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A NOTE ABOUT THE SUMS OF PRODUCTS OF BERNOULLI NUMBERS

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Abstract. This paper presents the formula to calculate the sums of products of the Bernoulli numbers in the form

$$\sum_{\substack{1\le k_1\le \left[\frac{n}{2}\right]\\ 1\le k_2\le \left[\frac{n-k_1}{2}\right]\\ \vdots \\ 1\le k_m\le \left[\frac{n-k_1-\cdots-k_{m-1}}{2}\right]}} A_{k_1,k_2,\ldots,k_m} B_{2k_1}\cdots B_{2k_m} B_{2(n-k_1-\cdots-k_m)}\,,$$

where $A_{k_1,k_2,\ldots,k_m}$ is a certain sequence of rational numbers.

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1. Introduction

The Bernoulli numbers B_n are defined by the generating function:

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \qquad (|t|<2\pi)\,.$$

Euler's identity involves the sum of products of two Bernoulli numbers as follows

$$\sum_{k=1}^{n-1}\binom{2n}{2k}B_{2k}B_{2n-2k} = -(2n+1)B_{2n} \qquad (n\ge 2)\,.$$

Eie [4] and Sitaramachandrarao and Davis [10] considered the sum of products of 3 and 4 Bernoulli numbers. Dilcher [3] proved for $m\ge 2$

$$\sum_{\substack{j_1+\cdots+j_m=n\\ j_1,\ldots,j_m\ge 0}}\binom{2n}{2j_1,\ldots,2j_m}B_{2j_1}\cdots B_{2j_m}=$$

$$=\begin{cases}\frac{(2n)!}{(2n-m)!}\sum\limits_{k=0}^{\left[\frac{m-1}{2}\right]} b_k^{(m)}\frac{B_{2n-2k}}{2n-2k} & 2n>m\,,\\ \frac{(2n)!}{4^n}+\sum\limits_{k=0}^{n-1}\frac{(2n)!}{2n-2k}b_k^{(2n)}B_{2n-2k} & 2n=m\,,\\ (-1)^{m-1}(m-2n-1)!(2n)!\,b_n^{(m)} & 0\le n\le\left[\frac{m-1}{2}\right]\,.\end{cases}$$

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where

$$\binom{2n}{2j_1, ..., 2j_m} := \frac{(2n)!}{(2j_1)! \cdots (2j_m)!}$$

is the multinomial coefficient and $b_k^{(m)}$ is the sequence of rational numbers defined by

$$b_0^{(1)} := 1, \qquad b_k^{(m+1)} := -\frac{1}{m} b_k^{(m)} + \frac{1}{4} b_{k-1}^{(m-1)} ,$$

with $b_k^{(m)} = 0$ for $k < 0$ and for $k > [(m-1)/2]$. Here $[x]$ denotes the integer part of x.

Finally, Petojević [8] established the following sums of product of $m \in \mathbb{N}$ Bernoulli numbers

$$\sum_{\substack{k_1+\cdots+k_{m+1}=n \\ k_1,...,k_m \geq 0 \\ k_{m+1} \geq 1}} \binom{n}{k_1, k_2, ..., k_{m+1}} B_{k_1} B_{k_2} \cdots B_{k_m} \frac{B_{2k_{m+1}+k_1}}{2k_{m+1}+k_1} =$$

$$\begin{cases} c_n & m = 1 \\ \sum_{k=0}^{n} \binom{n}{k} a_{k,m-1} c_{n-k} & m > 1 , \end{cases} \tag{1.1}$$

and for $m, n \in \mathbb{N}_0$

$$\sum_{\substack{k_1+\cdots+k_{2m+1}=n \\ k_1,...,k_{2m} \geq 0 \\ k_{2m+1} \geq 1}} \binom{n}{k_1, k_2, ..., k_{2m+1}} B_{k_1} B_{k_2} \cdots B_{k_{m+1}} \frac{B_{2k_{2m+1}+k_1}}{2k_{2m+1}+k_1} =$$

$$\begin{cases} \frac{1}{3n} B_n B_{3n} & n > m = 0 \\ \sum_{k=0}^{n} \binom{n}{k} c_{n-k} (-1)^k \binom{m-1}{k}^{-1} s(m-1, m-k-1) & 0 \leq n < m \\ \sum_{k=0}^{m-1} \binom{n}{k} c_{n-k} (-1)^k \binom{m-1}{k}^{-1} s(m-1, m-k-1) & \\ + \frac{(-1)^{m+1}}{(m-1)!} \sum_{k=m}^{n} \binom{n}{k} c_{n-k}\, k! \sum_{r=0}^{m-1} \frac{s(m-1,r)}{(k-m+r+1)\,(k-m)!} B_{k-m+r+1} & n \geq m > 0 \end{cases}$$

