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# CONVERGENCE OF A TRAJECTORY OF A VECTOR SUBSPACE UNDER THE ACTION OF A LINEAR MAP: GENERAL CASE

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**Abstract.** The behavior of a linear subspace $V$ of a finite-dimensional space under the action of the iterations of a linear mapping $A$ is considered. We get the conditions for the subspace $V$ under which there exists the limit of the sequence of subspaces $A^n(V)$. The explicit form of this limit is found using standard techniques of linear algebra.

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## 1. Introduction

Let $A$ be an arbitrary linear invertible operator of an $m$-dimensional complex space $L$. Under the action of $A$, any $d$-dimensional vector subspace $V \subset L$ transforms into a vector subspace of the same dimension. Under the action of powers of this map one gets the trajectory of a subspace $V$ — the sequence $A^n(V)$ of subspaces of the same dimension. The subject of investigation in this paper is the behavior of this sequence of subspaces.

One can look at this problem from another point of view. The set of all $d$-dimensional subspaces of a space $L$ has a natural structure of a smooth manifold, which is called Grassmann manifold and denoted by $G(m, d)$. From the previous discussion it follows that the operator $A$ induces bijective continuous map $\varphi : G(m, d) \to G(m, d)$. Therefore, this problem relates to the typical problems in the theory of dynamical systems: this is the problem of dynamical properties of the map $\varphi$, i.e. of the description of the behavior of trajectories of points under the action of the iterates $\varphi^k$ of the given map.

The need for the description of the properties of the map $\varphi$ and its dynamics arises in various problems in the theory of dynamical systems, operator theory, matrix factorization, normal forms of differential and functional operators.

The basic problem may be formulated as follows: for which subspaces $V$ there exists the limit:

$$\lim_{n\to\infty} A^n(V) = V_0 \qquad (1)$$

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and how it can be found in the explicit form? Let us note that such a limit $V_0$ is a fixed point of $\varphi$, i.e. an invariant subspace for $A$.

In the paper [3] it was shown that in the so-called case of Perron (when all of the eigenvalues of the linear map $A$ have different absolute values), for any subspace $V$ the limit (1) exists and the rule for obtaining this limit was found. It turns out that in that case is possible to construct such a basis $\{v_j\}$ of $V$ that there exist and are linearly independent vectors which are limits of the sequences of unit vectors of the form $\nu_{n,j}A^n v_j$, where $\nu_{n,j}$ are certain constants. These limits then form a basis in the limit space. Let us note that the conditions for the existence of the limit of the sequence of the vectors of the form $\nu_{n,j}A^n x$ are known for an arbitrary vector $x$ [5], [1], and that limit can be found in the form which is explicit enough.

On the other hand, in the general case, it can happen that the limits of sequence of the form $\nu_{n,j}A^n v_j$ either do not exist or such limits are linearly dependent vectors for the given basis. In particular, it can happen that all sequences of unit basis vectors converge to the same limit. Therefore, in the general case, the limits mentioned above do not determine the limit subspace and the question of behavior of the trajectories of subspaces does not reduce to the simple discussion of the trajectories of the basis vectors, but it requires a more detailed investigation and the different approach.

The crucial step towards the solution of the problem at hand is the analysis of the case in which the operator $A$ has only one eigenvalue, but it has many Jordan cells of different dimension. In [3], it was shown that in this case for any subspace there existed the limit of trajectories and the description of that limit had been obtained. For the solution of the problem in that case all the terms of the asymptotic expansion of the trajectories of an arbitrary vector were necessary and such an expansion wasobtained in [7]. The precise formulation of the results from [3] is given below. In this paper, the solution of the problem in the general case, for any operator $A$ is obtained. In sections 2 and 3, the necessary background material is collected and the main result is given in section 4.

## 2. Particular subspaces and expansions

We assume that the matrix of the operator $A$ is reduced to the Jordan form. In that case for vectors of the basis in which the matrix has the Jordan form it is convenient to use the following special numeration involving four indices.

