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A WORD ON n-INFINITE FORCING[1]

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Abstract. It is shown: the properties of Robinson's infinite forcing are naturally transmitted to the so called n-infinite forcing.

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1. Preliminaries

Throughout the article L is a first order language. The basic logical symbols will be $\neg$ (negation), $\wedge$ (conjunction) and $\exists$ (existential quantifier); the others being defined by the basic ones in the standard way. The choice of the logical symbols is irrelevant, but we kept the choice made in the case of (n-) finite forcing (see [2]). For a theory T of the language L, $\mu(T)$ will be the class of all its models (as usual, by a theory we assume a consistent deductively closed set of sentences – thus, $T \vdash \varphi$ means $\varphi \in T$). By Σ_n-formula we mean any formula equivalent to a formula in prenex normal form whose prenex consists of n blocks of quantifiers, the first one being the block of existential quantifiers (Π_n-formulas are defined analogously). The models (of the language L) will be denote by $\mathbf{A}$, $\mathbf{B}$, ..., while their domains will be $A, B, \ldots$. For a model $\mathbf{A}$, $Diag_n(\mathbf{A})$ is the set of all Σ_n- , Π_n-sentences of the language $L(A)$ (the simple expansion of the language L obtained by adding a new set of constants which is in one-to-one correspondence with domain A) which hold in $\mathbf{A}$. In particular, for $n = 0$, $Diag_0(\mathbf{A})$ is not the diagram of $\mathbf{A}$ in the sense in which it is used in model theory, but this difference is of no importance for the text (the same situation we have when dealing with the generalization of finite forcing). As usual, we will not distinguish an element a from A and the constant corresponding to it. If $\mathbf{A}$ is a submodel of $\mathbf{B}$ and $(\mathbf{B}, a)_{a \in A} \models Diag_n(\mathbf{A})$, we say that $\mathbf{A}$ is an n-elementary submodel of $\mathbf{B}$ (i.e. $\mathbf{B}$ is an n-elementary extension of $\mathbf{A}$), in notation $\mathbf{A} \prec_n \mathbf{B}$. In general, $\mathbf{A}$ is n-embedded in $\mathbf{B}$ if, for some embedding f of $\mathbf{A}$ into $\mathbf{B}$, $f(\mathbf{A})$ is an n-elementary submodel of $\mathbf{B}$. A Σ_{n+1}-chain of models is a chain of models $\mathbf{A}_0 < \mathbf{A}_1 < \ldots < \mathbf{A}_\alpha < \ldots$, $\alpha < \gamma$, where for each $\alpha < \beta$ ($< \gamma$), $\mathbf{A}_\alpha$ is an n-elementary submodel of $\mathbf{A}_\beta$; we use $\mathbf{A} < \mathbf{B}$ to denote that $\mathbf{A}$ is a submodel of $\mathbf{B}$, therefore $<$ is "equal" to $\prec_0$.

Remark. We are following mainly [7], in fact, in almost all of the given propositions we use the same proof patterns as in the case of infinite forcing. Thus

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a routine and tedious job is in question, and in that sense this paper does not bring anything essentially new. Its aim is primarily to introduce the definitions and present the basic facts considering n-infinite forcing which are to be used in the further research of the topic. Through a set of circumstances the paper appears with a great delay and after publishing some articles which (continued the examination of n-infinite forcing) announced it in the references there in; see [3], [4], [5].

2. n-infinite forcing relation

In the sequel we will assume that the considered class $\mathcal{K}$ of models of the language L is closed under unions of Σ_{n+1}-chains. In keeping with the standard terminology, we will say that the class $\mathcal{K}$ is n-inductive.

Definition 2.1. *For a model* $\mathbf{A}$ *from* $\mathcal{K}$ *and a sentence* φ *of the language* $L(A)$ *the relation:* $\mathbf{A}$ *n-infinitely forces* φ *(with respect to the class* $\mathcal{K}$*), in notation* $\mathbf{A} \Vdash_n \varphi$*, is defined inductively:*

(1) *if* φ *is an atomic sentence, then* $\mathbf{A} \Vdash_n \varphi$ *iff* $\mathbf{A} \models \varphi$*;*

(2) *if* $\varphi \equiv \phi \wedge \psi$*, then* $\mathbf{A} \Vdash_n \phi \wedge \psi$ *iff* $\mathbf{A} \Vdash_n \phi$ *and* $\mathbf{A} \Vdash_n \psi$*;*

(3) *if* $\varphi \equiv \neg\phi$*, then* $\mathbf{A} \Vdash_n \varphi$ *iff no n-elementary extension of* $\mathbf{A}$ *in* $\mathcal{K}$ *n-infinitely forces* ψ*;*

(4) *if* $\varphi \equiv \exists v \psi(v)$*, then* $\mathbf{A} \Vdash_n \varphi$ *iff, for some* $a \in A$*,* $\mathbf{A} \Vdash_n \psi(a)$*.*

