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SOME FOURTH-ORDER METHODS FOR NONLINEAR EQUATIONS[1]

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Abstract. We present three new fourth-order methods for solving nonlinear equations. These methods are modifications of Newton's method. Several numerical examples are given to illustrate the performance of the presented methods.

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1. Introduction

In this paper we consider three fourth-order iterative methods for finding a simple root of the nonlinear equation $f(x) = 0$. We assume that the function f satisfies

$$f \in C^4[a,b],\ f(a)f(b) < 0,\ f'(x) \neq 0 \text{ for } x \in [a,b]. \tag{1.1}$$

Under these assumptions the function has a unique root α. Newton's method is a well-known iterative method for computing approximation of α by using

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)},\quad k = 0,1,\ldots, \tag{1.2}$$

for some appropriate start value x_0. Newton's method converges quadratically in some neighbohood of α if $f'(\alpha) \neq 0$, [3].

There are many modifications of Newton's method which have order of convergence greater than 2, see for example [1], [2], [4] and reference therein. Here we also consider an improvement of Newton's method in order to obtain fourth-order methods.

Let us define

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(1.3) $$F(x) = x - \frac{f(x)}{f'(x)} G(x)$$

and

(1.4) $$x_{k+1} = F(x_k),\ k = 0, 1, \ldots,$$

Our aim is to choose an appropriate function G such that the iterative method (1.4) converges to the solution α of the equation $f(x) = 0$ with order four. The order of a method with iteration function F is determined by the value of the derivatives of F at α. A method is of a qth order if

(1.5) $$F'(\alpha) = F''(\alpha) = \cdots F^{(q-1)}(\alpha) = 0$$

and $F^{(q)}(\alpha) \neq 0$. For such a method, $|x_{k+1} - \alpha|$ becomes proportional to $|x_k - \alpha|^q$ as $k \to \infty$. Newton's method is at least of second order for simple roots.

In this paper we present new fourth-order modifications of Newton's method. These methods are based on the construction function G with the properties

(1.6) $$G(\alpha) = 1,\ \ G'(\alpha) = \frac{f''(\alpha)}{2f'(\alpha)},\ \ G''(\alpha) = \frac{-3f''(\alpha)^2 + 4f'(\alpha) f'''(\alpha)}{6f'(\alpha)^2},$$

where we shall use one evaluation of the function f and two evaluations of its first derivative. If we consider the definition of efficiency as $p^{\frac{1}{m}}$, where p is the order of the method and m is the number of function evaluations per iteration required by the method, we have that our methods have efficiency index equal to $4^{\frac{1}{3}} \approx 1.5874$, which is better than the one of Newton's method and two-step Newton's method $\sqrt{4} \approx 1.4142$.

2. Main result

If we define F by (1.3) we obtain an iterative method of fourth order if G satisfies (1.6). For the definition of the function G we need the knowledge of the zero α. Since α is unknown, we can use appropriate approximations for G. We shall consider three approximations of the function G, such that the resulting iterative method (1.4) has an order of convergence equal to four. In these approximations we use only two evaluations of f'. We suggest the following three approximations of the function G:

$$G_1(x) = \frac{1}{2} - \frac{f'(x)}{f'(x) - 3f'\left(x - \frac{2}{3}\frac{f(x)}{f'(x)}\right)},$$

$$G_2(x) = 1 + \frac{3f'(x)}{2f'\left(x - \frac{2}{3}\frac{f(x)}{f'(x)}\right)} - \frac{3f'(x)}{f'(x) + f'\left(x - \frac{2}{3}\frac{f(x)}{f'(x)}\right)},$$

$$G_3(x) = \frac{9f'(x)}{10f'\left(x - \frac{2}{3}\frac{f(x)}{f'(x)}\right)} + \frac{f'(x)}{25f'(x) - 15f'\left(x - \frac{2}{3}\frac{f(x)}{f'(x)}\right)}.$$

Obviously, using similar approximations one can also obtain new fourth order iterative methods.

Let us consider the iterative procedure (1.4), where F is given by (1.3). Our conditions imply that f has exactly one root in (a, b).

Theorem 2.1. *Let us assume that the function f satisfies* (1.1)*. Then the iterative method* (1.4)*, where*

$$F(x) = x - \frac{f(x)}{f'(x)} G(x), \quad G \in \{G_1, G_2, G_3\},$$

converges to the unique solution α of $f(x) = 0$ in a neighborhood of α. The order of covergence of is four.