Let us denote by $q$ the number of different absolute values of eigenvalues of our matrix and numerate these absolute values in the increasing sequence:

$$0 < r_1 < r_2 < \cdots < r_q.$$

Let $q(k)$ be a number of different eigenvalues of the absolute value $r_k$. Let us numerate all different eigenvalues with a given absolute value $r_k$ using two indices:

$$\lambda_{kj}, 1 \le k \le q, 1 \le j \le q(k).$$

Let $q(k,j)$ be a number of Jordan cells corresponding to the eigenvalue $\lambda_{kj}$; let us denote these cells by $J(k,j,i)$ assuming that they are numerated for fixed $k$ and $j$ in the order of nondecreasing dimensions:

$$1 \leq k \leq q, 1 \leq j \leq q(k), 1 \leq i \leq q(k,j).$$

Let $q(k,j,i)$ be the dimension of the cell $J(k,j,i)$. We denote the basis vectors corresponding to this cell by the index $l$:

$$e(k,j,i,l), 1 \leq k \leq q, 1 \leq j \leq q(k), 1 \leq i \leq q(k,j), 1 \leq l \leq q(k,j,i).$$

In this way one gets numeration of the basis vectors using four indices which correspond to the different properties of the basis vector. This numeration enables the simplification of the notation of the constructions discussed below.

Without the loss of generality one may assume that there is an inner product in which the basis vectors form an orthonormal basis. Let us note that there are many bases in which the matrix of the map has the Jordan form and the given inner product depends on the choice of the basis and it is not canonical.

Let us introduce some families of subspaces generated by the basis vectors with given index values.

Let $L(k)$ be the subspace spanned by all basis vectors $e(k,j,i,l)$ for a given $k$. By $L(k,j)$ we denote the subspace spanned by vectors $e(k,j,i,l)$ for the given $k$ and $j$. It is obvious that all of these subspaces are invariant and that one has

$$L = \bigoplus_k L(k) = \bigoplus_{k=1}^{q} \bigoplus_{j=1}^{q(k)} L(k,j).$$

In this decomposition we actually have an orthogonal direct sum of subspaces. In the following, $\bigoplus$ stands for direct sums of subspace which need not be orthogonal.

By $P(k)$ and $P(k,j)$ we denote the orthogonal projectors to these subspaces. We also need the following chain of subspaces

$$S(k) = \bigoplus_{1}^{k} L(k), 1 \leq k \leq q. \tag{2}$$

Our problem is to investigate the behavior of trajectories of an arbitrary $d$-dimensional subspace $V$. Let us first discuss one particular decomposition of such a subspace which is related to the chain of subspaces (2).

**Lemma 1.** *Let $V$ be an arbitrary $d$-dimensional subspace of $L$ and suppose there exists a chain of increasing subspaces*

$$0 = S_0 \subset S_1 \subset S_2 \subset \cdots S_q = L.$$

*Then there exist the subspaces $W(k)$ in $S_k$ for which $W(k) \cap S_{k-1} = 0$ and such that $V$ may be represented in the form of a direct sum (not orthogonal in* *general)*

$$V = \bigoplus_{1}^{q} W(k). \tag{3}$$

*If we denote by $Q(k)$ a projector onto the subspace $W(k)$, we get the representation of an arbitrary vector in $V$:*

$$x = \sum_{k=1}^{p} w(k), \ \text{where } w(k) = Q(k)x \in W(k). \tag{4}$$

As $W(k)$ one can take any subspace of $S_k \cap V$ which is a complement to $S_{k-1} \cap V$. To be more precise, we may assume that it is the orthogonal complement with respect to the given inner product.

Note that some of the subspaces $W(k)$ may be null.

Here and in what follows we write the number of a subspace and of a projector in parenthesis. This notation is useful when one has a family of objects which are numerated by several parameters. By writing some of these parameters as subscripts and partly in parenthesis, one gets expressions which are easier to handle. Such notation is particularly useful in the case when one has to distinguish an object whose number is given by some expression.

If $W(k)$ are subspaces from the decomposition (3) with respect to the subspaces $S(k)$ of the form (2), then for the projection $P(i)W(k)$ the following properties obviously hold.

i) $P(i)W(k) = 0$ for $i > k$,

ii) $P(k)W(k) = P(k)V$,

iii) $\dim P(k)W(k) = \dim W(k) = \dim P(k)V$.

