Novi Sad J. Math.
Vol. 37, No. 2, 2007, 259-267

QUASIASYMPTOTIC IN $\mathcal{D}'_{L^q}(\mathbb{R}^n)$[1]

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Abstract. The multiresolution expansion $\{E_j f\}_{j\in\mathbb{N}}$, $f \in \mathcal{D}'_{L^q}(\mathbb{R}^n)$, $1 \le q \le \infty$, is defined via a scaling function which order of regularity is equal to the order of f. Abelian and Tauberian type theorems for the quasiasymptotic behavior at infinity of distributions from $\mathcal{D}'_{L^q}(\mathbb{R}^n)$ related to the quasiasymptotic behavior at infinity of its projections $E_j f, j \in \mathbb{N}$, are given.

AMS Mathematics Subject Classification (2000): 62E20

Key words and phrases: multiresolution expansion, $\mathcal{D}'_{L^p}$, quasiasymptotics

1. Introduction

In general, a distribution does not have a value at a point ([5]) and at infinity ([10, 16]). This fact is the motivation for the generalized asymptotic analysis of a distribution at a point and at infinity, see [3, 10, 16]. We refer to [3, 10, 12, 14, 15, 16] for the use of asymptotic analysis of distribution in the analysis of integral transforms, PDE and mathematical physics, in problems where the use of classical asymptotics does not give enough informations. Especially the quasiasymptotic behavior of tempered distributions is studied in [10, 11, 12, 16, 18]. Note that some functions have quasiasymptotic behavior different from their classical asymptotic behavior or classical behavior does not exist at all. So it turned out that the quasiasymptotic behavior is appropriate for the Abelian and Tauberian type theorems for integral transforms such as Fourier, Laplace, Stieltjes and Mellin transform.

Spaces $\mathcal{D}'_{L^q}(\mathbb{R}^n), 1 \le q \le \infty$ are introduced in [15]. We refer to [1, 7, 8, 17] for these spaces and their application, for example, in solving of special nonhomogeneous convolution equations.

Notions of multiresolution analysis (MRA) and of multiresolution expansion ($\{E_j f\}_{j\in\mathbb{Z}}, f \in L^2(\mathbb{R}^n)$) come from a wavelet theory, see [2, 6, 20]. The convergence of wavelet and multiresolution expansions in different type of spaces is studied in many papers, see for example [4, 6, 12, 19, 20]. The MRA in spaces $H^{m,p}(\mathbb{R}^n)$, $m \in \mathbb{N}_0$, $1 \le p < \infty$ and spaces of distributions $\mathcal{D}'_{L^q}(\mathbb{R}^n)$, $\frac{1}{p} + \frac{1}{q} = 1$ is studied in [13] (for $p = \infty$ it is observed in $\dot{H}^{m,\infty}(\mathbb{R}^n)$ and $\mathcal{D}'_{L^1}(\mathbb{R}^n)$).

[1]This paper is part of the scientific research project no. 144016, supported by the Ministry of Science, Republic of Serbia
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In this paper we consider the quasiasymptotics at ∞ in $\mathcal{D}'_{L^q}(\mathbb{R}^n), 1 \le q \le \infty$. More preciously, we study relation between quasiasymptotic behavior at ∞ of $f \in \mathcal{D}'_{L^q}(\mathbb{R}^n), 1 \le q \le \infty$ and its projections $E_j f, j \in \mathbb{Z}$.

In Section 2 are given notions and notation. Also, Section 2 contains statements related to multiresolution expansion in $\mathcal{D}'_{L^q}(\mathbb{R}^n), 1 \le q \le \infty$ which are used in Section 3 for the characterization of quasiasymptotic behavior at ∞ of an $f \in \mathcal{D}'_{L^q}(\mathbb{R}^n)$ throughout the quasiasymptotic behavior at ∞ of its projections $E_j f, j \in \mathbb{N}$ (Theorem 6), and conversly for the characterization of quasiasymptotic behavior at ∞ of projections $E_j f$ via the quasiasymptotic behavior at ∞ of $f \in \mathcal{D}'_{L^q}(\mathbb{R}^n)$ (Theorem 5). In Theorems 7 and 8 is studied relations between the quasiasymptotic boundness of elements $f \in \mathcal{D}'_{L^1}(\mathbb{R}^n)$ and the quasiasymptotic boundness of their projections $E_j f, j \in \mathbb{N}$.

