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A GENERALIZATION OF THE PSEUDO-LAPLACE TRANSFORM

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Abstract. This paper gives a generalization of the Pseudo-Laplace transform. In the special cases of semirings, the pseudo-exchange formula is proved. Also, for these semirings the forms of the Pseudo-Laplace transform and inverse operator are given. The results can be applied in dynamical programming for finding the maximum and minimum of the utility functions.

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1. Introduction

The notion of the pseudo-integral transform $\mathcal{L}^{\oplus}$ is based on the generalized pseudo-character defined by the so called pseudo-operations (see [4],[5]). It is generalization of the pseudo-Laplace transform (see [6]). In [8], the notions of generalized $(\oplus, \odot)$ Laplace transform and distorted generalized $(\oplus, \odot)$ Laplace transform are given, which are also the generalization of the pseudo-Laplace transform. In these transforms, the kernel is represented by pseudo-operations, while the kernel of the pseudo-integral transform is a generalized pseudo-character.

To define this transform the notion of generalized pseudo-character will be introduced, and its representation for special cases will be given.

The corresponding analogue of the exchange formula will be proved for the already introduced pseudo-convolution.

For special cases, the corresponding inverse of pseudo-integral transform will also be presented.

Finally, the pseudo-integral transform for finding the maximum or minimum of the utility functions in dynamical programming will be applied.

2. Preliminaries

We briefly present some notions from the pseudo-analysis ([5], [9]).

Let the order $\preceq$ be defined on a set $I \neq \emptyset$, and $\emptyset \neq I^* \subset I$.

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The binary operation $*: I \times I \to I$ is a **pseudo-operation**, if it is commutative, associative, nondecreasing on I^* (i.e. $x \preceq y \Rightarrow x * u \preceq y * u$, for $u \in I^*$) and has a neutral element.

The element $u \in I$ is the **null element** of the operation $* : I^2 \to I$ if for any $x \in I,\ \ x * u = u * x = u$ holds. Pseudo-operation $*$ is **idempotent** if for any $x \in I,\ \ x * x = x$ holds.

Let $\oplus$ be the pseudo-operation defined on the ordered set $(I, \preceq)$, such that $I^{\oplus} = I$, with a neutral element $\mathbf{0}$, and $\odot$ be the pseudo-operation defined on $(I, \preceq)$, such that $I^{\odot} = \{x \in I : \mathbf{0} \preceq x\}$, with a neutral element $\mathbf{1}$. If $\odot$ is a distributive operation with respect to the pseudo-operation $\oplus$, and $\mathbf{0}$ is a null element of the operation $\odot$, we say that the triplet $(I, \oplus, \odot)$ is a **semiring**. The semiring $(I, \oplus, \odot)$ will be denoted by $I^{\oplus,\odot}$.

Let I be a subinterval of $[-\infty, +\infty]$ (we will take usually closed subintervals $[a, b]$). Then we name the operations $\oplus$ and $\odot$ as **pseudo-addition** and **pseudo-multiplication**.

Here we consider semirings with the following continuous operations:

A)

$$[a, b]^{\min,\odot}$$

Here $\mathbf{0} = b$. The idempotent operation min induces a partial (full) order in the following way: $x \preceq y$ if and only if $\min(x, y) = y$. Hence this order is opposite to the usual order. Neutral elements of the operations $\oplus = \min$ and $\odot$ are respectively $\mathbf{0} = b$ and $\mathbf{1}$.

B)

$$[a, b]^{\max,\odot}$$

Neutral elements of the operations $\oplus = \max$ and $\odot$ are respectively $\mathbf{0} = a$ and $\mathbf{1}$. The order is the usual one.

Sub-cases

a) the operation $\odot$ in **A**) has the multiplicative generator g, i.e. $x \odot y = g^{-1}(g(x) \cdot g(y))$,

b) the operation $\odot$ in **B**) has the multiplicative generator g.