Note that in the general case the subspace $W(k)$ may not be chosen as contained in $L(k)$. Therefore the projection $P(i)W(k)$ may be nonzero for $i < k$. That leads to the fact that although for $x \in W(k)$ the equality $x = \sum_{i \leq k} P(i)x$ holds, in general

$$W(k) \neq \bigoplus_{i \leq k} P(i)W(k).$$

This is related to the fact that the projections $P(i)x$ are dependent and this dependence is described by the following lemma.

**Lemma 2.** *For $x \in W(k)$ the projections $P(i)x$ are uniquely determined by the projection $P(k)x$ and there exists a constant $C$ such that*

$$\|P(i)(x)\| \leq C\|P(k)x\| \tag{5}$$

*for all $x \in W(k)$ and all $i$ and $k$.*

*Proof.* Since $S(k-1) \subset \ker P(k)$ and $W(k) \cap S(k-1) = 0$, the projector $P(k)$ bijectively maps the subspace $W(k)$ onto its image $P(k)W(k)$ and the inverse operator $R(k) : P(k)W(k) \to W(k)$ is defined, and it is bounded. Therefore, $P(i)x = [P(i)R(k)]P(k)x$, from which $\|P(i)x\| \le \|[P(i)R(k)]\|\|P(k)x\|$. If we put $C = \max_{i,k} \|P(i)R(k)\|$, we get the inequality (5). □

Let us also consider the projections $P(i,j)W(k) \subset L(i,j)$. These subspaces are orthogonal, but in the general case the equality

$$W(k) = \bigoplus_{ij} P(i,j)W(k) \tag{6}$$

may not be satisfied and, one only has the inclusion

$$W(k) \subset \bigoplus_{ij} P(i,j)W(k).$$

A necessary and sufficient condition for the equation (6) to hold is

$$\dim V = \sum_{kj} \dim P(k,j)W(k).$$

The decompositions constructed above in particular help to get the description of invariant subspaces.

**Lemma 3.** *The subspace $V$ is invariant with respect to the operator $A$ if and only if it is representable in the form of direct sum of subspaces*

$$V = \bigoplus_{k=1}^{q} \bigoplus_{j=1}^{q(k)} P(k,j)V \tag{7}$$

*and every $P(k,j)V$ is an invariant subspace in $L(k,j)$.*

Equality (7) is equivalent to the fact that there exists the decomposition (3) of the subspace $V$ for which the following holds:

$$P(k)W(k) = W(k), \quad P(i)W(k) = 0 \text{ for } i \ne k,$$
$$\dim W(k) = \sum_{j} \dim P(k,j)W(k).$$

This statement is contained in [6], and another proof may be found in [4] .

Let $A(k)$ be the restriction of the operator $A$ to the subspace $L(k)$. The spectrum of this operator consists of the eigenvalues $\lambda(k,j)$ for which $|\lambda(k,j)| = r(k)$.

**Lemma 4.** *For a given operator $A$ there exist constants $C$ and $p$ such that for all operators $A(k)$, the following estimate holds*

$$\|A(k)^n\| \le Cr(k)^n n^p.$$

This lemma follows from, for example, the decomposition of the trajectory of the vector obtained in [7].

## 3. The existence of the limit of the trajectory in the case of one eigenvalue

Let us now pass to the discussion of the basic question — for which subspaces there exist the limit of the trajectory and how one can construct that limit for the given subspace. Convergence of the sequence of subspaces is understood as the convergence in the respect to any of the equivalent natural metrics on the set of subspaces. If we concentrate on the subset consisting only of subspaces of the given dimension, then this is the metric on the Grassmann manifold.

We use metric on the set of subspaces given by the formula

$$\rho(V_1, V_2) = \max\{d(V_1, V_2), d(V_2, V_1)\},$$

$$d(V_1, V_2) = \max_{y \in V_1, \|y\|=1} \min_{z \in V_2} \|y - z\|.$$

Let us note that the element $z_y \in V_2$ at which the minimum $\min_{z \in V_2} \|y - z\|$ is achieved is the orthogonal projection of $y$ to the subspace $V_2$. Therefore, if $P_k$ is the orthogonal projector onto the subspace $V_k$, then $d(V_1, V_2) = \|(I - P_2)P_1\|$.