Proof. It is well known that the iterative method (1.4) is fourth-order convergent if F satisfies (1.1) with $q = 4$. Differentiating F we get

$$F'(x) = 1 - u'(x)G(x) - u(x)G'(x),$$

$$F''(x) = -u''(x)G(x) - 2u'(x)G'(x) - u(x)G''(x),$$

$$F'''(x) = -u'''(x)G(x) - 3u''(x)G'(x) - 3u'(x)G''(x) - u(x)G'''(x).$$

where

$$u(x) = \frac{f(x)}{f'(x)}.$$

It is easy to see that for all our functions G it holds $G(\alpha) = 1$. After simple calculations one can obtain that

$$G'(\alpha) = \frac{f''(\alpha)}{2f'(\alpha)}$$

and

$$G''(\alpha) = \frac{-3f''(\alpha)^2 + 4f'(\alpha) f'''(\alpha)}{6f'(\alpha)^2}.$$

We have

$$u(\alpha) = 0, \quad u'(\alpha) = 1.$$

Now, we can see that $F(\alpha) = \alpha$ and $F'(\alpha) = 0$. Since

$$u''(\alpha) = -\frac{f''(\alpha)}{f'(\alpha)}, \quad u'''(\alpha) = \frac{3f''(\alpha)^2 - 2f'(\alpha) f'''(\alpha)}{f'(\alpha)^2}$$

we conclude that

$$F''(\alpha) = -u''(\alpha)G(\alpha) - 2u'(\alpha)G'(\alpha) - u(\alpha)G''(\alpha) = -\frac{f''(\alpha)}{f'(\alpha)} - 2\frac{f''(\alpha)}{2f'(\alpha)} = 0$$

and

$$\begin{aligned} F'''(\alpha) &= -u'''(\alpha)G(\alpha) - 3u''(\alpha)G'(\alpha) - 3u'(\alpha)G''(\alpha) - u(\alpha)G'''(\alpha) \\ &= -\frac{3f''(\alpha)^2 - 2f'(\alpha)f'''(\alpha)}{f'(\alpha)^2} + \frac{3f''(\alpha)^2}{2f'(\alpha)^2} - \frac{-3f''(\alpha)^2 + 4f'(\alpha)f'''(\alpha)}{2f'(\alpha)^2} \\ &= 0 \end{aligned}$$

which is sufficient to complete the proof. □

3. Numerical results

The obtained theoretical results are confirmed by numerical experiments. We present some numerical test results for our fourth-order methods and Newton's method. The methods with iteration functions F were compared, where

$$F(x) = x - \frac{f(x)}{f'(x)}G(x)$$

and G is some of our functions G_1, G_2 and G_3. So, we have the following three iterative functions:

$$F_1(x) = x - f(x)\left(\frac{1}{2f'(x)} - \frac{1}{f'(x) - 3f'\left(x - \frac{2}{3}\frac{f(x)}{f'(x)}\right)}\right),$$

$$F_2(x) = x - f(x)\left(\frac{1}{f'(x)} + \frac{3}{2f'\left(x - \frac{2}{3}\frac{f(x)}{f'(x)}\right)} - \frac{3}{f'(x) + f'\left(x - \frac{2}{3}\frac{f(x)}{f'(x)}\right)}\right),$$

$$F_3(x) = x - f(x)\left(\frac{9}{10f'\left(x - \frac{2}{3}\frac{f(x)}{f'(x)}\right)} + \frac{1}{25f'(x) - 15f'\left(x - \frac{2}{3}\frac{f(x)}{f'(x)}\right)}\right).$$

We also consider Newton's method and the corresponding iterative function written as

$$F_0(x) = x - \frac{f(x)}{f'(x)}.$$

The order of convergence COC can be approximated using the formula

$$COC = \frac{\ln \left|\left(x_{k+1} - \alpha\right) / \left(x_k - \alpha\right)\right|}{\ln \left|\left(x_k - \alpha\right) / \left(x_{k-1} - \alpha\right)\right|}.$$

The expected value of $COC = 4$.

All computations were performed in Mathematica 6.0. When *SetPrecision* is used to increase the precision of a number, we can choose number *prec* of digits in floating point arithmetics. In our tables we give the value of *prec*. We use the following stopping criteria in our calculations: $|x_k - \alpha| < \varepsilon$ and $|f(x_k)| < \varepsilon$, where α is exact solution of considered equation. With *it* we denote number of iteration steps. For numerical illustrations in this section we used the fixed stopping criteria $\varepsilon = 10^{-15}$ and $prec = 1000$.