Lemma 2.2. *For a model* $\mathbf{A}$ *of the class* $\mathcal{K}$ *and sentences* φ *and* ψ *of the language* $L(A)$ *it holds:*

(1) *the model* $\mathbf{A}$ *cannot n-infinitely force both* φ *and* $\neg\varphi$*;*

(2) *if* $\mathbf{B}$ *from* $\mathcal{K}$ *is an n-elementary extension of* $\mathbf{A}$ *and* $\mathbf{A} \Vdash_n \varphi$*, then also* $\mathbf{B} \Vdash_n \varphi$*;*

(3) *if* $\mathbf{A} \Vdash_n \varphi$ *or* $\mathbf{A} \Vdash_n \psi$*, then* $\mathbf{A} \Vdash_n \neg(\neg\varphi \wedge \neg\psi)$*, that is* $\mathbf{A} \Vdash_n \varphi \vee \psi$*;*

(4) *if* $\mathbf{A} \Vdash_n \neg\exists v \neg\varphi(v)$*, then, for any* $a \in A$*,* $\mathbf{A} \Vdash_n \neg\neg\varphi(a)$*.*

Proof. (1) Directly, by the very definition of n-infinite forcing relation.

(2) Simple inductive argument by the complexity of the sentence φ.

(3) and (4) are immediate consequences of (1) nad (2) (and definition of n-infinite forcing). □

Note. In Robinson's [8] and subsequent papers on infinite forcing as the basic logical symbol it was taken also disjunction ($\vee$) and it was defined: $\mathbf{A} \Vdash \varphi \vee \psi$ iff $\mathbf{A} \Vdash \varphi$ or $\mathbf{A} \Vdash \psi$. As a consequence of the fact that in our case disjunction is defined by conjunction and negation, in item (3) we do not have the inverse implication. So, for instance, if $\mathcal{K}$ is the class of linearly ordered sets in the language with equality and a binary relation $\leq$, and $\mathbf{A} = \langle \omega \cup \{\omega\} \; (= \omega^+), \leq \rangle$, then $\mathbf{A} \Vdash \neg(\neg\exists v\,(v < 0) \wedge \neg\exists v\,(v > \omega))$, while neither $\mathbf{A} \Vdash \exists v (v < 0)$ nor

$\mathbf{A} \Vdash \exists v(v > \omega)$ (where, of course, $v < u$ stands for $v \leq u \wedge v \neq u$). It holds as well: $\mathbf{A} \Vdash \neg\neg\exists v\,(v < 0)$.

Definition 2.3. *A model* $\mathbf{A}$ *from* $\mathcal{K}$ *is n-infinitely generic iff for any sentence* φ *of the language* $L(A)$ *either* $\mathbf{A} \Vdash_n \varphi$ *or* $\mathbf{A} \Vdash_n \neg\varphi$.

Lemma 2.4. *Any model of the class* $\mathcal{K}$ *is an n-elementary submodel of some n-infinitely generic model.*

Proof. Let $\{\varphi_\alpha \mid \alpha < \lambda = max\{|A|, |L|, \aleph_0\}$ be an enumeration of the sentences of the language $L(A)$. We construct inductively the sequence of models in the following way: $\mathbf{A}_0 = \mathbf{A}$. On the assumption that the models $\mathbf{A}_\gamma$, for all $\gamma < \beta$, have been "chosen", we distinguish the cases: $\beta = \alpha + 1$ and β is a limit ordinal. In the first case, if $\mathbf{A}_\alpha \Vdash_n \varphi_\alpha$ or $\mathbf{A}_\alpha \Vdash_n \neg\varphi_\alpha$, we put $\mathbf{A}_\beta = \mathbf{A}_\alpha$; in the oposite case there exists some n-elementary extension $\mathbf{B}$ of $\mathbf{A}_\alpha$ in $\mathcal{K}$ such that $\mathbf{B} \Vdash_n \varphi_\alpha$, and we put: $\mathbf{A}_\beta = \mathbf{B}$. When β is a limit ordinal, we take $\mathbf{A}_\beta = \bigcup_{\alpha<\beta} \mathbf{A}_\alpha$. In any case, $\mathbf{A}_\alpha$ is in $\mathcal{K}$, for this class is closed under unions of Σ_{n+1}-chains. Obviously, for any sentence φ of the language $L(A)$, the model $\mathbf{A}^1 = \bigcup_{\alpha<\lambda} \mathbf{A}_\alpha$ n-infinitely forces either φ or $\neg\varphi$. If we construct in the same way the model $\mathbf{A}^2$ starting now with the model $\mathbf{A}^1$ and continuing this process we will finally obtain the model $\mathbf{A}^\omega = \bigcup_{n\geq 1} \mathbf{A}^n$, which is certainly n-infinitely generic. □

