Novi Sad J. Math.
Vol. 37, No. 2, 2007, 35-40

SOME RESULTS ON 2-INNER PRODUCT SPACES

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Abstract. We onsider "Riesz Theorem" in the 2-inner product spaces and give some results in this field. Also, we give some characterizations about 2-inner product spaces in b-approximation theory.

AMS Mathematics Subject Classification (2000): 41A65, 41A15

Key words and phrases: b-Orthogonality, 2-Normed spaces, 2-Inner product, b-Proximinal subspaces, b-Best approximation

1. Introduction

The concept of linear 2-normed spaces has been investigated by S. Gähler (see [9]) and has been developed extensively in different subjects by many authors (see [1-8]).

Let X be a linear space of dimension greater than 1. Suppose $\|.,.\|$ is a real-valued function on $X \times X$ satisfying the following conditions:
a) $\|x, y\| = 0$ if and only if x and y are linearly dependent vectors.
b) $\|x, y\| = \|y, x\|$ for all $x, y \in X$.
c) $\|\lambda x, y\| = |\lambda|\|x, y\|$ for all $\lambda \in \mathbb{R}$ and all $x, y \in X$.
d) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ for all $x, y, z \in X$.
Then $\|.,.\|$ is called a 2-norm on X and $(X, \|.,.\|)$ is called a linear 2-normed space. Some of the basic properties of 2-norms are that they are non-negative and $\|x, y + \alpha x\| = \|x, y\|$ for all $x, y \in X$ and all $\alpha \in \mathbb{R}$.

Every 2-normed space is a locally convex topological vector space. In fact, for a fixed $b \in X$, $p_b(x) = \|x, b\|$, $x \in X$, is a seminorm and the family $P = \{p_b : \ b \in X\}$ of seminorms generates a locally convex topology on X.

Let $(X, \|.,.\|)$ be a 2-normed space and let W_1 and W_2 be two linear subspaces of X. A map $\Lambda : W_1 \times W_2 \to \mathbb{R}$ is called a bilinear 2-functional on $W_1 \times W_2$, whenever for all $x_1, x_2 \in W_1, y_1, y_2 \in W_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$;
a) $\Lambda(x_1 + x_2, y_1 + y_2) = \Lambda(x_1, y_1) + \Lambda(x_1, y_2) + \Lambda(x_2, y_1) + \Lambda(x_2, y_2)$,
b) $\Lambda(\lambda_1 x_1, \lambda_2 y_1) = \lambda_1 \lambda_2 \Lambda(x_1, y_1)$.

A bilinear 2-functional $\Lambda : W_1 \times W_2 \to \mathbb{R}$ is said to be bounded if there exists a non-negative real number M (called a Lipschitz constant for Λ) such that $|\Lambda(x, y)| \leq M\|x, y\|$ for all $x \in W_1$ and all $y \in W_2$. Also, the norm of a bilinear 2-functional Λ is defined by

$$\|\Lambda\| = \inf\{M \geq 0 : \ M \textit{ is a Lipschitz constant for } \Lambda\}.$$

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It is known that ([4])

$$\begin{aligned}\|\Lambda\| &= \sup\{|\Lambda(x,y)| : (x,y) \in W_1 \times W_2, \ \|x,y\| \leq 1\} \\ &= \sup\{|\Lambda(x,y)| : (x,y) \in W_1 \times W_2, \ \|x,y\| = 1\} \\ &= \sup\{|\Lambda(x,y)|/\|x,y\| : (x,y) \in W_1 \times W_2, \ \|x,y\| > 0\}.\end{aligned}$$

For a 2-normed space $(X, \|.,.\|)$ and $0 \neq b \in X$, by X_b^* is denoted the Banach space of all bounded bilinear 2-functionals on $X \times < b >$, where $< b >$ is the subspace of X generated by b.

Let $(X, \|.,.\|)$ be a 2-normed space and $x, y \in X$, then x is said to be orthogonal to y if and only if there exists $b \in X$ such that for all scalar α, $\|x, b\| \neq 0$ and $\|x, b\| \leq \|x + \alpha y, b\|$, in this case we write $x \perp^b y$. If M_1 and M_2 are subsets of X, we say that M_1 is orthogonal to M_2 if and only if there exists $b \in X$ such that $g_1 \perp^b g_2$ for all $g_1 \in M_1, g_2 \in M_2$. If M_1 is orthogonal to M_2, we write $M_1 \perp^b M_2$. (see [10])