In [5] (see also [11]) the **pseudo-integral** $\int_X^{\oplus} f \odot dm$ of a bounded measurable function $f : X \to [a, b]$ (based on $\sigma - \oplus -$decomposable measure) is defined.

A) For the semiring $[a, b]^{\min,\odot}$, we have inf-decomposable measure $m = m_h$, defined using the function h with $m(A) = \inf_{x \in A} h(x)$. In this case pseudo-integral is given by

$$\int_{\mathbb{R}}^{\oplus} f \odot dm = \inf_{x \in \mathbb{R}} (f(x) \odot h(x)).$$

B) For the semiring $[a, b]^{\max,\odot}$ we have the sup-decomposable measure $m = m_h$ defined using the function h with $m(A) = \sup_{x \in A} h(x)$. In this case the pseudo-integral is given by

$$\int_{\mathbb{R}}^{\oplus} f \odot dm = \sup_{x \in \mathbb{R}} (f(x) \odot h(x)).$$

3. The pseudo-convolution

The notion of the pseudo-convolution of functions is introduced in [7]. We shall consider functions whose domain will be a commutative group $(G, \boxplus)$, $G \subset \mathbb{R}$. Let e be a neutral element of the operation $\boxplus$, and t' the inverse element for t, $t \in G$. Let the order defined on a set G be the usual order $\leq$, such that the operation $\boxplus$ is monotonous in relation to it.

The **pseudo-convolution** of two functions f_1 and f_2 with respect to a $\oplus$-decomposable measure m is given in the following way

$$(f_1 \star f_2)(x) = \int_{[e,x]}^{\oplus} f_1(t) \odot f_2(x \boxplus t') \odot dm,$$

where $m$is the decomposable measure.

The pseudo-convolution is a commutative and associative operation (see [7]).

The pseudo-convolution can be observed when $(G, \boxplus)$ is a semigroup and when the pseudo-integral is taken over the whole set G.

For cases A) and B) we take "uniform idempotent measure" $m(A) = \mathbf{1}$.

1. Let $\boxplus = +$ and $G = \mathbb{R}$. Then the pseudo-convolutions have the following form:

A) $(f_1 \star f_2)(x) = \inf_{0 \leq t \leq x} (f_1(t) \odot f_2(x - t))$.

B) $(f_1 \star f_2)(x) = \sup_{0 \leq t \leq x} (f_1(t) \odot f_2(x - t))$.

Sub-cases

a) $(f_1 \star f_2)(x) = \min_{0 \leq t \leq x} g^{-1}(g(f_1(t))g(f_2(x - t))) = g^{-1}(\min_{0 \leq t \leq x} [g(f_1(t))g(f_2(x - t))])$.

b) $(f_1 \star f_2)(x) = \max_{0 \leq t \leq x} g^{-1}(g(f_1(t))g(f_2(x - t))) = g^{-1}(\max_{0 \leq t \leq x} [g(f_1(t))g(f_2(x - t))])$.

2. Let $\boxplus = \cdot$ and $G = \mathbb{R} \setminus \{0\}$. Then the pseudo-convolutions have the following form:

A) $(f_1 \star f_2)(x) = \sup\limits_{1 \le t \le x} (f_1(t) \odot f_2(\frac{x}{t}))$,
B) $(f_1 \star f_2)(x) = \inf\limits_{1 \le t \le x} (f_1(t) \odot f_2(\frac{x}{t}))$,

Sub-cases
a) $(f_1 \star f_2)(x) = \min\limits_{1 \le t \le x} g^{-1}(g(f_1(t))g(f_2(\frac{x}{t}))) = g^{-1}(\min\limits_{1 \le t \le x} [g(f_1(t))g(f_2(\frac{x}{t}))])$.
b) $(f_1 \star f_2)(x) = \max\limits_{1 \le t \le x} g^{-1}(g(f_1(t))g(f_2(\frac{x}{t}))) = g^{-1}(\max\limits_{1 \le t \le x} [g(f_1(t))g(f_2(\frac{x}{t}))])$.