2. Notions and notation

As usual by $\mathbb{R}^n$ and $\mathbb{N}_0^n$, $n \in \mathbb{N}$, we denote the set of $n-$typles of real numbers and nonnegative integers, $\mathbb{Z}^n = \mathbb{N}_0^n \cup -\mathbb{N}^n$. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n, x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Then $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $f^{(\alpha)}(x) = \partial^\alpha f(x) = \frac{\partial^{\alpha_1}}{\partial x_1} \cdots \frac{\partial^{\alpha_n}}{\partial x_n} f(x)$. The space of compactly supported infinitely differentiable functions in $\mathbb{R}^n$ is denoted by $C_0^\infty(\mathbb{R}^n)$ and the space of $r-$times differentiable functions in $\mathbb{R}^n$ is denoted by $C^r(\mathbb{R}^n)$. By $H^{m,p}(\mathbb{R}^n), m \in \mathbb{N}_0, 1 \le p \le \infty$ are denoted Sobolev spaces. In particular $\dot{H}^{m,\infty}(\mathbb{R}^n), m \in \mathbb{N}_0$, are subspaces of $H^{m,\infty}(\mathbb{R}^n)$ consisting of those functions $\phi \in C^m(\mathbb{R}^n)$ such that $|\phi^{(\alpha)}(x)| \to 0$ as $|x| \to \infty, 0 \le |\alpha| \le m$.

The dual pairing between elements of a test function space X and elements of its dual X' is denoted by ${}_{X'}\langle \cdot, \cdot \rangle_X = \langle \cdot, \cdot \rangle$. As usual, $\mathcal{D}'(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ denote duals of $C_0^\infty(\mathbb{R}^n) = \mathcal{D}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$, respectively. $\mathcal{D}_{L^p}(\mathbb{R}^n)$, $1 \le p \le \infty$ is the space of $\phi \in C^\infty(\mathbb{R}^n)$ such that $\phi^{(\alpha)} \in L^p(\mathbb{R}^n), \alpha \in \mathbb{N}_0^n$. In particular $\dot{\mathcal{D}}_{L^\infty}(\mathbb{R}^n)$ is the subspace of $\mathcal{D}_{L^\infty}(\mathbb{R}^n)$ whose elements with all derivatives converge to zero as $|x| \to \infty$. The strong dual of $\mathcal{D}_{L^p}(\mathbb{R}^n), 1 \le p < \infty$, is the space of distributions $\mathcal{D}'_{L^q}(\mathbb{R}^n), q = \frac{p}{p-1}$ (if $p = 1$ then $q = \infty$). In particular, $\dot{\mathcal{D}}_{L^\infty}(\mathbb{R}^n)$ is a test space for $\mathcal{D}'_{L^1}(\mathbb{R}^n)$. The spaces $\mathcal{D}'_{L^q}(\mathbb{R}^n), 1 \le q \le \infty$, are subspaces of the space of tempered distributions.

Let $p \in [1, \infty)$. Since $\mathcal{D}_{L^p}(\mathbb{R}^n) = \cap_{m=0}^\infty H^{m,p}(\mathbb{R}^n)$, it follows that $\mathcal{D}'_{L^q}(\mathbb{R}^n) = \cup_{m=0}^\infty (H^{m,p}(\mathbb{R}^n))', 1 < q \le \infty$. So, for every $f \in \mathcal{D}'_{L^q}(\mathbb{R}^n), 1 < q \le \infty$, there is minimal $m_0 \in \mathbb{N}$ such that $f \in (H^{m_0,p}(\mathbb{R}^n))'$. We call m_0 the order of f.

If $p = \infty$ then $\dot{\mathcal{D}}_{L^\infty}(\mathbb{R}^n) = \cap_{m=0}^\infty \dot{H}^{m,\infty}(\mathbb{R}^n)$, hence for every $f \in \mathcal{D}'_{L^1}(\mathbb{R}^n)$ there is minimal $m_0 \in \mathbb{N}$ (order of f) such that $f \in (\dot{H}^{m_0,\infty}(\mathbb{R}^n))'$.