Lemma 2.5. (a) *A model* $\mathbf{A}$ *of the class* $\mathcal{K}$ *is n-infinitely generic iff for any sentence* φ *of the language* $L(A)$ *it holds:*

$$\mathbf{A} \Vdash_n \varphi \quad \textit{iff} \quad \mathbf{A} \models \varphi;$$

(b) *A model* $\mathbf{A}$ *is n-infinitely generic iff for any sentence* $\neg\varphi$ *of the language* $L(A)$ *it holds:*

$$\mathbf{A} \Vdash_n \neg\varphi \quad \textit{iff} \quad \mathbf{A} \models \neg\varphi.$$

Proof. (a) is proved by induction of the complexity of the formula φ; one implication in (b) follows directly from (a), as for the other, the given condition enables us to pass with the induction in checking that (a) holds. □

Corollary 2.6. (a) *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are n-infinitely generic models of the class* $\mathcal{K}$ *and if* $\mathbf{A}$ *is an n-elementary submodel of* $\mathbf{B}$, *then* $\mathbf{A}$ *is an elementary submodel of* $\mathbf{B}$;

(b) *every n-infinitely generic model of the class* $\mathcal{K}$ *is an n-existentially closed model in* $\mathcal{K}$.

Proof. (a) For a sentence φ of $L(A)$ we have: $\mathbf{A} \models \varphi \iff \mathbf{A} \Vdash_n \varphi \implies \mathbf{B} \Vdash_n \varphi \iff \mathbf{B} \models \varphi$.

(*b*) Let $\mathbf{A}$ be an n-infinitely generic model of the class $\mathcal{K}$, $\mathbf{A} \prec_n \mathbf{B} \in \mathcal{K}$ and let φ be a Σ_{n+1}-sentence of the language $L(A)$ which is satisfied in $\mathbf{B}$. If $\mathbf{C}$ is an n-infinitely generic model of $\mathcal{K}$, which is also an n-elementary extension of $\mathbf{B}$, then $\mathbf{C} \models \varphi$ and, because of $\mathbf{A} \prec \mathbf{C}$, it holds as well $\mathbf{A} \models \varphi$. □

Corollary 2.7. *The class $\mathcal{L}^n_{\mathcal{K}}$ of all n-infinitely generic models of the class $\mathcal{K}$ is closed under the unions of Σ_{n+1}-chains.*

Theorem 2.8. *The class $\mathcal{L}^n_{\mathcal{K}}$ is a unique subclass $\mathcal{C}$ of the class $\mathcal{K}$ satisfying the following:*

(1) *$\mathcal{C}$ is n-mutually-consistent or, in other words, n-model-consistent with $\mathcal{K}$ (which means in fact that any model of $\mathcal{K}$ is an n-elementary submodel of some model from $\mathcal{C}$);*

(2) *$\mathcal{C}$ is n-model-complete;*

and

(3) *$\mathcal{C}$ contains any other subclass of $\mathcal{K}$ which satisfies the conditions* (1) *and* (2).

Proof. It has already been proved that $\mathcal{L}^n_{\mathcal{K}}$ satisfies the conditions (1) and (2). Let $\mathcal{D}$ be the subclass of $\mathcal{K}$ which also satisfies these conditions and let $\mathbf{A} \in \mathcal{D}$. We show that for a sentence $\neg\varphi$ of the language $L(A)$ it holds: $\mathbf{A} \Vdash_n \neg\varphi$ iff $\mathbf{A} \models \neg\varphi$, that is that $\mathbf{A}$ is n-infinitely generic. Suppose $\mathbf{A} \Vdash_n \neg\varphi$ and let $\mathbf{B}$ be an n-infinitely generic model which is an n-elementary extension of $\mathbf{A}$. Then $\mathbf{B} \Vdash_n \neg\varphi$, thus $\mathbf{B} \models \neg\varphi$ too. We construct a countable chain of models in the following way. Let $\mathbf{A}_1$ be a model from $\mathcal{D}$ which is an n-elementary extension of $\mathbf{B}$, $\mathbf{B}_1$ an n-infinitely generic model which is an n-elementary extension of $\mathbf{A}_1$, and so on. Then $\mathbf{C} = \bigcup_{k\geq 1} \mathbf{A}_k = \bigcup_{k\geq 1} \mathbf{B}_k$ is n-infinitely generic and $\mathbf{A} \prec \mathbf{C}$, $\mathbf{B} \prec \mathbf{C}$ (since the chains $\mathbf{A} \prec_n \mathbf{A}_1 \prec \ldots \prec \mathbf{A}_k \prec_n \ldots$ and $\mathbf{B} \prec_n \mathbf{B}_1 \prec_n \ldots \prec_n \mathbf{B}_k \prec_n \ldots$ are elementary chains). But then $\mathbf{C} \models \neg\varphi$, whence also $\mathbf{A} \models \neg\varphi$. On the other hand, if $\mathbf{A}$ does not n-infinitely force $\neg\varphi$, then some n-elementary extension $\mathbf{B}$ of $\mathbf{A}$ n-infinitely forces φ. We can immediately assume that $\mathbf{B}$ is n-infinitely generic, and, as in the previous case, obtain a model $\mathbf{C}$, which is an elementary extension of both $\mathbf{A}$ and $\mathbf{B}$. Then, because of $\mathbf{B} \models \varphi$, it follows $\mathbf{C} \models \varphi$ and therefore $\mathbf{A} \models \varphi$, that is $\mathbf{A} \not\models \neg\varphi$. □