4. Integral transforms

Let $(G, \boxplus)$, $G \subset \mathbb{R}$ be a groupoid (group) and I be a semiring either of type **A)** or type **B)**.

We introduce the following version of the notion of character in pseudo-analysis.

Definition 4.1. *The* **generalized pseudo-character** *of the groupoid (group)* $(G, \boxplus)$*,* $G \subset \mathbb{R}$ *is a map* $\xi : G \to I$ *of the groupoid (group)* $(G, \boxplus)$ *in* $(I, \odot)$ *(where* $(I, \oplus, \odot)$ *is the semiring) with the property*

$$\xi(x \boxplus y) = \xi(x) \odot \xi(y), \quad x, y \in G. \tag{1}$$

It is obvious that the map $\xi \equiv \mathbf{0}$ or $\xi \equiv \mathbf{1}$ is a (trivial) generalized pseudo-character.

Theorem 4.1. *Let* $\xi : G \to I$ *be a continuous strictly increasing (decreasing) function which is a nontrivial generalized pseudo-character of the groupoid* $(G, \boxplus)$*,* $G \subset \mathbb{R}$ *and let* $\odot$ *be the pseudo-multiplication. Then* $\boxplus$ *is a pseudo-operation.*

Proof. Since, by the hypothesis, a continuous strictly increasing (decreasing) function ξ, the solution of (1), exists, ξ^{-1} must also exist, and hence

$$x \boxplus y = \xi^{-1}(\xi(x) \odot \xi(y)), \quad x, y \in G.$$

Because of commutativity and associativity of the operation $\odot$, it holds that

$$x \boxplus y = \xi^{-1}(\xi(x) \odot \xi(y)) = \xi^{-1}(\xi(y) \odot \xi(x)) = y \boxplus x,$$

$$(x \boxplus y) \boxplus z = \xi^{-1}(\xi(x \boxplus y) \odot \xi(z)) = \xi^{-1}((\xi(x) \odot \xi(y)) \odot \xi(z))$$
$$= \xi^{-1}(\xi(x) \odot (\xi(y) \odot \xi(z))) = \xi^{-1}(\xi(x) \odot \xi(y \boxplus z)) = x \boxplus (y \boxplus z),$$

i.e. $\boxplus$ is a commutative and associative operation.

The element $e = \xi^{-1}(\mathbf{1})$ is a neutral element, because for all $x \in G$ holds

$$x \boxplus e = \xi^{-1}(\xi(x) \odot \xi(e)) = \xi^{-1}(\xi(x) \odot \mathbf{1}) = \xi^{-1}(\xi(x)) = x.$$

If ξ is an increasing function, then it holds

$$x_1 \leq x_2 \Rightarrow \xi(x_1) \leq \xi(x_2) \Rightarrow \xi(x_1) \odot \xi(y) \leq \xi(x_2) \odot \xi(y)$$

$$\Rightarrow \xi^{-1}(\xi(x_1) \odot \xi(y)) \leq \xi^{-1}(\xi(x_2) \odot \xi(y)) \Rightarrow x_1 \boxplus y \leq x_2 \boxplus y,$$

i.e. $\boxplus$ is nondecreasing on $G^* = \{y \in G | \xi(y) \geq \mathbf{0}\}$. Analogously, $\boxplus$ is nondecreasing if ξ is a decreasing function. □

Theorem 4.2. *Let $\xi : G \to I$ be a continuous strictly increasing (decreasing) function which is a nontrivial generalized pseudo-character of the groupoid $(G, \boxplus)$, $G \subset \mathbb{R}$ and $\odot$ is the pseudo-multiplication with a multiplicative generator g. If $x \boxplus y$ is a polynomial, of degree greater than unity, then*

$$x \boxplus y = \frac{(px+q)(py+q)-q}{p}, \quad p \neq 0,\ p, q \in \mathbb{R},$$

and $(G \setminus \{-\frac{q}{p}\}, \boxplus)$ is the commutative group, and

$$\xi(x, c) = g^{-1}(|px+q|^c), \quad c \in \mathbb{R},$$

while for $x \boxplus y = x + y + r, \quad r \in \mathbb{R}$, is

$$\xi(x, c) = g^{-1}(e^{c(x+r)}).$$

Proof. From the previous theorem we have the commutativity and associativity of the operation $\boxplus$.