An MRA of $L^2(\mathbb{R}^n)$ consists of a sequence of closed subspaces $\{V_j\}_{j \in \mathbb{Z}}$ of $L^2(\mathbb{R}^n)$ satisfying

(*i*) $V_j \subset V_{j+1}$, for all $j \in \mathbb{Z}$;

(*ii*) $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$, for all $j \in \mathbb{Z}$;

(*iii*) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$;

(*iv*) $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}^n)$;

(v) There exists $\phi \in V_0$ such that $\{\phi(x-k)\}_{k\in\mathbb{Z}^n}$ is an orthonormal basis for V_0. Then ϕ is called a scaling function.

An MRA is $r-$ regular, $r \in \mathbb{N}_0$, if and only if for every $m \in \mathbb{N}$ there exists $C_m > 0$ such that

$$|\phi^{(\alpha)}(x)| \leq \frac{C_m}{(1+|x|)^m},\ 0 \leq |\alpha| \leq r, x \in \mathbb{R}^n.$$

We assume in the sequel that MRA is $r-$ regular, $r \in \mathbb{N}$. Let ϕ be a scaling function for some MRA. Then the function $E(x,y) = \sum_{k\in\mathbb{Z}^n} \phi(x-k)\overline{\phi(y-k)}$ $x, y \in \mathbb{R}^n$, is the reproducing kernel of V_0, i.e. $E_j f = f$ for $f \in V_0$. The orthogonal projection $E_j f$ of $f \in L^2(\mathbb{R}^n)$ onto $V_j, j \in \mathbb{Z}$, is given by

$$(E_j f)(x) = \langle E_j(x,y), f(y)\rangle = \int_{\mathbb{R}^n} E_j(x,y) f(y)\, dy,\ x \in \mathbb{R}^n,$$

where $E_j(x,y) = 2^{nj}E(2^j x, 2^j y), x, y \in \mathbb{R}^n, j \in \mathbb{Z}$ denotes the kernel of the projection operator E_j. Functions $E_j(x,y), x, y \in \mathbb{R}^n$ are the reproducing kernels for the spaces $V_j, j \in \mathbb{Z}$.

The sequence of projections $\{E_j f\}_{j\in\mathbb{Z}}$ is called the multiresolution expansion of f. In a similar way it is defined a multiresolution expansion of elements of Sobolev spaces.

Also, we can define multiresolution expansion in spaces of distributions $\mathcal{D}'_{L^q}(\mathbb{R}^n), 1 \leq q \leq \infty$. Let $h \in \mathcal{D}'_{L^q}(\mathbb{R}^n), 1 < q \leq \infty$, and h has order m_0 $(h \in (H^{m_0,p}(\mathbb{R}^n))')$. Let $E_j(x,y), x, y \in \mathbb{R}^n, j \in \mathbb{Z}$ be the reproducing kernel for the corresponding m_0- regular MRA. The multiresolution expansion $\{E_j h\}_{j\in\mathbb{Z}}$ of h is given through duality between $(H^{m_0,p}(\mathbb{R}^n))'$ and $H^{m_0,p}(\mathbb{R}^n)$ as follows:

$$\langle E_j h, f\rangle = \langle h, E_j f\rangle,\ f \in H^{m_0,p}(\mathbb{R}^n).$$

In particular, if $h \in \mathcal{D}'_{L^1}(\mathbb{R}^n)$ and h has order m_0 $(h \in (\dot{H}^{m_0,\infty}(\mathbb{R}^n))')$ then

$$\langle E_j h, f\rangle = \langle h, E_j f\rangle,\ f \in \dot{H}^{m_0,\infty}(\mathbb{R}^n).$$

In the following theorems are given statements about convergence of multiresolution expansion in Sobolev spaces and spaces $\mathcal{D}'_{L^q}(\mathbb{R}^n), 1 \leq q \leq \infty$.

Theorem 1. *[13] (i) Let $f \in H^{m,p}(\mathbb{R}^n), 0 \leq m \leq r, 1 \leq p < \infty$. Then the sequence $\{E_j f\}_{j\in\mathbb{N}}$ converges to f in $H^{m,p}(\mathbb{R}^n)$.*

(ii) Let $f \in \dot{H}^{m,\infty}(\mathbb{R}^n), 0 \leq m \leq r$. Then the sequence $\{E_j f\}_{j\in\mathbb{N}}$ converges to f in $\dot{H}^{m,\infty}(\mathbb{R}^n)$.

