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$\mathbb{R}$-COMPLEX FINSLER SPACES WITH (α, β)-METRIC

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Abstract. In this paper we introduce the class of $\mathbb{R}$-complex Finsler spaces with (α, β)-metrics and study some important exemples: $\mathbb{R}$-complex Randers spaces, $\mathbb{R}$-complex Kropina spaces. The metric tensor field of a $\mathbb{R}$-complex Finsler space with (α, β)-metric is determined (§2). A special approach is dedicated to the $\mathbb{R}$-complex Randers spaces (§3).

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1. $\mathbb{R}$-complex Finsler spaces

In a previous paper [14], we expanded the known definition of a complex Finsler space ([1, 2, 13, 17]), reducing the scalars to $\lambda \in \mathbb{R}$. The outcome was a new class of Finsler space called by us the $\mathbb{R}$-complex Finsler spaces, ([14]). Our interest in this class of Finsler spaces issues from the fact that the Finsler geometry means, first of all, a distance, and this refers to the curves depending on the real parameter.

In the present papers, following the ideas from real Finsler spaces with (α, β)-metrics ([6, 18, 10, 11, 12]), we introduce a similar notion on $\mathbb{R}$-complex Finsler spaces.

In this section we keep the general setting from [13, 14], and subsequently we recall only some necessary notions.

Let M be a complex manifold with $\dim_{\mathbb{C}} M = n$, $\left(z^k\right)$ be local complex coordinates in a chart (U, φ) and $T'M$ its holomorphic tangent bundle. It has a natural structure of complex manifold, $\dim_{\mathbb{C}} T'M = 2n$ and the induced coordinates in a local chart on $u \in T'M$ are denoted by $u = \left(z^k, \eta^k\right)$. The changes of local coordinates in u are given by the rules

$$z'^k = z'^k(z) \; ; \; \eta'^k = \frac{\partial z'^k}{\partial z^j}\eta^j. \tag{1.1}$$

The natural frame $\left\{\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \eta^k}\right\}$ of $T'_u(T'M)$ with the changes of Jacobi matrix of (1.1) changes into $\frac{\partial}{\partial z^k} = \frac{\partial z'^j}{\partial z^k}\frac{\partial}{\partial z'^j} + \frac{\partial^2 z'^j}{\partial z^k \partial z^h}\eta^h \frac{\partial}{\partial \eta'^j}$; $\frac{\partial}{\partial \eta^k} = \frac{\partial z'^j}{\partial z^k}\frac{\partial}{\partial \eta'^j}$.

A complex nonlinear connection, (briefly, c.n.c.), is a supplementary distribution $H(T'M)$ to the vertical distribution $V(T'M)$ in $T'(T'M)$. The vertical distribution is spanned by $\left\{\frac{\partial}{\partial \eta^k}\right\}$ and an adapted frame in $H(T'M)$ is

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$\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}$, where N_k^j are the coefficients of the (c.n.c.) and they have a certain rule of change in (1.1). The dual adapted basis of $\left\{\delta_k, \dot{\partial}_k\right\}$ are $\left\{dz^k, \delta\eta^k = d\eta^k + N_j^k dz^j\right\}$ and $\left\{d\bar{z}^k, \delta\bar{\eta}^k\right\}$ theirs conjugates.

We recall that the homogeneity of the metric function of a complex Finsler space ([1, 2, 13, 17]) is with respect to all complex scalars and the metric tensor of the space is one Hermititian. In [14] we changed a bit the definition of a complex Finsler metric.

A $\mathbb{R}$-*complex Finsler metric* on M is a continuous function $F : T'M \longrightarrow \mathbb{R}_+$ satisfying:

i) $L := F^2$ is smooth on $\widetilde{T'M}$ (except the 0 sections);

ii) $F(z, \eta) \geq 0$, the equality holds if and only if $\eta = 0$;

iii) $F(z, \lambda\eta, \bar{z}, \lambda\bar{\eta}) = |\lambda| F(z, \eta, \bar{z}, \bar{\eta}), \forall \lambda \in \mathbb{R}$;

It follows that L is $(2, 0)$ homogeneous with respect to the real scalars λ, and in [14] we proved that the following identities are fulfilled:

$$\begin{aligned} \frac{\partial L}{\partial \eta^i}\eta^i + \frac{\partial L}{\partial \bar{\eta}^i}\bar{\eta}^i &= 2L; \quad g_{ij}\eta^i + g_{\bar{j}i}\bar{\eta}^i = \frac{\partial L}{\partial \eta^j}; \\ \frac{\partial g_{ik}}{\partial \eta^j}\eta^j + \frac{\partial g_{ik}}{\partial \bar{\eta}^j}\bar{\eta}^j &= 0; \quad \frac{\partial g_{i\bar{k}}}{\partial \eta^j}\eta^j + \frac{\partial g_{i\bar{k}}}{\partial \bar{\eta}^j}\bar{\eta}^j = 0; \\ 2L &= g_{ij}\eta^i\eta^j + g_{\bar{i}\bar{j}}\bar{\eta}^i\bar{\eta}^j + 2g_{i\bar{j}}\eta^i\bar{\eta}^j, \end{aligned} \tag{1.2}$$

where

$$g_{ij} := \frac{\partial^2 L}{\partial \eta^i \partial \eta^j}\ ; \quad g_{i\bar{j}} := \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}; \quad g_{\bar{i}\bar{j}} := \frac{\partial^2 L}{\partial \bar{\eta}^i \partial \bar{\eta}^j} \tag{1.3}$$

are the metric tensors of the space.