If $x \boxplus y$ is a polynomial of degree n in x and of degree m in y, then $n = m$ from commutativity. Then the left side of $(x \boxplus y) \boxplus z = x \boxplus (y \boxplus z)$ is a polynomial of degree n in z, while the right side is of degree n^2 in z. Thus $n = 1$ and $x \boxplus y$ is a symmetric polynomial of degree 1 in x, and y and can be written as

$$x \boxplus y = pxy + q(x+y) + r. \tag{2}$$

To find p, q, and r, we substitute in the associative condition and we have:

$$p(pxy + q(x+y) + r)z + q(pxy + q(x+y) + r + z) + r$$

$$= px(pyz + q(y+z) + r) + q(x + pyz + q(y+z) + r) + r.$$

Equating coefficients of like products of variables, we find everything is an identity except for the coefficients of x and z. In both cases we get $pr + q = q^2$.

If $p = 0$, we find $q = 0$ or 1. If $q = 0$, (1) has only the trivial solution $\xi \equiv 0$. If $q = 1$, the equation (1) is in this case reduced to $\xi(x+y+r) = \xi(x) \odot \xi(y)$, i.e. by the representation of the operation $\odot$ on the following functional equation

$$\xi(x + y + r) = g^{-1}(g(\xi(x)) \cdot g(\xi(y))),$$

i.e. $g(\xi(x+y+r)) = g(\xi(x)) \cdot g(\xi(y))$. Hence, for $x = u-r$, $y = v-r$, we obtain

$$(g \circ \xi)(u+v-r) = (g \circ \xi)(u-r) \cdot (g \circ \xi)(v-r),$$

i.e.

$$h(u+v) = h(u) \cdot h(v),$$

where $h(u) = g(\xi(u-r))$, which has the nontrivial solution $h(u) = e^{cu}$, (see [1]) i.e. $\xi(x,c) = g^{-1}(e^{c(x+r)})$.

It is now easy to show that $e = \frac{1-q}{p}$ is a neutral element of $\boxplus$ and that each $x \in G \setminus \{-\frac{q}{p}\}$, has an inverse element $x' = \frac{1-q^2-pqx}{p^2x+pq}$. As

$$x \boxplus y = -\frac{q}{p} \Leftrightarrow x = -\frac{q}{p} \vee y = -\frac{q}{p},$$

therefore $(G \setminus \{-\frac{q}{p}\}, \boxplus)$ is a groupoid and hence also a group.

To obtain something essentially new, we require $p \neq 0$. Then $r = \frac{q^2-q}{p}$, and

$$x \boxplus y = \frac{(px+q)(py+q)-q}{p}.$$

Replacing $x = \frac{u-q}{p}$, $y = \frac{v-q}{p}$, the equality (1) becomes

$$\xi(\frac{(px+q)(py+q)-q}{p}) = g^{-1}(g(\xi(x)) \cdot g(\xi(y)))$$

i.e.

$$g(\xi(\frac{uv-q}{p})) = g(\xi(\frac{u-q}{p})) \cdot g(\xi(\frac{v-q}{p})).$$

$$h(uv) = h(u) \cdot h(v),$$

where $h(u) = g(\xi(\frac{u-q}{p}))$, which has the trivial solution $h(u) = 0$, and $h(u) = 1$, the nontrivial solution $h(u) = (g \circ \xi)(\frac{u-q}{p}) = |u|^c$, or $h(u) = (g \circ \xi)(\frac{u-q}{p}) = |u|^c \operatorname{sgn} u$, $c \in \mathbb{R}$ (see [3]). Since the domain of g^{-1} is $\mathbb{R}_0^+$, then $h(u) = |u|^c$, $\xi(x,c) = g^{-1}(|px+q|^c)$, $c \in \mathbb{R}$. □