Corollary 2.9. (*a*) *The class $\mathcal{L}^n_{\mathcal{K}}$ is a unique subclass $\mathcal{C}$ of $\mathcal{K}$ satisfying the first two conditions from the previuos theorem and*

(3)′ *if a model $\mathbf{A}$ from $\mathcal{K}$ is an elementary submodel of any model from $\mathcal{C}$, which is its n-elementary extension, then $\mathbf{A} \in \mathcal{C}$;*

(*b*) *On the condition that $\mathcal{K}$ is a generalized elementary class (that is $\mathcal{K} = \mu(Th(\mathcal{K}))$) the condition* (3)′ *can be replaced by*

(3)″ *if a model $\mathbf{A}$ from $\mathcal{K}$ is an elementary submodel of some model $\mathbf{B}$ from $\mathcal{C}$ which is its n-elementary extension, then $\mathbf{A} \in \mathcal{C}$.*

Proof. (a) We prove firstly that $\mathcal{L}^n_{\mathcal{K}}$ satisfies the third condition. So let $\mathbf{A}$ be an elementary submodel of any n-infinitely generic model which is its n-elementary extension. Suppose $\mathbf{A} \| \!=_n \neg\varphi$, where $\neg\varphi$ is defined in $\mathbf{A}$, and let $\mathbf{B}$ be an n-infinitely generic model, which is an n-elementary extension of $\mathbf{A}$. Then, $\mathbf{A} \prec \mathbf{B} \models \neg\varphi$, hence $\mathbf{A} \models \neg\varphi$. If $\mathbf{A}$ does not n-infinitely force $\neg\varphi$, we can find an n-infinitely generic model which n-infinitely forces φ and is an n-elementary extension of $\mathbf{A}$. It follows: $\mathbf{A} \prec \mathbf{B} \models \varphi$, thus $\mathbf{A} \models \varphi$, i.e. $\mathbf{A} \not\models \neg\varphi$.

On the other hand, let $\mathcal{D}$ be a subclass of $\mathcal{K}$ satisfying the conditions of the corollary. By the theorem, $\mathcal{D} \subseteq \mathcal{L}^n_{\mathcal{K}}$. But the inverse inclusion holds as well. For let $\mathbf{A}$ be n-infinitely generic and let $\mathbf{B}$ from $\mathcal{D}$ be its elementary extension. we just constructed a chain $\mathbf{A} \prec_n \mathbf{B} \prec_n \mathbf{A}_1 \prec_n \mathbf{B}_1 \prec_n \ldots \prec_n \mathbf{A}_k \prec_n \mathbf{B}_k \prec_n \ldots$, where $\mathbf{A}_i \in \mathcal{L}^n_{\mathcal{K}}$, $\mathbf{B}_i \in \mathcal{D}$, $i \geq 1$. Now, $\mathbf{C} = \bigcup_{k\geq 1} \mathbf{A}_k = \bigcup_{k \geq 1} \mathbf{B}_k$ is an elementary extension of both $\mathbf{A}$ and $\mathbf{B}$, thus $\mathbf{A}$ is an elementary submodel of $\mathbf{B}$, and by $(3)'$ $\mathbf{A}$ is in $\mathcal{D}$ (as for model $\mathbf{B}$ no restriction was made).

(b) By (a), we are to show that if a class $\mathcal{D}$ satisfies the conditions (1), (2) and (3)" and the model $\mathbf{A}$ is an elementary submodel of some model $\mathbf{B}$ from $\mathcal{D}$, which is its n-elementary extension, then $\mathbf{A}$ is an elementary submodel of any model from $\mathcal{D}$ whose an n-elementary submodel it is. So let $\mathbf{A} \prec_n \mathbf{C} \in \mathcal{D}$. It is easy to see that there exists a model $\mathbf{D}$ (in $\mathcal{K}$) into which $\mathbf{B}$ and $\mathbf{C}$ are n-embeddable. Because of the n-model-consistency we can assume that $\mathbf{D}$ is from $\mathcal{D}$ and because of the n-model completness of the class $\mathcal{D}$ both $\mathbf{B}$ and $\mathbf{C}$ are elementary submodels of $\mathbf{D}$. It follows that $\mathbf{A}$ is an elementary submodel of $\mathbf{C}$. □