2. $\mathbb{R}$-complex Finsler spaces with (α, β)-metric

Following the ideas from the real case, [6, 18, 10, 11, 12], we shall introduce $\mathbb{R}$-complex Finsler spaces with (α, β)-metrics. Let us consider $z \in M$, and $\eta \in T'_z M$, $\eta = \eta^i \frac{\partial}{\partial z^i}$, a section in a holomorphic tangent space.

Definition 2.1. *A* $\mathbb{R}$*-complex Finsler space* (M, F) *is called* (α, β)*-metric if the fundamental function* F *is* $\mathbb{R}$*-homogeneous by means of the functions* $\alpha(z, \eta, \bar{z}, \bar{\eta})$ *and* $\beta(z, \eta, \bar{z}, \bar{\eta})$,

$$F(z, \eta, \bar{z}, \bar{\eta}) = F\left(\alpha(z, \eta, \bar{z}, \bar{\eta}), \beta(z, \eta, \bar{z}, \bar{\eta})\right) \tag{2.1}$$

where

$$\begin{aligned} \alpha^2 &:= \frac{1}{2}\left(a_{ij}\eta^i\eta^j + a_{\bar{i}\bar{j}}\bar{\eta}^i\bar{\eta}^j + 2a_{i\bar{j}}\eta^i\bar{\eta}^j\right) = Re\{a_{ij}\eta^i\eta^j + a_{i\bar{j}}\eta^i\bar{\eta}^j\}; \\ \beta &:= \frac{1}{2}\left(b_i\eta^i + b_{\bar{i}}\bar{\eta}^i\right) = Re\{b_i\eta^i\}, \end{aligned} \tag{2.2}$$

with $a_{ij} = a_{ij}(z)$, $a_{i\bar{j}} = a_{i\bar{j}}(z)$, *both invertible or one of them being zero, and* $b := b_i dz^i$, $b_i = b_i(z)$ *a differential* 1*-form on* M.

If $a_{ij}=0$ and $(a_{i\bar{j}})$ invertible, then the space is said to be of *Hermitian type.* If $a_{i\bar{j}}=0$ and (a_{ij}) invertible, then the space is called *non-Hermitian.* Moreover, $a_{ij}=\partial^2\alpha^2/\,\partial\eta^i\partial\eta^j$ and $a_{i\bar{j}}=\partial^2\alpha^2/\,\partial\eta^i\partial\bar{\eta}^j$.

Indeed, α and β are homogeneous with respect to η and $\bar{\eta}$, i.e. $\alpha(z,\lambda\eta,\bar{z},\lambda\bar{\eta})=\lambda\alpha(z,\eta,\ \bar{z},\bar{\eta})$, $\beta(z,\lambda\eta,\bar{z},\lambda\bar{\eta})=\lambda\beta(z,\eta,\bar{z},\bar{\eta})$ for any $\lambda\in\mathbb{R}_+$, thus $L(z,\lambda\eta,\bar{z},\lambda\bar{\eta})=\lambda^2L(z,\eta,\bar{z},\lambda\bar{\eta})$ for any $\lambda\in\mathbb{R}$, and so the homogeneity property implies

$$
\begin{aligned}
(2.3)\qquad \frac{\partial\alpha}{\partial\eta^i}\eta^i+\frac{\partial\alpha}{\partial\bar{\eta}^j}\bar{\eta}^j &= \alpha\ ;\ \ \frac{\partial\beta}{\partial\eta^i}\eta^i+\frac{\partial\beta}{\partial\bar{\eta}^j}\bar{\eta}^j=\beta;\\
\alpha L_\alpha+\beta L_\beta &= 2L\ ;\ \ \alpha L_{\alpha\alpha}+\beta L_{\alpha\beta}=L_\alpha;\\
\alpha L_{\alpha\beta}+\beta L_{\beta\beta} &= L_\beta\ ;\ \ \alpha^2L_{\alpha\alpha}+2\alpha\beta L_{\alpha\beta}+\beta^2L_{\beta\beta}=2L,
\end{aligned}
$$

where $L_\alpha:=\frac{\partial L}{\partial\alpha}$, $L_\beta:=\frac{\partial L}{\partial|\beta|}$, $L_{\alpha\alpha}:=\frac{\partial^2L}{\partial\alpha^2}$, etc.