Corollary 1. *If*

1. $\boxplus = +$ *and* $G = \mathbb{R}$, *then we have* $\xi(x,c) = g^{-1}(e^{cx})$, $c \in \mathbb{R}$.
2. $\boxplus = \cdot$ *and* $G = \mathbb{R} \setminus \{0\}$, *then we have* $\xi(x,c) = g^{-1}(|x|^c)$, $c \in \mathbb{R}$.

This follows (as a consequence of the previous theorem), if we put $p = 0, q = 1, r = 0$ in the first case, i.e. in the second case $p = 1, q = 0, r = 0$, in the equation (2).

Definition 4.2. Pseudo-integral transform $\mathcal{L}^{\oplus}(f)$ *of a measurable function* f *is defined by*

$$(\mathcal{L}^{\oplus} f)(\xi)(z) = \int_{G_+}^{\oplus} \xi(x,-z) \odot dm_f,$$

where ξ *is the continuous generalized pseudo-character for* $z \in \mathbb{R}$, *for which the right side is meaningful.*

We consider also the pseudo-integral transform replacing in the pseudo-integral the whole G *instead of* $G_+ = \{x \in G : e \leq x\}$.

If $G = \mathbb{R}$ and $\boxplus = +$ the pseudo-integral transform becomes the pseudo-Laplace transform (see [6]).

In the special cases, the pseudo-integral transform gets the following forms:

1.

$$\textbf{A}) \ \mathcal{L}^{\oplus}(f)(z) = \inf_{x \geq 0} (\xi(x,-z) \odot f(x)).$$

$$\textbf{B}) \ \mathcal{L}^{\oplus}(f)(z) = \sup_{x \geq 0} (\xi(x,-z) \odot f(x)).$$

$$\textbf{a}) \ \mathcal{L}^{\oplus}(f)(z) = \min_{x \geq 0} g^{-1}(e^{-zx} g(f(x))) = g^{-1}(\min_{x \geq 0} [e^{-zx} g(f(x))]).$$

$$\textbf{b}) \ \mathcal{L}^{\oplus}(f)(z) = \max_{x \geq 0} g^{-1}(e^{-zx} g(f(x))) = g^{-1}(\max_{x \geq 0} [e^{-zx} g(f(x))]).$$

2.

$$\textbf{A}) \ \mathcal{L}^{\oplus}(f)(z) = \inf_{x \geq 1} (\xi(x,-z) \odot f(x)).$$

$$\textbf{B}) \ \mathcal{L}^{\oplus}(f)(z) = \sup_{x \geq 1} (\xi(x,-z) \odot f(x)).$$

$$\textbf{a}) \ \mathcal{L}^{\oplus}(f)(z) = \min_{x \geq 1} g^{-1}(x^{-z} g(f(x))) = g^{-1}(\min_{x \geq 1} [x^{-z} g(f(x))]).$$

$$\textbf{b}) \ \mathcal{L}^{\oplus}(f)(z) = \max_{x \geq 0} g^{-1}(x^{-z} g(f(x))) = g^{-1}(\max_{x \geq 1} [x^{-z} g(f(x))]).$$

The pseudo-integral transform is pseudo-linear in the general case (see [6]), i.e.

$$\mathcal{L}^{\oplus}(\lambda_1 \odot f_1 \oplus \lambda_2 \odot f_2) = \lambda_1 \odot \mathcal{L}^{\oplus}(f_1) \oplus \lambda_2 \odot \mathcal{L}^{\oplus}(f_2),$$

$f_1, f_2 \in B(G, I)$, and $\lambda_1, \lambda_2 \in I$.