Theorem 2. *[13] Let $h \in \mathcal{D}'_{L^q}(\mathbb{R}^n)$, $1 \leq q \leq \infty$. Then, there exists $r \in \mathbb{N}$, such that every $r-$ regular MRA give a multiresolution expansion of h which converges to h in $\mathcal{D}'_{L^q}(\mathbb{R}^n)$.*

Theorem 3. *[13] (i) Let $f \in H^{m,p}(\mathbb{R}^n), 0 \le m \le r, 1 \le p < \infty$. Then, as $k \to \infty$,*

$$\langle k^n E_j(k\cdot, ky), f(y)\rangle \to f(\cdot) \text{ in } H^{m,p}(\mathbb{R}^n) \text{ uniformly for } j \in \mathbb{N}.$$

(ii) Let $f \in \dot{H}^{m,\infty}(\mathbb{R}^n), 0 \le m \le r$. Then, as $k \to \infty$,

$$\langle k^n E_j(k\cdot, ky), f(y)\rangle \to f(\cdot) \text{ in } \dot{H}^{m,\infty}(\mathbb{R}^n) \text{ uniformly for } j \in \mathbb{N}.$$

3. Quasiasymptotic in $\mathcal{D}'_{L^q}(\mathbb{R}^n)$

Recall, a measurable function $\rho : (a, \infty) \to \mathbb{R}^+$, $a > 0$ is regularly varying at infinity if there exists $\alpha \in \mathbb{R}$, such that $\lim_{k\to\infty} \frac{\rho(\lambda k)}{\rho(k)} = \lambda^\alpha$, for all $\lambda > 0$.

Let $f \in \mathcal{D}'(\mathbb{R}^n)$ and let $\rho : (a, \infty) \to (0, \infty), a > 0$, be a regularly varying function. We say that f has the quasiasymptotics at ∞ (in $\mathcal{D}'(\mathbb{R}^n)$) related to ρ, if there exists $g \in \mathcal{D}'(\mathbb{R}^n), g \neq 0$, such that $\frac{f(k\cdot)}{\rho(k)} \to g$ in $\mathcal{D}'(\mathbb{R}^n)$, as $k \to \infty$. It is well known that the limit g is a homogeneous distribution with degree ν, i.e. $g(ax) = a^\nu g(x), a > 0$. Also, every distribution which has quasiasymptotic at ∞ is a tempered distribution ([10]), so, in dealing with quasiasymptotic at ∞ the space $\mathcal{D}'(\mathbb{R}^n)$ can be replaced by $\mathcal{S}'(\mathbb{R}^n)$.

Since, the weak and strong convergence are not equivalent in $\mathcal{D}'_{L^q}(\mathbb{R}^n), 1 \le q \le \infty$, we will use only the strong convergence.

Definition 1. *A distribution $f \in \mathcal{D}'_{L^q}(\mathbb{R}^n), 1 < q \le \infty$ has quasiasymptotic at ∞ in $\mathcal{D}'_{L^q}(\mathbb{R}^n)$ related to a continuous function $c : (a, \infty) \to (0, \infty), a > 0$, if there exists $m_0 \in \mathbb{N}$ and a distribution $g \in (H^{m_0,p})'(\mathbb{R}^n), g \neq 0$, such that for every bounded set $B \subset H^{m_0,p}(\mathbb{R}^n)$ it holds*

$$\lim_{k\to\infty} \sup_{\phi\in B} \Big(\Big\langle \frac{f(kx)}{c(k)}, \phi(x)\Big\rangle - \langle g(x), \phi(x)\rangle\Big) = 0.$$

If $q = 1$, then instead of $H^{m_0,\infty}(\mathbb{R}^n)$ one has to observe $\dot{H}^{m_0,\infty}(\mathbb{R}^n)$.

Since, it is known that g is a homogeneous distribution, we give the structural theorem for homogeneous distributions in $\mathcal{D}'_{L^q}(\mathbb{R}^n)$.