In the following, we propose to determine the metric tensors of a $\mathbb{R}$-complex Finsler space with (α,β) metric, i.e. $g_{ij}:=\partial^2L(z,\eta,\bar{z},\lambda\bar{\eta})/\,\partial\eta^i\partial\eta^j$; $g_{i\bar{j}}:=\partial^2L(z,\eta,\bar{z},\lambda\bar{\eta})/\,\partial\eta^i\partial\bar{\eta}^j$, each of these being of interest in the following. We consider

$$
\begin{aligned}
(2.4)\qquad \frac{\partial\alpha}{\partial\eta^i} &= \frac{1}{2\alpha}\left(a_{ij}\eta^j+a_{i\bar{j}}\bar{\eta}^j\right)=\frac{1}{2\alpha}l_i;\ \ \frac{\partial\beta}{\partial\eta^i}=\frac{1}{2}b_i;\\
\frac{\partial\rho_0}{\partial\eta^j} &= \rho_{-2}l_j+\rho_{-1}b_j;\ \ \frac{\partial\rho_1}{\partial\eta^i}=\rho_{-1}l_j+\mu_0b_i;
\end{aligned}
$$

where

$$
\begin{aligned}
(2.5)\qquad l_i\ &:\ =a_{ij}\eta^j+a_{i\bar{j}}\bar{\eta}^j;\ l_i\eta^i+l_{\bar{j}}\bar{\eta}^j=2\alpha^2;\ b^k:=a^{jk}b_j+a^{\bar{j}k}b_{\bar{j}};\\
b_l\ &:\ =b^ka_{kl}+b^{\bar{k}}a_{l\bar{k}};\ \varepsilon:=b_j\eta^j;\ \omega:=b_jb^j;\ \varepsilon+\bar{\varepsilon}=2\beta;\\
\eta_i\ &:\ =\frac{\partial L}{\partial\eta^i}=\rho_0l_i+\rho_1b_i;\ \ \rho_0:=\frac{1}{2}\alpha^{-1}L_\alpha;\ \ \rho_1:=\frac{1}{2}L_\beta;\\
\rho_{-2}\ &:\ =\frac{\alpha L_{\alpha\alpha}-L_\alpha}{4\alpha^3};\ \ \rho_{-1}:=\frac{L_{\alpha\beta}}{4\alpha};\ \ \mu_0:=\frac{L_{\beta\beta}}{4}
\end{aligned}
$$

and their conjugates, with $\left(a^{jk}\right)$ and $\left(a^{\bar{j}k}\right)$ are the inverse of (a_{ij}) and $(a_{i\bar{j}})$, respectively.

The functions $\rho_0,\rho_1,\rho_{-2},\rho_{-1},\mu_0$ are called invariants of the $\mathbb{R}$-complex Finsler space with (α,β)-metric (as in [18]). Subscripts $-2,-1,0,1$ give us the degree of homogeneity of these invariants. Note that $\eta_i=\rho_0l_i+\rho_1b_i$ is uniquely represented in this form. Indeed, if $f(z,\eta)l_i+g(z,\eta)b_i=0$, contracting it by η^i, we obtain $f(z,\eta)\alpha^2+g(z,\eta)\beta=0$. Deriving the last relation with respect to β, it results $g(z,\eta)=0$, and from here $f(z,\eta)\alpha^2=0$. So, $\alpha\neq0$ leads to $f(z,\eta)=0$.

Theorem 2.1. *The tensor fields of $\mathbb{R}$-complex Finsler space with (α,β)-metric are given by*

$$
\begin{aligned}
(2.6)\qquad g_{ij} &= \rho_0a_{ij}+\rho_{-2}l_il_j+\mu_0b_ib_j+\rho_{-1}\left(b_jl_i+b_il_j\right),\\
g_{i\bar{j}} &= \rho_0a_{i\bar{j}}+\rho_{-2}l_il_{\bar{j}}+\mu_0b_ib_{\bar{j}}+\rho_{-1}\left(b_{\bar{j}}l_i+b_il_{\bar{j}}\right)
\end{aligned}
$$

or in the equivalent form

$$(2.7)\quad g_{ij} = \rho_0 a_{ij} + \left(\rho_{-2} - \frac{\rho_0\rho_{-1}}{\rho_1}\right) l_i l_j + \left(\mu_0 - \frac{\rho_{-1}\rho_1}{\rho_0}\right) b_i b_j + \frac{\rho_{-1}}{\rho_0\rho_1}\eta_i\eta_j,$$

$$g_{i\bar{j}} = \rho_0 a_{i\bar{j}} + \left(\rho_{-2} - \frac{\rho_0\rho_{-1}}{\rho_1}\right) l_i l_{\bar{j}} + \left(\mu_0 - \frac{\rho_{-1}\rho_1}{\rho_0}\right) b_i b_{\bar{j}} + \frac{\rho_{-1}}{\rho_0\rho_1}\eta_i\eta_{\bar{j}}.$$