where $s(n, k)$ are Stirling numbers of the first kind and α_n , c_n and $a_{n,m}$ are the sequences defined by

$$\alpha_n \;:=\; (-1)^n \binom{2n-1}{n}^{-1}, \; n \in \mathbb{N}\,.$$

$$c_n \;:=\; \begin{cases} 0 & n = 0\,, \\ \frac{1}{12} & n = 1\,, \\ \frac{\alpha_n}{4n} B_{2n} - \frac{1}{2n} B_n^2 & n \geq 2\,. \end{cases}$$

$$a_{n,m} \;:=\; \begin{cases} 1 & n = 0\,, \\ m\binom{n}{m} \sum_{k=0}^{m-1} (-1)^{m-k-1} s(m, m-k) \frac{B_{n-k}}{n-k} & n \geq m \in \mathbb{N}\,, \\ \frac{n!}{(m-1)!} \sum_{k=0}^{m-n-1} \frac{(m-n-1)!}{k!\, m^{m-n-k}} & \\ \times \sum_{r=0}^{m} (-1)^{r-k+1} s(m,r)(r)_k & m > n > 0\,. \end{cases}$$

2. New sum of products

The relation (1.1) produces

$$\sum_{\substack{k_1+k_2=n \\ k_1 \geq 0,\, k_2 \geq 1}} \binom{n}{k_1, k_2} B_{k_1} \frac{B_{2k_2+k_1}}{2k_2+k_1} = \begin{cases} \frac{1}{12}\,, & n = 1\,, \\ \frac{\alpha_n}{4n} B_{2n} - \frac{1}{2n} B_n^2\,, & n > 1\,, \end{cases}$$

or, it can be rewritten as

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k \frac{B_{2n-k}}{2n-k} = \begin{cases} \frac{1}{12}\,, & n = 1\,, \\ \frac{\alpha(n)}{4n} B_{2n} - \frac{1}{2n} B_n^2\,, & n > 1\,. \end{cases} \tag{2.2}$$

Let us specify new generalization of the relation (2.2). Before that, we will specify the notation: for $m \in \mathbb{N}$ and $n > 1$, $Q_m(n)$ is defined by

$$Q_m(n) = \sum_{\substack{1 \leq k_1 \leq \left[\frac{n}{2}\right] \\ 1 \leq k_2 \leq \left[\frac{n-k_1}{2}\right] \\ \vdots \\ 1 \leq k_m \leq \left[\frac{n-k_1-\cdots-k_{m-1}}{2}\right]}} A_{k_1,k_2,\ldots,k_m} B_{2k_1} \cdots B_{2k_m} B_{2(n-k_1-\cdots-k_m)}$$

where $A_{k_1,k_2,\ldots,k_m}$ is the sequence of rational numbers, defined as:

$$A_{k_1,k_2,\ldots,k_m} = \begin{bmatrix} n \\ 2k_1, \ldots, 2k_m \end{bmatrix} \frac{2 - \alpha_{n-k_1-\cdots-k_m}}{n-k_1-\cdots-k_m} f(n, k_1, \ldots, k_m).$$

Here

$$\begin{bmatrix} n \\ 2k_1,\dots,2k_m \end{bmatrix} = \frac{\binom{n}{2k_1}}{2-\alpha_{n-k_1}} \cdot \frac{\binom{n-k_1}{2k_2}}{2-\alpha_{n-k_1-k_2}} \cdots \frac{\binom{n-k_1-\cdots-k_{m-1}}{2k_m}}{2-\alpha_{n-k_1-\cdots-k_m}},$$

$$f(n,k_1,\dots,k_m) = \begin{cases} 1, & 2k_m \neq n-k_1-\cdots-k_{m-1} \text{ or } m=1 \\ \frac{1}{2}, & 2k_m = n-k_1-\cdots-k_{m-1} \text{ and } m\neq 1 \end{cases}$$