We present some numerical test results for our iterative methods in Table 1. We used the following functions:

$f_1(x) = \sin x - \frac{1}{2}$, $\alpha_1^* \approx 0.52359877559829887 31$,

$f_2(x) = x^3 - 10$, $\alpha_2^* \approx 2.1544346900318837218$,

$f_3(x) = e^x - x^2$, $\alpha_3^* \approx 0.9100075724887090607$,

$f_4(x) = x^3 + 4x^2 - 10$, $\alpha_4^* \approx 1.3652300134140968458$,

$f_5(x) = (x-1)^3 - 1$, $\alpha_5^* = 2$,

$f_6(x) = \sin x - \frac{x}{2}$, $\alpha_6^* \approx 1.8954942670339809471$.

We also display the approximation α^* of the exact root α for each equation. α^* is calculated with the precision *prec*, but only 20 digits are displayed.

As a convergence criterion it was required that the distance between two consecutive approximations δ for the zero be less than 10^{-15}. Also displayed are the number of iterations to approximate root (*it*), the computational order of convergence (COC), the value $f(x_{it})$ and $|x_{it} - \alpha|$.

	it	COC	$\lvert x_{it}-\alpha\rvert$	$f(x_{it})$	δ
f_1, $x_0 = 0.05$					
F_0	5	2	3.6×10^{-35}	-3.1×10^{-35}	1.1×10^{-17}
F_1	4	4	6.4×10^{-220}	-5.5×10^{-220}	3.1×10^{-55}
F_2	4	4	3.5×10^{-216}	-3.0×10^{-216}	2.5×10^{-54}
F_3	4	4	1.2×10^{-210}	-1.1×10^{-210}	5.5×10^{-53}
f_1, $x_0 = 1.00$					
F_0	6	2	2.8×10^{-45}	-2.4×10^{-45}	9.8×10^{-23}
F_1	4	4	2.3×10^{-146}	-2.0×10^{-146}	7.6×10^{-37}
F_2	4	4	4.3×10^{-127}	-3.7×10^{-127}	4.7×10^{-32}
F_3	4	4	1.9×10^{-64}	-1.7×10^{-64}	2.0×10^{-16}
f_2, $x_0 = 2.20$					
F_0	7	2	1.5×10^{-215}	2.2×10^{-214}	5.8×10^{-108}
F_1	4	4	1.9×10^{-445}	2.6×10^{-444}	1.3×10^{-111}
F_2	4	4	1.0×10^{-414}	1.4×10^{-413}	5.0×10^{-104}
F_3	5	4	0.0×10^{-999}	0.0×10^{-999}	5.5×10^{-388}
f_3, $x_0 = 1.27$					
F_0	6	2	2.3×10^{-51}	-6.8×10^{-51}	6.2×10^{-26}
F_1	4	4	1.8×10^{-188}	-5.3×10^{-188}	1.6×10^{-47}
F_2	4	4	3.4×10^{-176}	-1.0×10^{-175}	1.6×10^{-44}
F_3	4	4	2.2×10^{-163}	-6.6×10^{-163}	2.3×10^{-41}
f_4, $x_0 = 1.00$					
F_0	6	2	2.4×10^{-44}	4.0×10^{-43}	2.2×10^{-22}
F_1	4	4	1.5×10^{-187}	2.5×10^{-186}	3.6×10^{-47}
F_2	4	4	7.6×10^{-154}	1.3×10^{-152}	7.9×10^{-39}
F_3	4	4	2.8×10^{-97}	4.7×10^{-96}	9.2×10^{-25}
f_5, $x_0 = 1.80$					
F_0	6	2	9.6×10^{-42}	2.9×10^{-41}	3.1×10^{-21}
F_1	4	4	2.2×10^{-181}	6.5×10^{-181}	7.6×10^{-46}
F_2	4	4	1.1×10^{-144}	3.4×10^{-144}	9.3×10^{-37}
F_3	4	4	6.4×10^{-80}	1.9×10^{-79}	1.2×10^{-20}
f_6, $x_0 = 2.30$					
F_0	6	2	3.0×10^{-48}	-2.5×10^{-48}	2.3×10^{-24}
F_1	4	4	2.7×10^{-182}	-2.2×10^{-182}	5.9×10^{-46}
F_2	4	4	6.9×10^{-168}	-5.7×10^{-168}	2.0×10^{-42}
F_3	4	4	5.1×10^{-154}	-4.2×10^{-154}	5.1×10^{-39}

Table 1.

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