Let $(X, \|.,.\|)$ be a 2-normed space, $x \in X$, A be a linear subspace of X and $b \in X \backslash x - A$. $y_0 \in A$ is b-best approximation for $x \in X$, if $x - y_0 \perp^b A$. Therefore, $y_0 \in A$ is a b-best approximation of x if for all $y \in A$

$$\|x - y_0, b\| \leq \|x - y, b\|,$$

then $\|x - y_0, b\| = \inf_{y \in A} \|x - y, b\| = \|x - A, b\|$. The set of all b-best approximations of x in A is denoted by $P_A^b(x)$. A is called b-proximinal if for every $x \in X \backslash (A + < b >)$ there exist $y_0 \in A$ such that $y_0 \in P_A^b(x)$. Also, A is called b-Chebyshev if for every $x \in X \backslash (A + < b >)$, there exists a unique $y_0 \in A$ such that $y_0 \in P_A^b(x)$.

Let X be a linear space of dimension greater than 1 over the field $\mathbb{K} = \mathbb{R}$ of real numbers or the field $\mathbb{K} = \mathbb{C}$ of complex numbers. Suppose that $(.,.|.)$ is a $\mathbb{K}$-valued function defined on $X \times X \times X$ satisfying the following conditions:

a) $(x, x|z) \geq 0$ and $(x, x|z) = 0$ if and only if x and z are linearly dependent;
b) $(x, x|z) = \underline{(z, z|x)}$;
c) $(y, x|z) = \overline{(x, y|z)}$;
d) $(\alpha x, y|z) = \alpha(x, y|z)$ for any scalar $\alpha \in \mathbb{K}$;
e) $(x + \acute{x}, y|z) = (x, y|z) + (\acute{x}, y|z)$.

$(.,.|.)$ is called a 2-inner product on X and $(X, (.,.|.))$ is called a 2-inner product space. Some basic properties of 2-inner products $(.,.|.)$ can be immediately obtained [1-3].

Let $(X, (.,.|.))$ be a 2-inner product space. We can define a 2-norm on $X \times X$ by

$$\|x, y\| = \sqrt{(x, x|y)}.$$

Let $(X, (.,.|.))$ be a 2-inner product space, $b \in X$ and $x, y \in X \backslash < b >$. Then

$$x \perp^b y \Leftrightarrow (x, y|b) = 0.$$

Using the above properties, we can prove the Cauchy-Schwartz inequality

$$(x, y|b)^2 \leq \|x, b\|^2 \|y, b\|^2$$

for every $x, y \in X$. Moreover, the equality holds in this inequality if and only if x and y are linearly dependent. Also, we have the parallelogram-law

$$\|x + y, b\|^2 + \|x - y, b\|^2 = 2\|x, b\|^2 + 2\|y, b\|^2$$

for every $x, y \in X$ (For more details about 2-inner product space see [1-3]).

2. Main results

In this section we shall obtain some characterization of 2-inner product spaces.

Theorem 2.1. *Let $(X, (., .|.))$ be a 2-inner product space, $b \in X$ and $\Lambda \in X_b^*$. If the set $M = \{x \in X : (x, b) \in ker\Lambda\}$ is b-proximinal, then there exists a $y \in X$ such that*

$$\Lambda(x, b) = (x, y|b) \qquad \forall x \in X.$$

Proof. If $\Lambda = 0$, put $y = 0$.
If $\Lambda \neq 0$, there exists $x_1 \in X$ such that $\Lambda(x_1, b) \neq 0$. Since M is a b-proximinal, there exists $m \in M$ such that $x_2 = x_1 - m \perp^b M$ and $\|x_1 - m, b\| \neq 0$. Therefore, $(x_2, y|b) = 0$ for all $y \in M$. Put $z = \frac{x_2}{\|x_2, b\|}$. Then, $(z, y|b) = 0$ and $\|z, b\| = 1$.

For all $x \in X$, we set $u = \Lambda(x, b)z - \Lambda(z, b)x$. Then, $\Lambda(u, b) = \Lambda(x, b)\Lambda(z, b) - \Lambda(z, b)\Lambda(x, b) = 0$. It follows that $u \in M$, therefore, $(z, u|b) = 0$. Now

$$\begin{aligned} 0 &= (z, u|b) = (\Lambda(x, b)z - \Lambda(z, b)x, z|b) \\ &= \Lambda(x, b)(z, z|b) - \Lambda(z, b)(x, z|b). \end{aligned}$$

Hence, $(z, z|b)\Lambda(x, b) = \Lambda(z, b)(x, z|b)$ and $\Lambda(x, b) = (x, y|b)$, where $y = z\Lambda(z, b)$. □