Theorem 4. *[9] Let f_α, $\alpha \in \mathbb{R}$, $\alpha \neq -n, -n-1, \ldots$ be a homogeneous distribution on $\mathbb{R}^n$ of degree α, $f_\alpha \neq 0$. Then*
(i) $f_\alpha \in \mathcal{D}'_{L^q}(\mathbb{R}^n)$ if and only if $\alpha \in (-\infty, -\frac{n}{q})$, $1 \le q < \infty$.
(ii) $f_\alpha \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ if and only if $\alpha \in (-\infty, 0]$.

Remark. Let f_α be a homogeneous distribution of degree $\alpha = -n - m$, $m \in \mathbb{N}_0$. Then $f_\alpha \in \mathcal{D}'_{L^q}(\mathbb{R}^n)$, $1 < q \le \infty$. In particular, a homogeneous distribution g_α of degree $\alpha = -n - m$, $m \in \mathbb{N}$ belongs to $\mathcal{D}'_{L^1}(\mathbb{R}^n)$.

We say that a distribution $f \in \mathcal{D}'_{L^1}(\mathbb{R}^n)$ is quasiasymptotically bounded in $\mathcal{D}'_{L^1}(\mathbb{R}^n)$ related to a continuous function $c : (a,\infty) \to (0,\infty)$, $a > 0$, if for every bounded set $B \subset \dot{H}^{m_0,\infty}(\mathbb{R}^n)$ there exists a constant C_B, such that

$$\Big|\Big\langle \frac{f(kx)}{c(k)}, \phi(x)\Big\rangle\Big| \le C_B,\ k \in \mathbb{N}, \phi \in B.$$

Theorem 5. *Let* $f \in \mathcal{D}'_{L^q}(\mathbb{R}^n)$, $1 \le q \le \infty$ *has the quasiasymptotics at* ∞ *(in* $\mathcal{D}'_{L^q}(\mathbb{R}^n)$*) related to a regularly varying function* ρ *equal to* $g \in (H^{m_0,p}(\mathbb{R}^n))'$*. Then projections* $E_j f$, $j \in \mathbb{Z}$ *have the quasiasymptotics at* ∞ *(in* $\mathcal{D}'_{L^q}(\mathbb{R}^n)$*) related to* ρ *equal to* $g \in (H^{m_0,p}(\mathbb{R}^n))'$ $(g \in (\dot{H}^{m_0,\infty}(\mathbb{R}^n))'$ *if* $p = \infty)$.

Proof. We will give the proof only for $1 \le p < \infty$. Let B be an arbitrary bounded set in $H^{m_0,p}(\mathbb{R}^n)$. By Theorem 1 to the related multiresolution expansion $\{E_j f\}_{j\in\mathbb{Z}} \in (H^{m_0,p}(\mathbb{R}^n))'$, for all $\phi \in B$ and $k > 0$ we have

$$\Big\langle \frac{(E_j f)(kx)}{\rho(k)}, \phi(x)\Big\rangle = \Big\langle \frac{\langle f(y), E_j(kx,y)\rangle}{\rho(k)}, \phi(x)\Big\rangle$$

$$= \Big\langle \frac{f(y)}{\rho(k)}, \langle E_j(kx,y), \phi(x)\rangle\Big\rangle = \Big\langle \frac{f(ky)}{\rho(k)}, \langle k^n\, E_j(kx,ky), \phi(x)\rangle\Big\rangle$$

$$= \Big\langle \frac{f(ky)}{\rho(k)}, \langle k^n\, E_j(kx,ky), \phi(x)\rangle - \phi(y)\Big\rangle + \Big\langle \frac{f(ky)}{\rho(k)}, \phi(y)\Big\rangle. \tag{1}$$

Definition 1 implies

$$\lim_{k\to\infty} \sup_{\phi\in B} \Big(\Big\langle \frac{f(kx)}{\rho(k)}, \phi(x)\Big\rangle - \langle g(x), \phi(x)\rangle\Big) = 0.$$

It remains to show that the first addend in (1) tends to zero uniformly on B. Recall, a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $(H^{m,p}(\mathbb{R}^n))'$ is strongly bounded if and only if for every bounded set $B \subset H^{m,p}(\mathbb{R}^n)$ there exist $C_B > 0$ such that

$$|\langle f_n,\, \phi\rangle| \le C_B \sup_{0\le|\alpha|\le m, \phi\in B} \|\, \phi^{(\alpha)}(x)\, \|_{L^p(\mathbb{R}^n)},\ \phi \in B, n \in \mathbb{N}. \tag{2}$$