Corollary 2.10. (a) *Let* $\mathbf{A}$ *be a model of the class* $\mathcal{K}$ *and* φ *some* Σ_n*- or* Π_n*-sentence defined in* $\mathbf{A}$*. Then it holds (compare with* 1.3 *in* [2]*):*

$$\mathbf{A} \models \varphi \quad \textit{iff} \quad \mathbf{A} \| \!=_n \neg\neg\varphi.$$

(b) *If* $\varphi \equiv \exists \tilde{v} \psi(\tilde{v})$ *is a* Σ_{n+1}*-sentence defined in* $\mathbf{A} \in \mathcal{K}$*, then from* $\mathbf{A} \models \varphi$ *it follows* $\mathbf{A} \| \!=_n \neg\neg\varphi$*; on the other hand, if* $\mathbf{A} \| \!=_n \neg\neg\varphi$*, then some* n*-extension of* $\mathbf{A}$ *in* $\mathcal{K}$ *satisfies* φ*.*

Proof. (a) Suppose $\mathbf{A} \models \varphi$ but that $\mathbf{A}$ does not n-infinitely force $\neg\neg\varphi$. Then for some n-extension $\mathbf{B}$ of $\mathbf{A}$ in $\mathcal{K}$, $\mathbf{B} \| \!=_n \neg\varphi$. But, if $\mathbf{C}$ is an n-infinitely generic model, which is an n-extension of $\mathbf{B}$, it follows $\mathbf{C} \| \!=_n \neg\varphi$, while also $\mathbf{C} \models \varphi$, contradictory to 2.5. It is clear now that $\mathbf{A} \| \!=_n \neg\neg\varphi$ implies $\mathbf{A} \models \varphi$.

(b) Suppose $\mathbf{A} \models \psi(\tilde{a})$. By (a), $\mathbf{A} \| \!=_n \neg\neg\psi(\tilde{a})$, whence obviously also $\mathbf{A} \| \!=_n \neg\neg\exists\tilde{v}\psi(\tilde{v})$. If $\mathbf{A} \| \!=_n \neg\neg\varphi$, then any n-infinitely generic n-extension of $\mathbf{A}$ satifies φ. By the note given after 2.2, the model $\mathbf{A}$ itself does not have to satisfy the sentence φ. □

Corollary 2.11. *Let* $\mathbf{A}$ *be a model of the class* $\mathcal{K}$ *and let* $\varphi(v_1, \ldots, v_{k-1})$ *and* $\psi(v_1, \ldots, v_{k-1})$ *be formulas of the language* L *such that* $\vdash \varphi(v_1, \ldots, v_{k-1}) \Longrightarrow$

$\psi(v_1,\dots,v_{k-1})$. *Then, for any element* $a_1,\dots,a_{k-1}$ *from* $\mathbf{A}$ *it holds: if* $\mathbf{A}\,\|{=}_n\,\varphi(a_1,\dots,a_{k-1})$ *then* $\mathbf{A}\,\|{=}_n\,\neg\neg\psi(a_1,\dots,a_{k-1})$.

Proof. If we assume that $\mathbf{A}\,\|{=}_n\,\varphi(\tilde a)$ but not $\mathbf{A}\,\|{=}_n\,\neg\neg\psi(\tilde a)$, we can find an n-infinitely generic model $\mathbf{B}$ which is an n-extension of $\mathbf{A}$ and which n-infinitely forces $\varphi(\tilde a)$ and $\neg\psi(\tilde a)$. But then $\mathbf{B}\models\varphi(\tilde a)\wedge\neg\psi(\tilde a)$, a contradiction. □

Lemma 2.12. *Let* $\mathcal{K}$ *be a generalized elementary class,* $\mathbf{A}$ *and* $\mathbf{B}$ *its members and* $a_1,\dots,a_{k-1}$, $b_1,\dots,b_{k-1}$ *the elements of* $\mathbf{A}$ *and* $\mathbf{B}$*, respectively. Then the following conditions are equivalent:*

(1) *there is an* n*-elementary extension,* $\mathbf{C}$*, of* $\mathbf{A}$ *in* $\mathcal{K}$ *and the* Σ_{n+1}*-existential type of* $b_1,\dots,b_{k-1}$ *in* $\mathbf{B}$ *(i.e. the set of* Σ_{n+1}*-formulas* $\varphi(v_1,\dots,v_{k-1})$ *for which it holds* $\mathbf{B}\models\varphi[b_1,\dots,b_{k-1}]$*) is contained in the* Σ_{n+1}*-existential type of* $a_1,\dots,a_{k-1}$ *in* $\mathbf{C}$*;*