Let $\xi : G \to I$ be a nontrivial continuous generalized pseudo-character of the group $(G, \boxplus)$, $G \subset \mathbb{R}$. For case **B)**, we give the proof of the following exchange formula

$$\mathcal{L}^{\oplus}(f_1 \star f_2) = \mathcal{L}^{\oplus}(f_1) \odot \mathcal{L}^{\oplus}(f_2),$$

assuming that the pseudo-integral transforms of the measurable functions f_1, f_2 and $f_1 \star f_2$, exist.

Proof.

$$\begin{aligned}
\mathcal{L}^{\oplus}(f_1 \star f_2)(z) &= \sup_{x\geq e}[\xi(x,-z)\odot(f_1\star f_2)(x)] \\
&= \sup_{x\geq e}[\xi(x,-z)\odot \sup_{e\leq y\leq x}[f_1(y)\odot f_2(x\boxplus y')]] \\
&= \sup_{x\geq e}\sup_{e\leq y\leq x}[\xi(x,-z)\odot f_1(y)\odot f_2(x\boxplus y')] \\
&= \sup_{y\geq e}\sup_{x\geq y}[\xi(x,-z)\odot f_1(y)\odot f_2(x\boxplus y')] \\
&= \sup_{y\geq e}\sup_{x\geq y}[\xi(y\boxplus(x\boxplus y'),-z)\odot f_1(y)\odot f_2(x\boxplus y')] \\
&= \sup_{y\geq e}\sup_{x\geq y}[\xi(y,-z)\odot\xi(x\boxplus y',-z)\odot f_1(y)\odot f_2(x\boxplus y')] \\
&= \sup_{y\geq e}[\xi(y,-z)\odot f_1(y)\odot \sup_{x\geq y}[\xi(x\boxplus y',-z)\odot f_2(x\boxplus y')]] \\
&= \sup_{y\geq e}[\xi(y,-z)\odot f_1(y)\odot \sup_{x\boxplus y'\geq e}[\xi(x\boxplus y',-z)\odot f_2(x\boxplus y')]] \\
&= \sup_{y\geq e}[\xi(y,-z)\odot f_1(y)\odot \sup_{w\geq e}[\xi(w,-z)\odot f_2(w)]] \\
&= \sup_{y\geq e}[\xi(y,-z)\odot f_1(y)]\odot \sup_{w\geq e}[\xi(w,-z)\odot f_2(w)] \\
&= \mathcal{L}^{\oplus}(f_1)(z)\odot\mathcal{L}^{\oplus}(f_2)(z).
\end{aligned}$$

We have used the equality $\sup_s[\varphi(t)\odot\psi(s)] = \varphi(t)\odot\sup_s\psi(s)$, (this is implied by monotonicity and continuity of $\odot$) and that the operation sup is invariant with respect to translation, i.e. $\sup_x f(x) = \sup_{x\boxplus y'} f(x\boxplus y')$. We also used the property $y\leq x \Leftrightarrow e\leq x\boxplus y'$ (e is a neutral element of the operation $\boxplus$, and y' is the inverse of the element $y\in G$). □

Let $f\in B(G,I)$, where I is the semiring from the cases **A)** and **B)**. The problem of the existence and uniqueness of the inverse operator $\mathcal{L}^{\oplus^{-1}}$ is complex. We are interesting here only in the forms of inverse operator when existence and uniqueness are satisfied.

Theorem 4.3. *If for $\mathcal{L}^{\oplus}(f) = F$, there exists $(\mathcal{L}^{\oplus})^{-1}(F)$, then it has the following form for the cases* **a**), **b**)*:*

1. **a**) $((\mathcal{L}^{\oplus})^{-1}(F))(x) = \max_{z\geq 0} g^{-1}(e^{xz}g(F(z))) = g^{-1}(\max_{z\geq 0} e^{xz}g(F(z)))$,

b) $((\mathcal{L}^{\oplus})^{-1}(F))(x) = \min_{z\geq 0} g^{-1}(e^{xz}g(F(z))) = g^{-1}(\min_{z\geq 0} e^{xz}g(F(z)))$.