The set $\{\frac{f(k\cdot)}{\rho(k)}, k \in \mathbb{N}\}$ is bounded in $(H^{m_0,p}(\mathbb{R}^n))'$. Also, by Theorem 3, for every $\phi \in B$ the set

$$\{\langle k^n E_j(kx,ky), \phi(x)\rangle - \phi(y), k \in \mathbb{N}\}$$

is uniformly bounded in $H^{m_0,p}(\mathbb{R}^n)$. Hence, by (2) it follows that there exists $C > 0$ not depending on $\phi \in B$, such that for every $\phi \in B$

$$\Big|\Big\langle \frac{f(ky)}{\rho(k)}, \langle k^n\, E_j(kx,ky), \phi(x)\rangle - \phi(y)\Big\rangle\Big|$$

$$\le\ C \sup_{0\le|\alpha|\le m_0,\, k\in\mathbb{N}} \|\, (\langle k^n\, E_j(kx,ky), \phi(x)\rangle - \phi(y))^{(\alpha)}\, \|_{L^p(\mathbb{R}^n)}\,. \tag{3}$$

Theorem 3 implies that for every $0 \leq |\alpha| \leq m_0$

$$\lim_{k\to\infty} \sup_{\phi\in B} \parallel (\langle k^n\, E_j(kx,ky), \phi(x)\rangle - \phi(y))^{(\alpha)} \parallel_{L^p(\mathbb{R}^n)}= 0.$$

Thus, by (3),

$$\lim_{k\to\infty} \sup_{\phi\in B} \Big| \Big\langle \frac{f(ky)}{\rho(k)}, \langle k^n\, E_j(kx,ky), \phi(x)\rangle - \phi(y) \Big\rangle \Big| = 0.$$

□

Theorem 6. *Let $f \in \mathcal{D}'_{L^q}(\mathbb{R}^n), 1 \leq q \leq \infty$ and let the projections $E_j f, j \in \mathbb{N}$ have the quasiasymptotics at ∞ (in $\mathcal{D}'_{L^q}(\mathbb{R}^n)$) related to a regularly varying function ρ equal to distributions $g_j \in (H^{m_0,p}(\mathbb{R}^n))', j \in \mathbb{N}$ and let $g_j \to g$ as $j \to \infty$, in $(H^{m_0,p}(\mathbb{R}^n))'$ in the strong topology. Moreover, suppose that $\left\{\frac{f(ky)}{\rho(k)}, k \in \mathbb{N}\right\}$ is bounded in $(H^{m_0,p})'(\mathbb{R}^n)$. Then f has the quasiasymptotics at ∞ (in $\mathcal{D}'_{L^q}(\mathbb{R})$) related to ρ equal to $g \in (H^{m_0,p}(\mathbb{R}^n))'$.*

Proof. Let B be a bounded set in $H^{m_0,p}(\mathbb{R}^n)$ and let $\varepsilon > 0$. There exists $j_0 \in \mathbb{N}$, such that for every $j > j_0$ and $\phi \in B$

$$|\langle g_j(x) - g(x), \phi(x)\rangle| < \frac{\varepsilon}{4}.$$

Also, for every $j \in \mathbb{N}$ there exists $k_0(j) \in \mathbb{N}$, such that for $k > k_0(j)$ and $\phi \in B$

$$\Big| \Big\langle \frac{(E_j f)(ky)}{\rho(k)} - g_j(y), \phi(y) \Big\rangle \Big| < \frac{\varepsilon}{4}.$$

Let $j > j_0, k > k_0(j)$ and $\phi \in B$. Then

$$\Big| \Big\langle \frac{(E_j f)(ky)}{\rho(k)} - g(y), \phi(y) \Big\rangle \Big| = \Big| \Big\langle \frac{(E_j f)(ky)}{\rho(k)} - g(y) + g_j(y) - g_j(y), \phi(y) \Big\rangle \Big|$$

$$\leq \Big| \Big\langle \frac{(E_j f)(ky)}{\rho(k)} - g_j(y), \phi(y) \Big\rangle \Big| + \Big| \Big\langle g_j(y) - g(y), \phi(y) \Big\rangle \Big| < \frac{\varepsilon}{2}.$$