Definition 2.2. *Let $(X, (., .|.))$ be a 2-inner product space, $b \in X$.*
a) *A sequence $\{x_n\}$ in X is a b-Cauchy sequence if*

$$\forall \epsilon > 0\ \exists N > 0, \text{ such that } \forall\ m,\ n\ \geq\ N\ 0 < \|x_m - x_n, b\| < \epsilon$$

b) *X is b-Hilbert if every b-Cauchy sequence is converges in the seminormed $(X, \|., b\|)$.*
c) *If a subset A in X is closed in the space $(X, \|., b\|)$, then we say that A is b-closed in the seminormed $(X, \|., b\|)$.*

Theorem 2.3. *Let $(X, (., .|.))$ be a 2-inner product space, A be a convex set in X and $b \in X$. Then each $x \in X \backslash A+ < b >$ has at most one b-best approximation in A.*

Proof. Suppose $x \in X \backslash A + < b >$ and $y_1, y_2 \in P_A^b(x)$. By convexity A, $\frac{1}{2}(y_1 + y_2) \in A$. Therefore

$$\begin{aligned} \|x - A, b\| &\leq \|x - \frac{1}{2}(y_1 + y_2), b\| \\ &= \|\frac{1}{2}(x - y_1) + \frac{1}{2}(x - y_2), b\| \\ &\leq \frac{1}{2}\|x - y_1, b\| + \frac{1}{2}\|x - y_2, b\| \\ &= \frac{1}{2}\|x - A, b\| + \frac{1}{2}\|x - A, b\| \\ &= \|x - A, b\|. \end{aligned}$$

Hence equality must hold throughout these inequalities. By the condition of equality and b-strictly convex in the triangle inequality, $x - y_1 = k(x - y_2)$ for some $k \geq 0$. But, $\|x - y_1, b\| = \|x - A, b\| = \|x - y_2, b\|$ implies $k = 1$, and hence $y_1 = y_2$. □

Theorem 2.4. *Every nonempty b-closed, convex A in a b-Hilbert space X with $A \cap < b >= \phi$, is b-Chebyshev.*

Proof. Suppose $x \in X \backslash A + < b >$, put $E = x - A$ and $\delta = \inf_{y \in E} \|y, b\|$. Then E is a b-closed and convex set in X. Suppose $y', y \in E$, since E is convex and $\frac{y' + y}{2} \in E$, therefore using the parallelogram-law:

$$\frac{1}{2}\|y' - y, b\|^2 \leq \|y', b\|^2 + \|y, b\|^2 - 2\delta^2. \qquad (*)$$

Let $\{y_n\}$ be a sequence in E, where $\|y_n, b\| \longrightarrow \delta$. From $(*)$, since for all $m, n \geq 1$, $\|y_n - y_m, b\| \neq 0$ then $\{y_n\}$ is a b-Cauchy sequence. Since X is b-Hilbert, there exist $y_0 \in X$ such that $y_n \longrightarrow y_0$ as $n \to \infty$, also E is b-closed and $y_n \in E$, therefore $y_0 \in E$. It follows that $\|y_0, b\| = \delta$. Therefore, there exists $a_0 \in A$ such that $\|x - a_0, b\| = \|x - A, b\|$. That is, a_0 is a b-best approximation for x.

For uniqueness, if $a_1, b_1 \in A$ are b-best approximations for x. Put $y_1 = x - a_1, y_2 = x - b_1$, then $\|y_1, b\| = \|y_2, b\| = \delta$. If we apply the inequality $(*)$, it follows that $y_1 = y_2$, therefore $a_1 = b_1$. □

Let $(X, (.,.|.))$ be a 2-inner product space, A be a subspace of X and $b \in X$. Put

$$A_b^{\perp} = \{x \in X : \ (x, g|b) = 0, \forall g \in A\}.$$

Theorem 2.5. *Let A be a b-Chebyshev subspace of the 2-inner product space $(X, (.,.|.))$, $b \in X$ (e.g. a closed subspace of a b-Hilbert space) and $A \bigcap < b >=$*

$\emptyset$. *Then $A_b^{\perp}$ is a b-Chebyshev subspace, and the following statements are true:*
a) $X = A_b^{\perp} \oplus A$
b) $A = \{x \in X : \; \|x, b\| = \|x - A_b^{\perp}, b\|\}$
c) $A_b^{\perp} = \{x \in X : \; \|x, b\| = \|x - A, b\|\}$
d) $\|x, b\|^2 = \|g, b\|^2 + \|g_0, b\|^2$, *for all $x \in X \backslash < b >$, where $x = g + g_0$, $g \in A$ and $g_0 \in A^{\perp}$*