Proof. Taking into account (2.3), we have

$$g_{ij} = \frac{\partial^2 L}{\partial\eta^i\partial\eta^j} = \frac{\partial}{\partial\eta^j}\left(\frac{\partial L}{\partial\eta^i}\right) = \frac{\partial}{\partial\eta^j}(\rho_0 l_i + \rho_1 b_i) = \frac{\partial\rho_0}{\partial\eta^j} l_i + \rho_0\frac{\partial l_i}{\partial\eta^j} + \frac{\partial\rho_1}{\partial\eta^j} b_i$$

and

$$g_{i\bar{j}} = \frac{\partial^2 L}{\partial\eta^i\partial\bar{\eta}^j} = \frac{\partial}{\partial\bar{\eta}^j}\left(\frac{\partial L}{\partial\eta^i}\right) = \frac{\partial}{\partial\bar{\eta}^j}(\rho_0 l_i + \rho_1 b_i) = \frac{\partial\rho_0}{\partial\bar{\eta}^j} l_i + \rho_0\frac{\partial l_i}{\partial\bar{\eta}^j} + \frac{\partial\rho_1}{\partial\bar{\eta}^j} b_i.$$

From here, immediately results (2.6).

To prove (2.7), we compute $\eta_i\eta_j$, and obtain $l_i b_j + l_j b_i = \frac{1}{\rho_0\rho_1}(\eta_i\eta_j - \rho_0^2 l_i l_j - \rho_1^2 b_i b_j)$. □

As in real case, we may consider the following examples of $\mathbb{R}$-complex Finsler spaces with (α,β)-metric:

1. $\mathbb{R}$-complex Randers spaces: $L(\alpha,\beta) = (\alpha+\beta)^2$ with

$$(2.8)\quad L_\alpha = L_\beta = 2(\alpha+\beta) = 2F,\ L_{\alpha\alpha} = L_{\beta\beta} = L_{\alpha\beta} = 2.$$

2. $\mathbb{R}$-complex Kropina spaces: $L(\alpha,\beta) = \left(\frac{\alpha^2}{\beta}\right)^2$, $(\beta \neq 0)$ with

$$(2.9)\quad \begin{aligned} L_\alpha &= \frac{4\alpha^3}{\beta^2};\ L_\beta = -\frac{2\alpha^4}{\beta^3};\ \alpha L_\alpha + \beta L_\beta = 2L \\ L_{\alpha\alpha} &= \frac{12\alpha^2}{\beta^2};\ L_{\alpha\beta} = -\frac{8\alpha^3}{\beta^3};\ L_{\beta\beta} = \frac{6\alpha^4}{\beta^4}. \end{aligned}$$

3. $\mathbb{R}$-complex Randers spaces

As noticed in [14], an $\mathbb{R}$-complex Finsler space produces two tensor fields g_{ij} and $g_{i\bar{j}}$. For a properly Hermitian geometry $g_{i\bar{j}}$ to be invertible is mandatory requirement, but from some physicist point of view, for which Hermitian condition is an impediment; it seems more attractive that g_{ij} be an invertible metric tensor. These problems lead us in [14] to speak about $\mathbb{R}$-complex Hermitian Finsler spaces (i.e. $\det(g_{i\bar{j}}) \neq 0$) and $\mathbb{R}$-complex non-Hermitian Finsler spaces (i.e. $\det(g_{ij}) \neq 0$). The present section applies our results to $\mathbb{R}$-complex Randers spaces, better illustrating the interest for this work.

A first question is when a $\mathbb{R}$-complex Randers metric is positively defined on $\widetilde{T'M}$, i.e. $\alpha > -\beta$, equivalently with

$$(3.1)\quad \left[Re\{a_{ij}\eta^i\eta^j + a_{i\bar{j}}\eta^i\bar{\eta}^j\}\right]^{\frac{1}{2}} > Re\{b_i\eta^i\},\ \text{for all } \eta \neq 0.$$

The answer comes namely. If we suppose that (3.1) is true, then by substituting $\eta^i = -b^i$ into (3.1) we obtain $\frac{\omega+\bar{\omega}}{2} =: ||b|| \in (0,1)$. Conversely, $||b|| \in (0,1)$ imply that $||b||(1-||b||) < 0$. Setting $b^i = -\lambda\eta^i$, $\lambda \in \mathbb{R}$, it results (3.1). So, we have proved the following proposition.