**Theorem 1.** *Let the spectrum of the operator $A$ consist of one point $\lambda$. Then for any subspace $V$ there exists $\lim_{n\to\infty} A^n(V)$. Besides, one can construct, using effectively standard operations of linear algebra, an operator $\Psi_V : V \to L$ such that*

$$\lim_{n\to\infty} A^n(V) = \Psi_V(V). \tag{8}$$

Detailed proof of this theorem is given in [3].

In some sense, the problem consists in finding the limit of the sequence $A^n$ although in the usual sense this sequence does not have the limit, or it has zero limit. If for any $n$ one chooses an operator $B_n$ which bijectively maps the subspace $V$ to itself, then $A^n(V) = A^n B_n(V)$, i.e. one gets the same sequence of subspace by applying the operators $A^n B_n$. In [3], the following problem was discussed: given a subspace $V$, construct a sequence of operators $B_n$ such that there exists the limit $\Psi$ of the sequence $A^n B_n$ and that limit injectively acts on the subspace $V$. If such a sequence is constructed, then (8) holds.

The proof of Theorem 1 from [3] contains description of the method for the construction of the sequence $B_n$ and the method for the calculation of the limit of the sequence $A^n B_n$. These constructions are dictated by the asymptotic behavior of the sequence of images $A^n x$. In the expression for $A^n x$ one finds terms which have different growth speed. These terms are multiplied by specially chosen sequences of numbers which make the growth speeds equal, i.e. one performs a certain gauging of the terms in the expression for $A^n x$. The resulting operators $B_n$ are obtained as a result of a couple of subsequent gauging.

In such a way the proof of Theorem 1 in [3] contains the following more detailed proposition.

**Theorem 2.** *Let the spectrum of the operator $A$ consist of one point $\lambda$. For any subspace $V$ there exists a sequence of invertible operators $B_n : V \to V$ and the operator $\Psi : V \to L$ whose image $\Psi(V)$ is a $d$-dimensional subspace such that*

$$\rho\ (A^n B_n(V), \Psi(V)) \leq \frac{C_1}{n} \to 0,$$

*from which follows that*

$$\lim A^n(V) = \Psi(V).$$

*For the sequence of operators $B_n$ there exist constants $C_2$ and $\mu$ such that the following estimate holds*

$$\|B_n\| \leq C_2 |\lambda|^{-n} n^{\mu}. \tag{9}$$

## 4. The main theorem

Let us consider a problem concerning the existence of the limit of the trajectory of an arbitrary subspace and forms of this limit in the case of an arbitrary operator $A$.

**Theorem 3.** *Let $A$ be any invertible linear operator of the complex vector space $L$ and $V$ be a subspace in $L$. Then, there exists a limit of the trajectory $A^n(V)$ of this subspace if and only if for all $k$ the following equality holds*

$$P(k)W(k) = \bigoplus_j P(k,j)W(k) \tag{10}$$

*where $W(k)$ are components in the decomposition (3) of the subspace $V$, where $P(k)$ and $P(k,j)$ are the projectors defined above.*

*If the condition (10) is satisfied then*

$$\begin{aligned} \lim_{n\to\infty} A^n(V) &= \lim_{n\to\infty} A^n \left( \bigoplus_k P(k)W(k) \right) \\ &= \bigoplus_k \bigoplus_j \lim_{n\to\infty} P(k,j)W(k) \\ &= \bigoplus_k \bigoplus_j \Psi_{kj} P(k,j)W(k), \end{aligned}$$

*where $\Psi_{kj}$ are the operators constructed in Theorem 1.*

*Proof.* Sufficiency. Let us consider the subspace

$$\widehat{V} = \bigoplus_k P(k)W(k).$$

From the equality (10) we have

$$\widehat{V} = \bigoplus_{kj} P(k,j)W(k), \quad \dim \widehat{V} = \dim V.$$

The subspace $P(k,j)W(k)$ is contained in the subspace $L(k,j)$, and such subspaces are pairwise orthogonal. The restriction $A(k,j)$ of the operator $A$ to the subspace $L(k,j)$ is a linear operator whose spectrum consists of only one element $\lambda(k,j)$. According to Theorem 1, there exists the limit $\widetilde{V}(k,j)$ of the trajectory of the subspace $P(k,j)W(k)$. From Theorem 2 it follows that there exists such a linear operator $\Psi_{k,j} : P(k,j)W(k) \to \widetilde{V}(k,j)$ and there exists a sequence of invertible operators $B_n(k,j)$ of $P(k,j)W(k)$ such that