Also, for every $j \in \mathbb{N}$,

$$\Big| \Big\langle \frac{f(ky)}{\rho(k)} - g(y), \phi(y) \Big\rangle \Big| = \Big| \Big\langle \frac{f(ky)}{\rho(k)} - g(y) + \frac{(E_j f)(ky)}{\rho(k)} - \frac{(E_j f)(ky)}{\rho(k)}, \phi(y) \Big\rangle \Big|$$

$$\leq \Big| \Big\langle \frac{(E_j f)(ky)}{\rho(k)} - g(y), \phi(y) \Big\rangle \Big| + \Big| \Big\langle \frac{f(ky) - (E_j f)(ky)}{\rho(k)}, \phi(y) \Big\rangle \Big|.$$

So, it remains to find $k_1 \in \mathbb{N}$, which does not depend on $j \in \mathbb{N}$, such that for $k > k_1$ and $\phi \in B$

$$\Big|\Big\langle \frac{f(ky)-(E_jf)(ky)}{\rho(k)}, \phi(y)\Big\rangle\Big| < \frac{\varepsilon}{2}. \tag{4}$$

For projections $E_jf, j \in \mathbb{N}$, we have

$$\Big\langle \frac{(E_jf)(kx)}{\rho(k)}, \phi(x)\Big\rangle = \Big\langle \langle \frac{f(y)}{\rho(k)}, E_j(kx,y)\rangle, \phi(x)\Big\rangle = \Big\langle \langle \frac{f(ky)}{\rho(k)}, k^n E_j(kx,ky)\rangle, \phi(x)\Big\rangle$$

$$= \Big\langle \frac{f(ky)}{\rho(k)}, \langle k^n E_j(kx,ky), \phi(x)\rangle\Big\rangle. \tag{5}$$

The sets $\{\frac{f(ky)}{\rho(k)}, k \in \mathbb{N}\}$ and $\{\phi(y) - \langle k^n E_j(kx,ky),\, \phi(x)\rangle, k \in \mathbb{N}\}$ are bounded in $(H^{m_0,p}(\mathbb{R}^n))'$ and $H^{m_0,p}(\mathbb{R}^n)$, for all $j \in \mathbb{N}$, hence by (2) and (5) we have that exists C (not depending on $\phi \in B$), such that for every $k \in \mathbb{N}, j \in \mathbb{N}$ and $\phi \in B$

$$\Big|\Big\langle \frac{f(ky)-(E_jf)(ky)}{\rho(k)}, \phi(y)\Big\rangle\Big| = \Big|\Big\langle \frac{f(ky)}{\rho(k)}, \phi(y) - \langle k^n E_j(kx,ky), \phi(x)\rangle\Big\rangle\Big|$$

$$\leq C \sup_{0\leq|\alpha|\leq m_0, k\in\mathbb{N}} \| \, (\phi(y) - \langle k^n E_j(kx,ky), \phi(x)\rangle)^{(\alpha)} \, \|_{L^p(\mathbb{R}^n)} \, .$$

By Theorem 3 it follows that for every α, $0 \leq |\alpha| \leq m_0$ there exists $k_\alpha \in \mathbb{N}$, such that, for every $k > k_\alpha$ and $j \in \mathbb{N}$

$$\| \, (\phi(y) - \langle k^n \, E_j(kx,ky), \phi(x)\rangle)^{(\alpha)} \, \|_{L^p(\mathbb{R}^n)} < \frac{\varepsilon}{2C}, \ \phi \in B.$$

We put $k_1 = \max\{k_\alpha : 0 \leq |\alpha| \leq m_0\}$. So, we proved (4), and furthermore

$$\Big|\Big\langle \frac{f(ky)}{\rho(k)} - g(y), \phi(y)\Big\rangle\Big| < \varepsilon, \ k > k_1, \phi \in B.$$

□

Theorem 7. *Let* $f \in \mathcal{D}'_{L^1}(\mathbb{R}^n)$ *be quasiasymptotically bounded in* $\mathcal{D}'_{L^1}(\mathbb{R}^n)$ *related to a positive and continuous function c. Then projections* $E_jf, j \in \mathbb{N}$ *are quasiasymptotically bounded in* $\mathcal{D}'_{L^1}(\mathbb{R}^n)$ *related to c.*