Proposition 3.1. *A $\mathbb{R}$-complex Randers metric is positively defined on $\widetilde{T'M}$ if and only if $||b|| \in (0,1)$.*

The invariants of a $\mathbb{R}$-complex Randers space are:

$$\begin{aligned} \rho_0 \ &: \ = \alpha^{-1}F; \ \ \rho_1 := F; \\ \rho_{-2} \ &: \ = -\frac{\beta}{2\alpha^3}; \ \ \rho_{-1} := \frac{1}{2\alpha}; \ \ \mu_0 := \frac{1}{2} \end{aligned} \tag{3.2}$$

and substituting them into (2.6) and (2.7), we obtain

Proposition 3.2. *The metric tensor fields of a $\mathbb{R}$-complex Randers space are given by*

$$\begin{aligned} g_{ij} &= \frac{\alpha}{F}a_{ij} - \frac{\beta}{2\alpha^3}l_il_j + \frac{1}{2}b_ib_j + \frac{1}{2\alpha}(b_jl_i + b_il_j); \\ g_{i\bar{j}} &= \frac{\alpha}{F}a_{i\bar{j}} - \frac{\beta}{2\alpha^3}l_il_{\bar{j}} + \frac{1}{2}b_ib_{\bar{j}} + \frac{1}{2\alpha}(b_{\bar{j}}l_i + b_il_{\bar{j}}) \end{aligned} \tag{3.3}$$

or equivalently

$$\begin{aligned} g_{ij} &= \frac{F}{\alpha}a_{ij} - \frac{F}{2\alpha^3}l_il_j + \frac{1}{2L}\eta_i\eta_j, \\ g_{i\bar{j}} &= \frac{F}{\alpha}a_{i\bar{j}} - \frac{F}{2\alpha^3}l_il_{\bar{j}} + \frac{1}{2L}\eta_i\eta_{\bar{j}}. \end{aligned} \tag{3.4}$$

The next objective is to obtain the determinant and the inverse of the tensor fields g_{ij} and $g_{i\bar{j}}$. For this we apply Proposition 11.2.1, p. 287 from [6] and Proposition 2.2 from [4] for an arbitrary non-singular Hermitian matrix $(Q_{i\bar{j}})$:

Proposition 3.3. *[4] Suppose:*

- *$(Q_{i\bar{j}})$ is a non-singular $n \times n$ complex matrix with inverse $(Q^{\bar{j}i})$;*
- *C_i and $C_{\bar{i}} := \overline{C_i}$, $i = 1,..,n$, are complex numbers;*
- *$C^i := Q^{\bar{j}i}C_{\bar{j}}$ and its conjugates; $C^2 := C^iC_i = \bar{C}^iC_{\bar{i}}$; $H_{i\bar{j}} := Q_{i\bar{j}} \pm C_iC_{\bar{j}}$*

Then

i) $\det(H_{i\bar{j}}) = (1 \pm C^2)\det(Q_{i\bar{j}})$

ii) Whenever $1 \pm C^2 \neq 0$, the matrix $(H_{i\bar{j}})$ is invertible and in this case its inverse is $H^{\bar{j}i} = Q^{\bar{j}i} \mp \frac{1}{1 \pm C^2}C^iC^{\bar{j}}$.

Proposition 3.4. *For the $\mathbb{R}$-complex non-Hermitian Randers space $F := \alpha + |\beta|$, (with $a_{i\bar{j}} = 0$), we have*

i) $g^{ij} = \frac{\alpha}{F}a^{ij} + \frac{\alpha(2\beta+\alpha\omega)}{FM}\eta^i\eta^j + \frac{\alpha^2\bar{\gamma}}{FM}b^ib^j - \frac{\alpha^2(\varepsilon+2\alpha)}{FM}(\eta^ib^j + b^i\eta^j)$;

ii) $\det(g_{ij}) = \left(\frac{F}{\alpha}\right)^n \frac{M}{4\alpha^2 F} \det(a_{ij})$,
where

$$\begin{aligned} \gamma &: = a_{jk}\eta^j\eta^k = l_k\eta^k;\ l_i = a_{ij}\eta^j;\ \gamma+\bar{\gamma} = 2\alpha^2; \\ b^k &= a^{jk}b_j;\ b_l = b^k a_{kl}; \\ M &: = \bar{\gamma}(\alpha\omega + 2\beta) + \alpha\left(2\alpha+\varepsilon\right)^2. \end{aligned} \tag{3.5}$$

Proof. Applying Proposition 11.2.1, p. 287 from [6] in two steps, we are yields g^{ij} and $\det(g_{ij})$. For the beginning we write g_{ij} in the form $g_{i\bar{j}} = \frac{F}{\alpha}\left(a_{ij} - \frac{1}{2\alpha^2}l_il_j + \frac{\alpha}{2F^3}\eta_i\eta_j\right)$.