(2) *there is a model* $\mathbf{D}$ *in* $\mathcal{K}$ *into which the models* $\mathbf{A}$ *and* $\mathbf{B}$ *are* n*-embeddable so that the images of the elements* a_i *and* b_i*,* $i=1,\dots,k-1$*, coincide.*

Proof. Suppose (1) is satisfied. If $ADiag_n(\mathbf{B})$ is the set of sentences obtained from $Diag_n(\mathbf{B})$ by replacing the constants $b_1,\dots,b_{k-1}$ by, respectively, $a_1,\dots,a_{k-1}$, by a simple compactness argument it follows that $Th(\mathcal{K})\cup Diag_n(\mathbf{A})$ $\cup ADiag_n(\mathbf{B})$ is consistent, and any model of this theory satisfies the second condition. □

Definition 2.13. *The modified rank of the formula* φ *of the language* L*, in notation* $m.r.(\varphi)$*, is defined by:*

$$m.r.(\varphi)=\begin{cases}1 & \text{if } \varphi \text{ is either atomic or } \neg\psi\\ m.r.(\psi)+m.r.(\theta) & \text{if } \varphi\equiv\psi\wedge\theta\\ m.r.(\psi)+1 & \text{if } \varphi\equiv\exists v\psi(v)\end{cases}$$

The existential degree of a formula φ *of the language* L*, in notation* $e.d.(\varphi)$*, is defined by:*

$$e.d.(\varphi)=\begin{cases}0 & \text{if } \varphi \text{ is atomic or } \neg\psi\\ e.d.(\psi)+e.d.(\theta) & \text{if } \varphi\equiv\psi\wedge\theta\\ e.d.(\psi)+1 & \text{if } \varphi\equiv\exists v\psi(v)\end{cases}$$

Corollary 2.14. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be models of the generalized elementary class* $\mathcal{K}$ *and let the elements* $a_1,\dots,a_{k-1}$ *and* $b_1,\dots,b_{k-1}$ *of, respectively,* $\mathbf{A}$ *and* $\mathbf{B}$*, have the same* Σ_{n+1}*-existential type in* $\mathbf{A}$*, that is* $\mathbf{B}$*. If* $\varphi(v_1,\dots,v_{k-1})$ *is a formula of the language* L *of modified rank* 1*, then*

$$\mathbf{A}\,\|{=}_n\,\varphi(a_1,\dots,a_{k-1}) \quad iff \quad \mathbf{B}\,\|{=}_n\,\varphi(b_1,\dots,b_{k-1}).$$

In particular, if θ *is either* Σ_n*- or* Π_n*-formula, then*

$$\mathbf{A}\,\|{=}_n\,\neg\neg\theta(a_1,\dots,a_{k-1}) \quad iff \quad \mathbf{B}\,\|{=}_n\,\neg\neg\theta(b_1,\dots,b_{k-1}).$$

Proof. Let $\Gamma(v_1,\ldots,v_{k-1})$ be the Σ_{n+1}-existential type of the elements $a_1,\ldots,a_{k-1}$ and $b_1,\ldots,b_{k-1}$ in, respectively, **A** and **B**. The second part of the corollary is a direct consequence of 2.10 – $\mathbf{A}\,\|=_n \neg\neg\theta(a_1,\ldots,a_{k-1})$ iff $\theta(v_1,\ldots,v_{k-1})\in\Gamma(v_1,\ldots,v_{k-1})$ iff $\mathbf{B}\,\|=_n \neg\neg\theta(b_1,\ldots,b_{k-1})$.

As for the first part, let us suppose that for a formula $\varphi\equiv\neg\psi$, $\mathbf{A}\,\|=_n \neg\psi(a_1,\ldots,a_{k-1})$, but that **B** does not n-infinitely force $\neg\psi(b_1,\ldots,b_{k-1})$. Then, for some n-extension **C** of **B** in $\mathcal{K}$, $\mathbf{C}\,\|=_n \psi(b_1,\ldots,b_{k-1})$, and, certainly, Σ_{n+1}-existential type of $b_1,\ldots,b_{k-1}$ in **C** contains $\Gamma(v_1,\ldots,v_{k-1})$. By 2.12, **A** and **C** are n-embeddable into some model **D** from $\mathcal{K}$ in such a way that the elements a_i and b_i, $i=1,\ldots,k-1$, have the same images – d_i. But then, **D** n-infinitely forces both $\neg\psi(d_1,\ldots,d_{k-1})$ and $\psi(d_1,\ldots,d_{k-1})$, a contradiction. □