$$\rho(A^n B_n P(k,j)W(k), \widetilde{V}(k,j)) \leq \frac{C}{n}.$$

The subspace $\widetilde{V}(k,j)$ is contained in $L(k,j)$; such subspaces are also pairwise orthogonal and

$$\dim \widetilde{V}(k,j) = \dim P(k,j)W(k),$$

$$\sum_j \dim \widetilde{V}(k,j) = \dim P(k)W(k) = \dim W(k).$$

From this we get that the limit of the trajectory of the subspace $\widehat{V}$ exists and that limit is the subspace $\widetilde{V} = \bigoplus_{kj} \widetilde{W}(k,j)$. In particular, the limit of the trajectory of the subspace $P(k)W(k) = \bigoplus_j P(k,j)W(k)$ also exists, and it is the subspace

$$\widetilde{W}(k) = \bigoplus_j \widetilde{V}(k,j).$$

Let us consider on $P(k)W(k)$ the sequence of operators $B_n(k) = \bigoplus_j B_n(k,j)$ and operator $\Psi_k = \bigoplus_j \Psi_{k,j}$. Then $\widetilde{W}(k) = \Psi_k W(k)$ and

$$\rho(A^n B_n(k)P(k)W(k), \widetilde{V}(k)) \leq \frac{C}{n}.$$

Let us show that the trajectory of the subspace $V$ has the same limit $\widetilde{V}$ as the trajectory of $\widehat{V}$. In order to do this it is enough to prove that the trajectory of the subspace $W(k)$ has the same limit $\widetilde{W}(k)$ as the trajectory of the subspace $P(k)W(k)$, i.e. that the limit of the trajectory of the subspace $W(k)$ depends only on the projection $P(k)W(k)$ and does not depend on the projection $P(i)W(k)$ for $i < k$.

Let us first estimate

$$d(\widetilde{W}(k), A^n W(k)) = \max_{y \in \widetilde{W}(k), \|y\|=1} \inf_{w \in A^n W(k)} \|y - w\|.$$

From Theorem 2 we get that the vector $y$ belongs to $\widetilde{W}(k)$ if and only if it has the form $y = \Psi_k v$, where $v \in P(k)W(k)$. The sequence $z_n = A^n B_n(k)(v) \in A^n P(k)W(k)$ converges to $y$ and the estimate $\|y - z_n\| \leq \frac{C}{n}\|y\|$ holds. Since $v = \Psi_k^{-1} y$, then

$$\|v\| \leq \|\Psi_k^{-1}\| \|y\|. \tag{11}$$

As was shown previously, in the proof of Lemma 2, the operator $P(k)$ maps $W(k)$ onto $P(k)W(k)$ injectively and there exists the inverse operator $R(k) : P(k)W(k) \to W(k)$. Let $x = R(k)v$. Consider the decomposition

$$x = \sum_{i \le k} P(i)x = \sum_{i \le k} P(i)R(k)v.$$

Then the vector

$$x_n = \sum_{i \le k} P(i)R(k)B_n(k)v \tag{12}$$

belongs to the subspace $W(k)$ and the vector $y_n = A^n x_n$ belongs to the subspace $A^n W(k)$. Therefore

$$\inf_{w \in A^n W(k)} \|y - w\| \le \|y - y_n\|.$$

Since

$$\|y - y_n\| \le \|y - z_n\| + \|z_n - y_n\| \le \frac{C}{n}\|y\| + \|z_n - y_n\|,$$

in order to get the estimate of $D(\widetilde{W}(k), A^n W(k))$ let us give an estimate of $\|z_n - y_n\|$.