1) In the first applications we set $Q_{ij} := a_{ij}$ and $C_i := \frac{1}{\alpha\sqrt{2}}l_i$. We obtain $Q^{ji} = a^{ji}$, $C^2 = \frac{\gamma}{2\alpha^2}$, $1 - C^2 = \frac{\bar{\gamma}}{2\alpha^2}$ and $C^i = \frac{1}{\alpha\sqrt{2}}\eta^i$. So, the matrix $H_{ij} = a_{ij} - \frac{1}{2\alpha^2}l_il_j$ is invertible with $H^{ji} = a^{ji} - \frac{1}{\bar{\gamma}}\eta^i\eta^j$ and $\det\left(a_{ij} - \frac{1}{2\alpha^2}l_il_j\right) = \frac{\bar{\gamma}}{2\alpha^2}\det(a_{ij})$.

2) Now, we consider $Q_{ij} := a_{ij} - \frac{1}{2\alpha^2}l_il_j$ and $C_i := \frac{\sqrt{\alpha}}{F\sqrt{2F}}\eta_i$. Hence $Q^{\bar{j}i} = a^{\bar{j}i} + \frac{1}{\bar{\gamma}}\eta^i\bar{\eta}^j$, $C^2 = \frac{\alpha}{2F^3}\left[a^{ji} + \frac{1}{\bar{\gamma}}\eta^i\eta^j\right]\eta_i\eta_j = \frac{M-2\bar{\gamma}F}{2\bar{\gamma}F}$, $1 + C^2 = \frac{M}{2\bar{\gamma}F} \neq 0$ and $C^i = \frac{\sqrt{\alpha}}{F\sqrt{2F}}\left[a^{ji} + \frac{1}{\bar{\gamma}}\eta^i\eta^j\right]\eta_j = \sqrt{\frac{\alpha}{2F}}\left(\frac{2\alpha+\varepsilon}{\bar{\gamma}}\eta^i + b^i\right)$. It results that the inverse of $H_{ij} = a_{ij} - \frac{1}{2\alpha^2}l_il_j + \frac{\alpha}{2F^3}\eta_i\eta_j$ exists. It is
$H^{ji} = a^{ji} + \frac{1}{\bar{\gamma}}\eta^i\eta^j - \frac{\alpha\bar{\gamma}}{M}\left(\frac{2\alpha+\varepsilon}{\bar{\gamma}}\eta^i + b^i\right)\left(\frac{2\alpha+\varepsilon}{\bar{\gamma}}\eta^j + b^j\right)$ and
$\det\left(a_{ij} - \frac{1}{2\alpha^2}l_il_j + \frac{\alpha}{2F^3}\eta_i\eta_j\right) = \frac{M}{2\bar{\gamma}F}\det\left(a_{ij} - \frac{1}{2\alpha^2}l_il_j\right) = \frac{M}{4\alpha^2F}\det(a_{ij})$.

Taking into account that $g_{ij} = \frac{F}{\alpha}H_{ij}$, with H_{ij} from 2), we obtain $g^{ji} = \frac{\alpha}{F}H^{ji}$ and $\det(g_{ij}) = \left(\frac{F}{\alpha}\right)^n\det(H_{ij})$. From here, immediately results *i)* and *ii)*. □

Example 1. We set α as

$$\alpha^2(z,\eta) := \frac{(1+\varepsilon|z|^2)\sum_{k=1}^n Re(\eta^k)^2 - \varepsilon Re < z,\eta >^2}{(1+\varepsilon|z|^2)^2}, \tag{3.6}$$

where $|z|^2 := \sum_{k=1}^n z^k\bar{z}^k$, $< z,\eta > := \sum_{k=1}^n z^k\bar{\eta}^k$, defined over the disk

$$\Delta_r^n = \left\{ z \in \mathbf{C}^n,\ |z| < r,\ \ r := \sqrt{\frac{1}{|\varepsilon|}} \right\}$$

if $\varepsilon < 0$, on $\mathbf{C}^n$ if $\varepsilon = 0$ and on the complex projective space $P^n(\mathbf{C})$ if $\varepsilon > 0$. By computation we obtain $a_{ij} = \frac{1}{1+\varepsilon|z|^2}\left(\delta_{ij} - \varepsilon\frac{\bar{z}^i\bar{z}^j}{1+\varepsilon|z|^2}\right)$ and $a_{i\bar{j}} = 0$ and so, $\alpha^2(z,\eta) = \frac{1}{2}\left(a_{ij}\eta^i\eta^j + a_{\bar{i}\bar{j}}\bar{\eta}^i\bar{\eta}^j\right)$. Now, taking $\beta(z,\eta) := Re\frac{<z,\eta>}{1+\varepsilon|z|^2}$, where $b_i := \frac{\bar{z}^i}{1+\varepsilon|z|^2}$, we obtain some examples of $\mathbb{R}$-complex non-Hermitian Randers metrics:

$$F_\varepsilon := \frac{\sqrt{(1+\varepsilon|z|^2)\sum_{k=1}^n Re(\eta^k)^2 - \varepsilon Re < z,\eta >^2}}{1+\varepsilon|z|^2} + Re\frac{< z,\eta >}{1+\varepsilon|z|^2}. \tag{3.7}$$