2. **a**) $((\mathcal{L}^{\oplus})^{-1}(F))(x) = \max_{z\geq 1} g^{-1}(x^{z}g(F(z))) = g^{-1}(\max_{z\geq 1} x^{z}g(F(z)))$,

b) $((\mathcal{L}^{\oplus})^{-1}(F))(x) = \min_{z\geq 1} g^{-1}(x^{z}g(F(z))) = g^{-1}(\min_{z\geq 1} x^{z}g(F(z)))$.

Proof. 2. **a)** We suppose that, because of the existence of $(\mathcal{L}^{\oplus})^{-1}(f)$, there is one-to-one correspondence between x and z values, so that $\mathcal{L}^{\oplus}(f)(z) = F(z) = \min_{z\geq 1} g^{-1}(x^{-z}g(f(x)))$.

By the definition of the pseudo-integral transform, for $x \geq 0$, we have

$$F(z) \geq g^{-1}(x^{-z}g(f(x))),$$

which implies that if g is increasing (decreasing):

$$g(F(z)) \geq x^{-z}g(f(x)) \quad (\ g(F(z)) \leq x^{-z}g(f(x))\ ),$$

i.e.

$$x^{z}g(F(z)) \geq g(f(x)) \quad (\ x^{z}g(F(z)) \leq g(f(x))\ ),$$

that is,

$$g^{-1}(x^{z}g(F(z))) \geq f(x) \quad (\ g^{-1}(x^{z}g(F(z))) \geq f(x)\ ).$$

Hence $\min_{z\geq 1} g^{-1}(x^{z}g(F(z))) \geq f(x)$, with the equality for one value (one-to-one correspondence between x and z). □

Let $\mathcal{L}^{\oplus}(f_1) = F_1$, $\mathcal{L}^{\oplus}(f_2) = F_2$, i.e. $(\mathcal{L}^{\oplus})^{-1}(F_1) = f_1$, $(\mathcal{L}^{\oplus})^{-1}(F_2) = f_2$. Then

$\mathcal{L}^{\oplus}(\lambda_1 \odot f_1 \oplus \lambda_2 \odot f_2) = \lambda_1 \odot \mathcal{L}^{\oplus}(f_1) \oplus \lambda_2 \odot \mathcal{L}^{\oplus}(f_2) = \lambda_1 \odot F_1 \oplus \lambda_2 \odot F_2$, giving

$$(\mathcal{L}^{\oplus})^{-1}(\lambda_1 \odot F_1 \oplus \lambda_2 \odot F_2) = \lambda_1 \odot f_1 \oplus \lambda_2 \odot f_2 = \lambda_1 \odot (\mathcal{L}^{\oplus})^{-1}(F_1) \oplus \lambda_2 \odot (\mathcal{L}^{\oplus})^{-1}(F_2),$$

i.e. the inverse of the pseudo-integral transform $(\mathcal{L}^{\oplus})^{-1}$ is also pseudo-linear.

5. Applications in dynamical programming

We shall find the maximum or minimum of the functions

$$U(x_1, x_2, ..., x_n) = f_1(x_1) \odot f_2(x_2) \odot \cdots \odot f_n(x_n)$$

on the domain $R = \{(x_1, x_2, ..., x_n) | x_1 + x_2 + ... + x_n = x,\ \ x_i \geq 0,\ i = 1, ..., n\}$, where the operation $\odot$ has the multiplicative generator g. Such problems often occur in the mathematical economy and operation research (see [2]). In the paper [6] we considered the utility function $U(x_1, x_2, ..., x_n)$, where $\odot = +$.