Next, for $m \in \mathbb{N}$ and $n > 1$ the sums $P_m(n)$ and $\tilde{P}_m(n)$ are defined by

$$P_m(n) = \sum_{\substack{1\le k_1\le\left[\frac{n}{2}\right] \\ 1\le k_2\le\left[\frac{n-k_1}{2}\right] \\ \vdots \\ 1\le k_m\le\left[\frac{n-k_1-\cdots-k_{m-1}}{2}\right]}} \begin{bmatrix} n \\ 2k_1,\dots,2k_m \end{bmatrix} \frac{2-\alpha_{n-k_1-\cdots-k_m}}{n-k_1-\cdots-k_m} \times$$

$$\times \quad B_{2k_1}\cdots B_{2k_m} B_{2(n-k_1-\cdots-k_m)},$$

and

$$\tilde{P}_m(n) = \sum_{\substack{1\le k_1\le\left[\frac{n}{2}\right] \\ 1\le k_2\le\left[\frac{n-k_1}{2}\right] \\ \vdots \\ 1\le k_m\le\left[\frac{n-k_1-\cdots-k_{m-1}}{2}\right]}} \begin{bmatrix} n \\ 2k_1,\dots,2k_m \end{bmatrix} B_{2k_1}\cdots B_{2k_m} \frac{B^2_{n-k_1-\dots-k_m}}{n-k_1-\cdots-k_m}.$$

Finally, we will define the sequence $d_k(n)$ $(n \geq 2)$:

$$d_1(n) = \left[\frac{n}{2}\right], \quad d_k(n) = \left[\frac{n-\sum_{j=1}^{k-1} d_j(n)}{2}\right],$$

so that, clearly, if $d_k(n) = 1$ and $d_{k+1}(n) = 0$ then $\max m = k$. Hence

$$1 \le m < 1 + \log_2 n.$$

After the appropriate substitution $B_{2(n-k_1)}, \dots, B_{2(n-k_1-k_2-\cdots-k_{m-1})}$ in formula (2.2) we get

$$P_m(n) = \frac{\alpha_n}{n(-2)^m} B_{2n} + \frac{1}{n(-2)^{m-1}} B_n^2 + \sum_{s=1}^{m-1} (-2)^{s+1-m} \tilde{P}_s(n),$$

or recursively

$$P_1(n) = -\frac{2-\alpha_n}{2n} B_{2n} + \frac{1}{n} B_n^2,$$

$$P_m(n) = -\frac{1}{2} P_{m-1}(n) + \tilde{P}_{m-1}(n).$$

For $2k_m \neq n - k_1 - \cdots - k_{m-1} > 1$, the equation $B_{n-k_1-\cdots-k_{m-1}} = 0$ produces

$$
\begin{aligned}
& P_m(n) - \tilde{P}_{m-1}(n) = \\
= & \sum_{\substack{1 \le k_1 \le \left[\frac{n}{2}\right] \\ 1 \le k_2 \le \left[\frac{n-k_1}{2}\right] \\ \vdots \\ 1 \le k_m \le \left[\frac{n-k_1-\cdots-k_{m-1}}{2}\right] \\ 2k_m \neq n-k_1-\cdots-k_{m-1}}} \begin{bmatrix} n \\ 2k_1, \ldots, 2k_m \end{bmatrix} \frac{2 - \alpha_{n-k_1-\cdots-k_m}}{n - k_1 - \cdots - k_m} \times \\
\times & B_{2k_1} \cdots B_{2k_m} B_{2(n-k_1-\cdots-k_m)} \\
+ & \sum_{\substack{1 \le k_1 \le \left[\frac{n}{2}\right] \\ 1 \le k_2 \le \left[\frac{n-k_1}{2}\right] \\ \vdots \\ 1 \le k_m \le \left[\frac{n-k_1-\cdots-k_{m-1}}{2}\right] \\ 2k_m = n-k_1-\cdots-k_{m-1}}} \begin{bmatrix} n \\ 2k_1, \ldots, 2k_m \end{bmatrix} \frac{2 - \alpha_{k_m}}{2k_m} \times \\
\times & B_{2k_1} \cdots B_{2k_{m-1}} B_{2k_m}^2 \\
= & Q_m(n)\,.
\end{aligned}
$$

That is how the following statement is proved.

Theorem 2.1. *For $n \geq 2$ and $1 \leq m < 1 + \log_2 n$ we have:*

$$
\begin{aligned}
Q_1(n) &= \frac{1}{n} B_n^2 + \frac{\alpha_n - 2}{2n} B_{2n}\,, \\
Q_m(n) &= -\frac{1}{2} P_{m-1}(n) \\
&= \frac{2 - \alpha_n}{n(-2)^m} B_{2n} + \frac{1}{n(-2)^{m-1}} B_n^2 + \sum_{s=1}^{m-2} (-2)^{s+1-m} \tilde{P}_s(n)\,.
\end{aligned}
$$

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