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A REMARK ON GROWTH RELATION

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Abstract. The aim of this note is to present the answer to a problem concerning the growth relation posed by G. R. Krause and T. H. Lenagan [2].

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1. Main result

Let Φ' be the set of all nondecreasing sequences of positive integers. In [1], the following growth relation $<^*$ in Φ' has been considered:

$\{a_n\} <^* \{b_n\}$ if and only if there is $s \in \mathbb{N}$ such that $a_n \leq b_{sn}$ for almost all $n \in \mathbb{N}$.

In the course of proving Theorem 2.2 of [1] the authors use the following statement (for $\{c_n\} \in \Phi'$)

If $\{n^2\} <^* \{c_n\}$ and $\neg(\{c_n\} <^* \{n^2\})$, then for any polynomial $p(n)$ of degree 2, such that $p(n) \geq p(n-1)$ for all $n > 1$ we have $\{c_n + p(n)\} <^* \{c_n\}$.

On page 10 of the book [2] the authors have asserted that they are not able to verify the statement. In this paper we show that the statement is not true. Namely, we prove the following

Theorem 1. *Let Φ be the set of all strictly increasing sequences of positive integers. If $\{a_n\}$ is a member of Φ such that $\neg(\{a_n\} <^* \{n\})$, then for every $\{b_n\} \in \Phi$ there exists a sequence $\{c_n\} \in \Phi$ such that $\{b_n\} <^* \{c_n\}$, $\neg(\{c_n\} <^* \{b_n\})$ and $\neg(\{c_n + a_n\} <^* \{c_n\})$.*

Proof. Let $\{b_n\} \in \Phi$. Let us put $c_1 = b_1 + 1$. Suppose that we have defined $c_1, c_2, \ldots, c_{n_k}$ in k steps. Then, in the $(k+1)$-th step of our procedure we define $c_{n_k+1}, c_{n_k+2}, \ldots, c_{n_k+t_k}$ by putting $c_{n_k+i} = M_k + i$, $1 \leq i \leq t_k$, where $j_k = \min\{s \in \mathbb{N} : k(n_k + s) < a_{n_k+s}\}$, $t_k = k(n_k + j_k)$ and $M_k = \max\{b_{k(n_k+t_k)}, c_{n_k}\}$. It follows from our hypothesis on $\{a_n\}$ that the number j_k is well defined. Since $c_{n_k+i} \geq M_k \geq b_{k(n_k+t_k)} \geq b_{n_k+i}$ for $1 \leq i \leq t_k$ it is clear that $\{b_n\} <^* \{c_n\}$. We show that for a fixed $s \in \mathbb{N}$ there exist infinitely many $n \in \mathbb{N}$ such that $b_{sn} \leq c_n$ and there exist infinitely many $n \in \mathbb{N}$ such that $c_{sn} \leq c_n + a_n$.

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Indeed, let $k > s$, and let $c_1, c_2, \dots, c_{n_k}$ be all the elements of the sequence $\{c_n\}$ which were defined in the first k steps of the procedure. Then we have

$$c_{n_k+1} = M_k + 1 \geq b_{k(n_k+t_k)} + 1 > b_{k(n_k+t_k)} \geq b_{s(n_k+1)}$$

and

$$\begin{aligned} c_{s(n_k+j_k)} &\leq c_{k(n_k+j_k)} = M_k + k(n_k + j_k) - n < M_k + k(n_k + j_k) \\ &< M_k + a_{n_k+j_k} < c_{n_k+j_k} + a_{n_k+j_k} \end{aligned}$$

where j_k, t_k and M_k are defined as above. This shows that $\{c_n\} <^* \{b_n\}$ and $\{c_n + a_n\} <^* \{c_n\}$ do not hold and the theorem follows. □

Remark 1. *Theorem 2.2 of [1] is true under an additional hypothesis (see for example [3] Theorem 1.8). Moreover in [3] there is an example of the sequences $\{c_n\} \in \Phi'$ for which the statement of Borho and Kraft is not true.*

References

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