We shall consider this problem in the following general form

$$f(x) = (f_1 \star f_2 \star \cdots \star f_n)(x), \tag{3}$$

where $\star$ is the pseudo-convolution for cases 1. a) and 1. b), i.e.

$f(x) = \min_{(x_1,...,x_n)\in R} U(x_1, ..., x_n)$ or $f(x) = \max_{(x_1,...,x_n)\in R} U(x_1, ..., x_n)$.

Applying the pseudo integral transform

$$F(z) = \mathcal{L}^{\oplus}(f)(z) = \mathcal{L}^{\oplus}(f_1 \star \cdots \star f_n)(z),$$

we obtain by the pseudo-exchange formula

$$\mathcal{L}^{\oplus}(f)(z) = \bigodot_{i=1}^{n} \mathcal{L}^{\oplus}(f_i)(z) = \mathcal{L}^{\oplus}(f_1)(z) \odot \cdots \odot \mathcal{L}^{\oplus}(f_n)(z)$$

Applying the inverse of the pseudo integral transform, we obtain the formal solution

$$\begin{aligned} f(x) &= ((\mathcal{L}^{\oplus})^{-1}(F(z)))(x) \\ &= ((\mathcal{L}^{\oplus})^{-1}(\bigodot_{i=1}^{n} \mathcal{L}^{\oplus} f_i(z)))(x) \end{aligned}$$

$$= ((\mathcal{L}^{\oplus})^{-1}(\bigodot_{i=1}^{n} F_i(z)))(x). \tag{4}$$

So, we have

$$\textbf{a)}\ \ f(x) = \max_{z \geq 0} g^{-1}(e^{xz} \prod_{i=1}^{n} \min_{x_i \geq 0} [e^{-x_i z} g(f_i(x_i))]),$$

$$\textbf{b)}\ \ f(x) = \min_{z \geq 0} g^{-1}(e^{xz} \prod_{i=1}^{n} \max_{x_i \geq 0} [e^{-x_i z} g(f_i(x_i))]).$$

Example 5.1. For $g(x) = x + 1$, there is the problem (3) of finding minimum (maximum) function (see [10])

$$U(x_1, x_2, ..., x_n) = \sum_{i_1, \ldots, i_n \in \{0,1\}} f_1^{i_1}(x_1) f_2^{i_2}(x_2) \cdots f_n^{i_n}(x_n)$$

on the domain R. The solution (4) can be expressed by

$$\textbf{a)}\ \ f(x) = \max_{z \geq 0} [e^{xz} \prod_{i=1}^{n} \min_{x_i \geq 0} [e^{-x_i z}(f_i(x_i) + 1)]],$$

$$\textbf{b)}\ \ f(x) = \min_{z \geq 0} [e^{xz} \prod_{i=1}^{n} \max_{x_i \geq 0} [e^{-x_i z}(f_i(x_i) + 1)]].$$

Now, we consider the problem (3) where $\star$ is the pseudo-convolution for cases 2. **a)** and 2. **b)**, i.e.

$$f(x) = \min_{(x_1,...,x_n)\in R} U(x_1, ..., x_n) \quad \text{or} \quad f(x) = \max_{(x_1,...,x_n)\in R} U(x_1, ..., x_n),$$

where the domain is $R = \{(x_1, x_2, ..., x_n)|x_1 \cdot x_2 \cdot ... \cdot x_n = x, \ \ x_i \geq 1, \ i = 1, ..., n\}$.

Formal solution is

$$\textbf{a)} \ \ f(x) = \max_{z\geq 1} g^{-1}(x^z \prod_{i=1}^{n} \min_{x_i\geq 1}[x_i^{-z} g(f_i(x_i))]),$$

$$\textbf{b)} \ \ f(x) = \min_{z\geq 1} g^{-1}(x^z \prod_{i=1}^{n} \max_{x_i\geq 1}[x_i^{-z} g(f_i(x_i))]).$$

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