Proposition 3.5. *For the $\mathbb{R}$-complex Hermitian Randers space $F := \alpha + |\beta|$, (with $a_{ij} = 0$), we have*

i) $g^{j\bar{\imath}} = \frac{\alpha}{F} a^{j\bar{j}} + \frac{2\beta+\alpha\omega}{FH}\eta^i\eta^j - \frac{\alpha^3}{FH} b^i \bar{b}^j - \frac{\alpha}{FH}\left[(\bar{\varepsilon}+2\alpha)\,\eta^i\bar{b}^j + (\varepsilon+2\alpha)\, b^i\bar{\eta}^j\right]$;

ii) $\det\left(g_{i\bar{j}}\right) = \left(\frac{F}{\alpha}\right)^n \frac{H}{4\alpha F}\det\left(a_{i\bar{j}}\right)$,

where

$$(3.8)\qquad \begin{aligned} \alpha^2 &= a_{j\bar{k}}\eta^j\bar{\eta}^k = l_{\bar{k}}\bar{\eta}^k;\ l_i = a_{i\bar{j}}\bar{\eta}^j;\\ b^k &= a^{\bar{j}k} b_{\bar{j}};\ b_{\bar{l}} = b^k a_{k\bar{l}};\\ H &:= \alpha(4F + 2\beta + \alpha\omega) + \varepsilon\bar{\varepsilon}. \end{aligned}$$

Proof. We apply Proposition 3.3 two times. Again, we write
$g_{i\bar{j}} = \frac{F}{\alpha}\left(a_{i\bar{j}} - \frac{1}{2\alpha^2} l_i l_{\bar{j}} + \frac{\alpha}{2F^3}\eta_i\eta_{\bar{j}}\right)$.

1) Setting $Q_{i\bar{j}} := a_{i\bar{j}}$ and $C_i := \frac{1}{\alpha\sqrt{2}} l_i$ we obtain $Q^{\bar{j}i} = a^{\bar{j}i}$, $C^2 = \frac{1}{2}$, $1 - C^2 = \frac{1}{2}$ and $C^i = \frac{1}{\alpha\sqrt{2}}\eta^i$. So, the matrix $H_{i\bar{j}} = a_{i\bar{j}} - \frac{1}{2\alpha^2} l_i l_{\bar{j}}$ is invertible with $H^{\bar{j}i} = a^{\bar{j}i} + \frac{1}{\alpha^2}\eta^i\bar{\eta}^j$ and $\det\left(a_{i\bar{j}} - \frac{1}{2\alpha^2} l_i l_{\bar{j}}\right) = \frac{1}{2}\det\left(a_{i\bar{j}}\right)$.

2) Considering $Q_{i\bar{j}} := a_{i\bar{j}} - \frac{1}{2\alpha^2} l_i l_{\bar{j}}$ and $C_i := \frac{\sqrt{\alpha}}{F\sqrt{2F}}\eta_i$, it gives $Q^{\bar{j}i} = a^{\bar{j}i} + \frac{1}{\alpha^2}\eta^i\bar{\eta}^j$, $C^2 = \frac{\alpha}{2F^3}\left[a^{\bar{j}i} + \frac{1}{\alpha^2}\eta^i\bar{\eta}^j\right]\eta_i\bar{\eta}_j = \frac{H-2\alpha F}{2\alpha F}$, $1 + C^2 = \frac{H}{2\alpha F} \neq 0$ and $C^i = \frac{\sqrt{\alpha}}{F\sqrt{2F}}\left[a^{\bar{j}i} + \frac{1}{\alpha^2}\eta^i\bar{\eta}^j\right]\bar{\eta}_j = \sqrt{\frac{\alpha}{2F}}\left(\frac{2\alpha+\bar{\varepsilon}}{\alpha^2}\eta^i + b^i\right)$. So, $H_{i\bar{j}} = a_{i\bar{j}} - \frac{1}{2\alpha^2} l_i l_{\bar{j}} + \frac{\alpha}{2F^3}\eta_i\eta_{\bar{j}}$ is invertible and its inverse is $H^{\bar{j}i} = a^{\bar{j}i} + \frac{1}{\alpha^2}\eta^i\bar{\eta}^j - \frac{\alpha^2}{H}\left(\frac{2\alpha+\bar{\varepsilon}}{\alpha^2}\eta^i + b^i\right)\left(\frac{2\alpha+\varepsilon}{\alpha^2}\bar{\eta}^j + \bar{b}^j\right)$ and $\det\left(a_{i\bar{j}} - \frac{1}{2\alpha^2} l_i l_{\bar{j}} + \frac{\alpha}{2F^3}\eta_i\eta_{\bar{j}}\right) = \frac{H}{2\alpha F}\det\left(a_{i\bar{j}} - \frac{1}{2\alpha^2} l_i l_{\bar{j}}\right) = \frac{H}{4\alpha F}\det\left(a_{i\bar{j}}\right)$. But $g_{i\bar{j}} = \frac{F}{\alpha} H_{i\bar{j}}$, with $H_{i\bar{j}}$ from 2), so that we obtain $g^{\bar{j}i} = \frac{\alpha}{F} H^{\bar{j}i}$ and $\det\left(g_{i\bar{j}}\right) = \left(\frac{F}{\alpha}\right)^n \det\left(H_{i\bar{j}}\right)$. It results in *i)* and *ii)*. □