Proof. Let B be a bounded set in $\dot{H}^{m_0,\infty}(\mathbb{R}^n)$. By the proof of Theorem 5 we know that for every $j \in \mathbb{N}$ and $\phi \in B$

$$\Big\langle \frac{(E_jf)(kx)}{c(k)}, \phi(x)\Big\rangle = \Big\langle \frac{f(ky)}{c(k)}, \phi(y)\Big\rangle + \Big\langle \frac{f(ky)}{c(k)}, \langle k^n \, E_j(kx,ky), \phi(x)\rangle - \phi(y)\Big\rangle,$$

and there are $C > 0$ and $r \in \mathbb{N}_0$ (not depending on $\phi \in B$), such that for every $j \in \mathbb{N}$ and $\phi \in B$

$$\Big|\Big\langle \frac{f(ky)}{c(k)}, \langle k^n E_j(kx,ky),\phi(x)\rangle - \phi(y)\Big\rangle\Big|$$

$$\leq C \sup_{0\leq|\alpha|\leq r, k\in\mathbb{N}} \| \left(\langle k^n E_j(kx,ky),\phi(x)\rangle - \phi(y)\right)^{(\alpha)} \|_{L^\infty(\mathbb{R}^n)} .$$

The last term tends to zero when $k \to \infty$, hence it is bounded. Since f is quasiasymptotically bounded we have

$$\Big|\Big\langle \frac{(E_j f)(kx)}{c(k)}, \phi(x)\Big\rangle\Big| \leq \tilde{C}_B,\ \phi\in B, k\in\mathbb{N}, j\in\mathbb{Z}.$$

□

Theorem 8. *Let $f\in\mathcal{D}'_{L^1}(\mathbb{R}^n)$ and projections $E_j f, j\in\mathbb{N}$ are quasiasymptotically bounded in $\mathcal{D}'_{L^1}(\mathbb{R}^n)$ related to a positive, continuous function c, i.e. for every bounded set $B\subset \dot{H}^{m_0,\infty}(\mathbb{R}^n)$ and $j\in\mathbb{N}$ there exists $C_{B,j}>0$ such that*

$$\Big|\Big\langle \frac{(E_j f)(kx)}{c(k)}, \phi(x)\Big\rangle\Big| \leq C_{B,j},\ k\in\mathbb{N}, \phi\in B.$$

Moreover, assume that the family $\{\frac{f(ky)}{c(k)}, k\in\mathbb{N}\}$ is bounded in $\mathcal{D}'_{L^1}(\mathbb{R}^n)$. Then f is quasiasymptotically bounded in $\mathcal{D}'_{L^1}(\mathbb{R}^n)$ related to c.

Proof. Let B be a bounded set in $\dot{H}^{m_0,\infty}(\mathbb{R}^n)$. By the proof of Theorem 6, for every $j\in\mathbb{N}$ and every $\phi\in B$ we have

$$\Big\langle \frac{f(ky)}{c(k)}, \phi(y)\Big\rangle = \Big\langle \frac{(E_j f)(ky)}{c(k)}, \phi(y)\Big\rangle + \Big\langle \frac{f(ky)-(E_j f)(ky)}{c(k)}, \phi(y)\Big\rangle.$$

Moreover, there are $C>0$ and $r\in\mathbb{N}_0$ (not depending on $\phi\in B$), such that for $j\in\mathbb{N}$ and $\phi\in B$

$$\Big|\Big\langle \frac{f(ky)-(E_j f)(ky)}{c(k)}, \phi(y)\Big\rangle\Big|$$

$$\leq C \sup_{0\leq|\alpha|\leq r, k\in\mathbb{N}} \| \left(\phi(y) - \langle k^n E_j(kx,ky),\phi(x)\rangle\right)^{(\alpha)} \|_{L^\infty(\mathbb{R})} .$$

The last term is bounded, because it converge to zero. Finally we obtain

$$\Big|\Big\langle \frac{f(ky)}{c(k)}, \phi(y)\Big\rangle\Big| \ \leq\ C_B,\ \phi\in B,\ k\in\mathbb{N}.$$

□

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Received by the editors December 10, 2007