Proof. If $x \in X$, then there exists $y \in A$ such that $x - y \perp^b A$. Put $y_0 = x - y$ then $x = y + y_0$ and $y_0 \in A_b^{\perp}$, implies $X = A + A_b^{\perp}$ and

$$\begin{aligned} \|x - A_b^{\perp}, b\| &= \|y - A_b^{\perp}, b\| \\ &\leq \|y, b\|. \end{aligned}$$

Now if $z \in A_b^{\perp}$ and $y \in A$, we have

$$\begin{aligned} \|x - A_b^{\perp}, b\|^2 &\geq \|z - y, b\|^2 \\ &= (z - y, z - y|b) \\ &= \|z, b\|^2 + \|y, b\|^2 \qquad (**) \\ &\geq \|y, b\|^2. \end{aligned}$$

Therefore $\|x - y_0, b\| = \|x - A_b^{\perp}, b\|$, i.e., $y_0 \in P^b_{A_b^{\perp}}(x)$.

If $g_0, y_0 \in P^b_{A_b^{\perp}}(x)$, then $x = y + y_0 = g + g_0$ for some $y, g \in A$. It follows that $y_0 - g_0 \in A_b^{\perp} \cap A = \{0\}$, hence $y_0 = g_0$. Therefore $A_b^{\perp}$ is b-Chebyshev, $X = A \oplus A_b^{\perp}$, $\|y - A_b^{\perp}, b\| = \|y, b\|$ and from $(**)$ we have

$$\|x, b\|^2 = \|y, b\|^2 + \|y_0, b\|^2.$$

If $x \in X$ and $\|x - A_b^{\perp}, b\| = \|x, b\|$, then $0 \in P^b_{A_b^{\perp}}(x)$ and $x = y + y_0$ for some $y \in A$ and $y_0 \in A_b^{\perp}$. Hence

$$\begin{aligned} \|x - A_b^{\perp}, b\| &= \|y - A_b^{\perp}, b\| \\ &= \|y, b\| \\ &= \|x - y_0, b\|. \end{aligned}$$

implies $y_0 \in P^b_{A_b^{\perp}}(x)$. Therefore $y_0 = 0$ and $x = y \in A$. Then

$$A = \{x \in X : \; \|x, b\| = \|x - A_b^{\perp}, b\|\}.$$

Finally, by paying attention to the definition of $A_b^{\perp}$ we have

$$A_b^{\perp} = \{x \in X : \; \|x, b\| = \|x - A, b\|\}.$$

$\square$

References

[1] Dragomir, S. S., Cho, Y. J., Kim, S. S., Sofo, A., Some Boas-Bellman type inequalities in 2-inner product spaces. J. Inequal. in Pure and Appl. Math. 6(2)55 (2005), 1-13.

[2] Cho, Y. J., Lin, P. C. S., Kim, S. S. Misiak, A., Theory of 2-inner product spaces. New York: Nova Science Publishes, Inc. 2001.

[3] Cho, Y. J., Matic, M., Pecaric, J. E., On Gram's determinant in 2-inner product spaces. J. Korean Math. Soc. 38 (6)(2001), 1125-1156.

[4] Lewandowska, Z., Linear operators on generalized 2-normed spaces. Bull. Math. Soc. Sci. Math. Roumanie (N.S.) 42(90) No. 4 (1999), 353-368.

[5] Lewandowska, Z., Generalized 2-normed spaces. Supskie Space Matemayczno Fizyczne 1 (2001), 33-40.

[6] Lewandowska, Z., On 2-normed sets. Glas. Mat. Ser. III 38(58) No. 1 (2003), 99-110.

[7] Lewandowska, Z., Banach-Steinhaus theorems for bounded linear operators with values in a generalized 2-normed space. Glas. Mat. Ser. III 38(58) No. 2, (2003), 329-340.

[8] Lewandowska, Z., Bounded 2-linear operators on 2-normed sets. Glas. Mat. Ser. III 39(59) (2004), 301-312.

[9] Gähler, S., Linear 2-normierte Räume. Math. Nachr. 28 (1964), 1-43.

[10] Mazaheri, H., Golestani, S., Some results on b-orthogonality in 2-normed linear spaces. Journal of Mathematical Analysis (2007) (to appear)

Received by the editors June 6, 2006