If we use the decomposition (12), we get the decomposition of the vector $y_n$:

$$y_n = \sum_{i \le k} A^n P(i)R(k)B_n(k)v. \tag{13}$$

Since the component with the number $k$ in the sum (13) is $z_n$, we get

$$y_n - z_n = \sum_{i < k} A^n P(i)R(k)B_n(k)v. \tag{14}$$

Components in (14) are pairwise orthogonal, so

$$\|y_n - z\|^2 = \sum_{i < k} \|A^n P(i)R(k)B_n(k)v\|^2. \tag{15}$$

In order to estimate arbitrary component in (15), we apply Lemma 4, Theorem 2 and Lemma 2. We get

$$\|A^n P(i)R(k)B_n(k)v\| \le \|A(i)^n\|\|R(k)\|\|B_n(k)\|\|v\| \le$$

$$\le Cr(k-1)^n n^p \|R(k)\| r(k)^{-n} n^\mu \|\Psi_k^{-1}\|\|y\| = C\left[\frac{r(k-1)}{r(k)}\right]^n n^l \|y\|. \tag{16}$$

This inequality allows us to estimate (15), which, if we include the condition $\|y\| = 1$, leads to the fact that with some constant $C$ the following inequality holds

$$d(\widetilde{W}(k), A^n(W(k)) \le C\left\{\frac{1}{n} + \left[\frac{r(k-1)}{r(k)}\right]^n n^p\right\}.$$

Since $\frac{r(k-1)}{r(k)} < 1$ we get $d(\widetilde{W}(k), A^n W(k)) \to 0$.

Similarly, one checks that $d(A^n W(k), \widetilde{W}(k)) \to 0$.

So, $\rho(A^n W(k), \widetilde{W}(k)) \to 0$, i. e. $\lim_{n\to\infty} A^n W(k) = \widetilde{W}(k)$. From this it follows that

$$\lim A^n V = \lim A^n \widehat{V} = \widetilde{V},$$

which is what we needed. Sufficiency of the condition is thus established.

Necessity. Let the trajectory of the subspace $V$ has the limit $V_0$. We use the following fact: if $Q$ is an arbitrary projector of the space $L$ and the sequence of subspaces $V_n$ converges to $V_0$ then

$$QV_n \to QV_0.$$

Since the decomposition (3) is given by some projectors $Q(k) : W(k) = Q(k)V$, then there exists the limit $\widetilde{W}(k)$ of the trajectory of the subspace $W(k)$ and there exist limits $\widetilde{W}(k,j)$ of trajectories of subspaces $P(k,j)W(k)$ with

$$\widetilde{W}(k,j) = P(k,j)\widetilde{W}(k), \quad \dim \widetilde{W}(k,j) = \dim P(k,j)W(k).$$

The subspace $\widetilde{W}(k)$ is invariant with respect the the operator $A$. Therefore, according to Lemma 3, one has the decomposition

$$\widetilde{W}(k) = \bigoplus_j P(k,j)\widetilde{W}(k).$$

In particular, $\dim \widetilde{W}(k) = \sum_j \dim(P(k,j)\widetilde{W}(k))$.

Therefore

$$\widetilde{W}(k,j) = \lim A^n[(P(k,j)W(k))] = P(k,j)\widetilde{W}(k).$$

So, the following equality holds

$$\dim W(k) = \dim \widetilde{W} = \sum_j \dim \widetilde{W}(k,j) = \sum_j \dim P(k,j)W(k),$$

which is equivalent to the equality (10) from the statement of the theorem.

Theorem is thus proved. □

**Example.** Let us consider a special case in which all of the eigenvalues of the operator $A$ have different absolute values. Then all of the subspaces $L(k)$ are one-dimensional and in the decomposition (3) of an arbitrary subspace $V$, all components $W(k)$ are either one-dimensional or null. Let

$$K(V) = \{k : W(k) \neq 0\}.$$

The number of elements in the set $K(V)$ is the number of non-zero components in the decomposition (3), and it is equal to the dimension $d$ of the subspace $V$.

The set $K(V)$ may be also characterized as follows

$$K(V) = \{k : S(k) \cap V \neq S(k-1) \cap V\}. \tag{17}$$

For $k \in K(V)$, we have $P(k)W(k) = L(k)$. The subspace $L(k)$ is invariant, its trajectory is stationary and it has the same subspace as a limit. This means that, in the notation of Theorem 3, the operator $\Psi_k$ is an identity operator. Therefore, as a special case of Theorem 3, we get the following proposition, which was previously captured in [2]

**Theorem 4.** *Let $A$ be such an operator that all of its $m$ eigenvalues have different absolute values. Then, for any subspace $V$ there exists the limit of its trajectory and*

$$\lim_{n\to\infty} A^n V = \bigoplus_{k\in K(V)} L(k),$$

*where the set of indices is given in (17).*

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