\{\psi=(\psi_\beta)\in\Gamma_d:\ \psi_\beta\in L^2(\mathbb{R};H),\ \psi_\beta\equiv 0,\ \beta\neq\alpha\}.$$

It is endowed with the natural norm such that it is isometric to $L^2(\mathbb{R};H)$, for every $\alpha\in\{0,1,...,d\}$. Let $\psi\in\Gamma_d$. Denote by $[\psi]_\alpha$ the corresponding element in $\Gamma_{d,\alpha}$. The βth coordinates of $[\psi]_\alpha$ are equal to zero for $\beta\neq\alpha$ and the αth coordinate is equal to ψ_α. Since F is a GRP (II), it follows that $F_\alpha = F|_{\Gamma_{d,\alpha}}$ is a GRP (II) on $\Omega\times\Gamma_{d,\alpha}$ i.e. on $\Omega\times L^2(\mathbb{R};H)$, for every α, $\alpha\leq d$.

By Theorem 3.1, for every $\alpha=0,1,\ldots,d$, there exists a function $f_\alpha:\Omega\times\mathbb{R}\to H$ such that $f_\alpha(\cdot,t)$ is measurable for every $t\in\mathbb{R}$, $f_\alpha(\omega,\cdot)\in L^2(\mathbb{R};H)$, $\omega\in\Omega$, and

$$F_\alpha(\omega,\varphi)=\int_{\mathbb{R}}\langle f_\alpha(\omega,t),\varphi(t)\rangle_H dt,\quad \varphi\in L^2(\mathbb{R};H),\ \omega\in\Omega.$$

Thus, if $\omega\in\Omega$ and $\psi=\theta(\varphi)$ for $\varphi\in\mathcal{K}\{M_p\}(H)$, then

$$F(\omega,\psi)=\sum_{\alpha=0}^{d}F_\alpha(\omega,[\psi]_\alpha)=\sum_{\alpha=0}^{d}\int_{\mathbb{R}^n}\langle f_\alpha(\omega,t),M_d(t)\psi^{(\alpha)}\rangle_H dt \tag{32}$$

and

$$||F(\omega,\cdot)||'_{\Gamma_d}=\sum_{\alpha=0}^{d}||f_\alpha(\omega,\cdot)||_{L^2(\mathbb{R};H)}\leq S(\omega)\leq d,\quad \omega\in\Omega,$$

where $||\cdot||'_{\Gamma_d}$ is the dual norm. Now, the assertion follows by (28).

The proof of the last assertion in a) follows by repeating the previous proof starting from relation (29). It follows that ξ is of the form (10) for every $\omega\in\Omega$.

b) Obviously, $C_\xi(\varphi,\psi)=E(\langle\xi(\cdot,\varphi),\overline{\xi(\cdot,\psi)}\rangle_H)$, $\varphi,\psi\in\mathcal{K}\{M_p\}(H)$ is bilinear. The continuity follows from

$$C_\xi(\varphi,\psi)=|E(\langle\xi(\cdot,\varphi),\overline{\xi(\cdot,\psi)}\rangle_H)|\leq\parallel\xi(\cdot,\varphi)\parallel_{Z^2}\parallel\xi(\cdot,\psi)\parallel_{Z^2}\leq$$

$$\leq\parallel\varphi\parallel_{d,2;H}\parallel\psi\parallel_{d,2;H}\sup\{\parallel\xi(\cdot,\varphi)\parallel_{Z^2},\varphi\in\mathcal{K}\{M_p\}(H),\parallel\varphi\parallel_{d,2;H}\leq 1\}$$

$$\cdot \sup\{\| \xi(\cdot,\psi) \|_{Z^2}, \psi \in \mathcal{K}\{M_p\}(H), \| \psi \|_{d,2;H} \leq 1\}.$$

Fubini's theorem implies

$$C_\xi(\varphi,\psi) = E(\langle \xi(\cdot,\varphi), \overline{\xi(\cdot,\psi)}\rangle_H) =$$

$$E((\sum_{\alpha=0}^{d}\int_{\mathbb{R}}\langle f_\alpha(\cdot,t), M_d(t)\varphi^{(\alpha)}(t)\rangle_H dt)(\sum_{\alpha=0}^{d}\int_{\mathbb{R}}\langle \overline{f_\alpha(\cdot,s)}, \overline{M_d(s)}\overline{\psi^{(\alpha)}(s)}\rangle_H ds))$$

$$= \sum_{\alpha=0}^{d}\sum_{\beta=0}^{d} E(\int_{\mathbb{R}}\int_{\mathbb{R}}\langle f_\alpha(\cdot,t), M_d(t)\varphi^{(\alpha)}(t)\rangle_H \langle \overline{f_\beta(\cdot,s)}, \overline{M_d(s)}\overline{\psi^{(\beta)}(s)}\rangle_H dtds)$$

$$= \sum_{\alpha=0}^{d}\sum_{\beta=0}^{d}(\int_{\mathbb{R}}\int_{\mathbb{R}} E(\langle f_\alpha(\cdot,t), \overline{f_\beta(\cdot,s)}\rangle_H) M_d(t)\overline{M_d(s)}\langle \varphi^{(\alpha)}(t), \overline{\psi^{(\beta)}(s)}\rangle_H dtds.$$

This proves the last assertion in b).
c) If ξ is a GRP on $\mathcal{K}\{M_p\}(H)$ which satisfies (12) and $C(\cdot) \in Z^2$, then ξ is a continuous mapping $\mathcal{K}\{M_p\}(H) \to Z^2$. Namely, for any sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathcal{K}\{M_p\}(H)$ such that $\varphi_n \to 0, n \to \infty$, it follows

$$\| \xi(\cdot,\varphi_n) \|_{Z^2} = E(|\xi(\cdot,\varphi_n)|^2) \leq E(C^2(\cdot)) \| \varphi_n \|^2_{d,2;H} \to 0.$$

The next example shows that $\xi : \mathcal{K}\{M_p\}(H) \to Z^2$ may be a continuous mapping from $\mathcal{K}\{M_p\}(H)$ to Z^2 although $C(\cdot) \notin Z^2$. □

Example 3.1. Let $\Omega = \mathbb{R}$, $H = \mathbb{R}$, $\mathcal{F}$ be the Borel field, $P(A) = \int_A \frac{dx}{\pi(1+x^2)}$ for $A \in \mathcal{F}$, and let, for $x \in \mathbb{R}$, $\varphi \in \mathcal{K}\{M_p\}$,

$$\xi(x,\varphi) = X(x)\varphi(X(x)) = X(x)\langle \delta(y - X(x)), \varphi(y)\rangle. \tag{33}$$

It is a GRP (II), and moreover it is a continuous mapping from $\mathcal{K}\{M_p\}$ to Z^2, i.e. it is also a GRP (I). For every $x \in \mathbb{R}$ we have

$$|\xi(x,\varphi)| = |X(x)\varphi(x)| \leq |X(x)| \| \varphi \|_{p,2}, \tag{34}$$

although $|X| \notin Z^2$.

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