Lemma 2.15. *Let $\mathcal{K}$ be a generalized elementary class and $\varphi(v_1,\ldots,v_{k-1})$ a formula of the language L of modified rank 1. Then there is a set $\mathcal{R}_\varphi$ of Σ_{n+1}-existential types such that for any model **A** from $\mathcal{K}$ and any element $a_1,\ldots,a_{k-1}$ from **A** it holds: $\mathbf{A}\,\|=_n \varphi(a_1,\ldots,a_{k-1})$ iff the Σ_{n+1}-existental type of the elements $a_1,\ldots,a_{k-1}$ in **A** is contained in $\mathcal{R}_\varphi$.*

Proof. Just put: $\mathcal{R}_\varphi=\{\Phi(v_1,\ldots,v_{k-1})\mid$ there exists a model **B** in $\mathcal{K}$ and its elements $b_1,\ldots,b_{k-1}$ such that **B** n-infinitely forces $\varphi(b_1,\ldots,b_{k-1})$ and $\Phi(v_1,\ldots,v_{k-1})$ is the Σ_{n+1}-existential type of $b_1,\ldots,b_{k-1}$ in $\mathbf{B}\}$. □

Theorem 2.16. *(Robinson's reduction theorem). Let $\mathcal{K}$ be a generalized elementary class and $\varphi(v_1,\ldots,v_{k-1})$ a formula of the language L of existential degree m. Then there is a set $\mathcal{R}_\varphi$ of Σ_{n+1}-existential types $\Phi(v_1,\ldots,v_{k-1}, v_k,\ldots,v_{k-1+m})$ such that for any model **A** from $\mathcal{K}$ and its elements $a_1,\ldots,a_{k-1}$ it holds:*

*$\mathbf{A}\,\|=_n \varphi(a_1,\ldots,a_{k-1})$ iff, for some elements $b_1,\ldots,b_m$ from **A**, the Σ_{n+1}-existential type of elements $a_1,\ldots,a_{k-1},b_1,\ldots,b_m$ in **A** is in $\mathcal{R}_\varphi$.*

Proof. By induction on the modified rank of the formula φ. The case $m.r.(\varphi)=1$ has been already considered (previous lemma).

Let $m.r.(\varphi)=r>1$ and $\varphi(v_1,\ldots,v_{k-1})\equiv\psi(v_1,\ldots,v_{k-1})\wedge\theta(v_1,\ldots,v_{k-1})$, and let s and t be existential degrees of ψ and θ respectively. By the inductive assumption, there are sets of Σ_{n+1}-existential types $\mathcal{R}_\psi=\{\Psi_\alpha(v_1,\ldots,v_{k-1},v_k,\ldots, v_{k-1+s})\mid\alpha<\kappa\}$ and $\mathcal{R}_\theta=\{\Theta_\beta(v_1,\ldots,v_{k-1},v_k,\ldots,v_{k-1+t})\mid\beta<\lambda\}$, which satisfy the conditions of the theorem for formulas ψ and θ. Then $\mathcal{R}_\varphi= \{\Psi_\alpha(v_1,\ldots,v_{k-1},\, v_k,\ldots,v_{k-1+s})\cup\Theta_\beta(v_1,\ldots,v_{k-1},v_{k+s},\ldots,v_{k+s+t-1})\mid\alpha< \kappa,\,\beta<\lambda\}$ is the "wanted" type for φ.

If $\varphi\equiv\exists v_k\psi(v_1,\ldots,v_{k-1},v_k)$, $r=e.d.(\psi)$ and $\mathcal{R}_\psi=\{\Psi_\alpha(v_1,\ldots,v_{k-1}, v_k,v_{k+1},\ldots,v_{k+r})\mid\alpha<\kappa\}$ the corresponding type for ψ, then we simply take: $\mathcal{R}_\varphi=\mathcal{R}_\psi$. The checking that this type satisfies the condition of the theorem for φ is routine, as in the previous case, and hence it is omitted. □

Corollary 2.17. (a) *Let $\mathcal{K}$ be a generalized elementary class, $\mathbf{A}$ an n-infinitely generic model and $\varphi(v_1, \ldots, v_{k-1})$ a formula of the language L. Then, for the elements $a_1, \ldots, a_{k-1}$ from $\mathbf{A}$, $\mathbf{A}$ n-infinitely forces $\varphi(a_1, \ldots, a_{k-1})$ iff the Σ_{n+1}-existential type of $a_1, \ldots, a_{k-1}$ in $\mathbf{A}$ is in $\mathcal{R}_{\neg\neg\varphi}$ (where $\mathcal{R}_{\neg\neg\varphi}$ is the type coresponding to the formula $\neg\neg\varphi$ from the previous theorem).*

(b) *Let $\mathcal{K}$ be a generalized elementary class, $\mathbf{A}$ an n-infinitely generic model and $a_1, \ldots, a_{k-1}$ some elements from $\mathbf{A}$. Then the complete type of $a_1, \ldots, a_{k-1}$ in $\mathbf{A}$ is uniquely determined by the Σ_{n+1}-existential type of these elements in $\mathbf{A}$.*

3. Some relevant classes of models

n-existentially complete models have already been introduced (in our previous papers). The definition of n-existentially universal model follows, of course, the definition of an existentially universal model – now existential types are replaced by Σ_{n+1}-existential types. Finally, we say that a model $\mathbf{A}$ is n-pregeneric in a class $\mathcal{K}$ iff whenever $\mathbf{A}$ is n-elementary sumodel of n-infinitely generic models $\mathbf{B}$ and $\mathbf{C}$ and φ is a sentence of the language $L(A)$, then $\mathbf{B} \models \varphi$ iff $\mathbf{C} \models \varphi$. Let us denote by $\mathcal{E}^n_{\mathcal{K}}$, $\mathcal{A}^n_{\mathcal{K}}$ and $\mathcal{P}^n_{\mathcal{K}}$, respectively the classes of all n-existentally complete, n-existentially universal and n-pregeneric models of the class $\mathcal{K}$. All these clases are (as well as $\mathcal{L}^n_{\mathcal{K}}$ – 2.4, 2.7) n-inductive and n-model-consistent with $\mathcal{K}$; we recall that $\mathcal{K}$ is n-inductive. On the analogy of the "standard case" we have

Lemma 3.1. (a) $\mathcal{L}^n_{\mathcal{K}} \subseteq \mathcal{P}^n_{\mathcal{K}} \cap \mathcal{E}^n_{\mathcal{K}}$;

$\mathcal{E}^n_{\mathcal{K}} \supseteq \mathcal{A}^n_{\mathcal{K}} \supseteq \mathcal{A}^n_{\mathcal{K}} \cap \mathcal{L}^n_{\mathcal{K}} \neq \emptyset$.

(b) *On the additional condition that $\mathcal{K}$ is a generalized elementary class it holds:*

$\mathcal{P}^n_{\mathcal{K}} \supseteq \mathcal{E}^n_{\mathcal{K}} \supseteq \mathcal{L}^n_{\mathcal{K}} \supseteq \mathcal{A}^n_{\mathcal{K}}$

and

$\mathcal{L}^n_{\mathcal{K}}$ *is the class of elementary substructures of the members of the class* $\mathcal{A}^n_{\mathcal{K}}$.

Proof. (b) Let $\mathbf{A}$ be an n-existentially complete model which is n-elementary submodel of n-infinitely generic models $\mathbf{B}$ and $\mathbf{C}$. Since the class $\mathcal{K}$ is generalized elementary there is a model $\mathbf{D}$ in it such that $\mathbf{B} \prec_n \mathbf{D}$ and $\mathbf{C} \prec \mathbf{D}$. Now for a sentence φ of the language $L(A)$ the assumption that, for instance, $\mathbf{B} \models \varphi$ and $\mathbf{C} \models \neg\varphi$ would imply $\mathbf{D} \Vdash_n \varphi \wedge \neg\varphi$, a contradiction.

In proving $\mathcal{L}^n_{\mathcal{K}} \supseteq \mathcal{A}^n_{\mathcal{K}}$ we use 2.9 (b) and the facts that both classes are n-inductive and that, for $\mathbf{A}, \mathbf{B} \in \mathcal{A}^n_{\mathcal{K}}$, from $\mathbf{A} \prec_n \mathbf{B}$ follows $\mathbf{A} \prec \mathbf{B}$. □

If L is a language with equality and $\mathcal{K}$ has finite models, then all these models are n-infinitely generic (in $\mathcal{K}$) for any $n \geq 1$ (obviously, if $\mathbf{A}$ is a finite model and $\mathbf{A} \prec_1 \mathbf{B}$, then $\mathbf{A} = \mathbf{B}$. This fact can be used in showing that in some cases the class $\mathcal{L}^n_{\mathcal{K}}$ is not generalized elementary. We offer one example.

Lemma 3.2. *Let $\mathcal{G}$ be the class of groups (defined in the standard language $\{\cdot, ^{-1}, e\}$). The class $\mathcal{L}^1_{\mathcal{G}}$ is not generalized elementary.*

Proof. If P is the set of all prime numbers, $\mathbf{C}_p$, $p \in P$, the cyclic group of order p and F a nonprincipal ultrafilter over P, then the ultraproduct $\prod_{p\in P} \mathbf{C}_p/F$ is not 1-existentially complete in $\mathcal{G}$; for the given group is isomorphic to the additive group of reals – $\mathbf{Re}$ and while we have $\mathbf{Re} \prec_1 \mathbf{Re} \times \mathbf{Z}$ (where $\mathbf{Z}$ is the additive group of the integers), it does not hold $\mathbf{Re} \prec_2 \mathbf{Re} \times \mathbf{Z}$ (for instance the sentence $\exists x \forall y (x \neq y + y)$ does not hold in $\mathbf{Re}$). □

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