Example 2. We consider α given by

$$(3.9)\qquad \alpha^2(z,\eta) := \frac{|\eta|^2 + \varepsilon\left(|z|^2|\eta|^2 - |< z,\eta >|^2\right)}{(1+\varepsilon|z|^2)^2},$$

defined over the disk

$$\Delta_r^n = \left\{ z \in \mathbf{C}^n,\ |z| < r,\ \ r := \sqrt{\frac{1}{|\varepsilon|}} \right\}$$

if $\varepsilon < 0$, on $\mathbf{C}^n$ if $\varepsilon = 0$ and on the complex projective space $P^n(\mathbf{C})$ if $\varepsilon > 0$, where $|< z,\eta >|^2 :=< z,\eta > \overline{< z,\eta >}$. By computation we obtain $a_{ij} = 0$ and $a_{i\bar{j}} = \frac{1}{1+\varepsilon|z|^2}\left(\delta_{i\bar{j}} - \varepsilon\frac{\bar{z}^i z^j}{1+\varepsilon|z|^2}\right)$ and so, $\alpha^2(z,\eta) = a_{i\bar{j}}(z)\eta^i\bar{\eta}^j$. Thus it determines a purely Hermitian metrics which have special properties. They are Kähler with constant holomorphic curvature $\mathcal{K}_\alpha = 4\varepsilon$. Particularly, for $\varepsilon = -1$ we obtain the *Bergman metric* on the unit disk $\Delta^n := \Delta_1^n$; for $\varepsilon = 0$ the *Euclidean metric* on $\mathbf{C}^n$, and for $\varepsilon = 1$ the *Fubini-Study metric* on $P^n(\mathbf{C})$. Setting $\beta(z,\eta)$ as in

Example 1, we obtain some examples of $\mathbb{R}$-complex Hermitian Randers metrics:

$$F_\varepsilon := \frac{\sqrt{|\eta|^2 + \varepsilon\left(|z|^2|\eta|^2 - |< z, \eta >|^2\right)}}{1 + \varepsilon|z|^2} + Re\frac{< z, \eta >}{1 + \varepsilon|z|^2}. \tag{3.10}$$

Similar considerations can be done for the $\mathbb{R}$-complex Kropina spaces. We are aware of the fact that the subject offers much other working paths with various applications.

References

[1] Abate, M., Patrizio, G., Finsler Metrics - A Global Approach. Lecture Notes in Math. 1591, Springer-Verlag, 1994.

[2] Aikou, T., Projective Flatness of Complex Finsler Metrics. Publ. Math. Debrecen 63 (2003), 343–362.

[3] Aldea, N., Complex Finsler spaces of constant holomorphic curvature. Diff. Geom. and its Appl., Proc. Conf. Prague 2004, Charles Univ. Prague (Czech Republic) 2005, 179-190.

[4] Aldea, N., Munteanu, G., On complex Finsler spaces with Randers metric. (submitted).

[5] Aldea, N., Munteanu, G., (α, β)-complex Finsler metrics. Proc. MENP-4, Bucharest 2006, Geom. Balk. Press 2007, 1-6.

[6] Bao, D., Chern, S. S., Shen, Z., An Introduction to Riemannian Finsler Geom. Graduate Texts in Math. 200, Springer-Verlag, 2000.

[7] Bao, D., Robles, C., On Randers spaces of constant flag curvature. Rep. Math. Phys. 51 (2003), 9-42.

[8] Fukui, Complex Finsler manifold. J. Math. Kyoto Univ. 29 No. 4 (1989), 609-624.

[9] Kobayashi, S., Horst, C., Wu H. H., Complex Differential Geometry, Birkhäuser Verlag, 1983.

[10] Matsumoto, M., Theory of Finsler spaces with (α, β)-metric. Rep. on Math. Phys. 31 (1991), 43-83.

[11] Miron, R., General Randers spaces. Kluwer Acad. Publ., FTPH No. 76 (1996), 126-140.

[12] Miron, R., Hassan, B. T., Variational problem in Finsler Spaces with (α, β)-metric. Algebras Groups and Geometries 20 (2003), 285-300.

[13] Munteanu, G., Complex Spaces in Finsler, Lagrange and Hamilton Geometries. Kluwer Acad. Publ. FTPH 141, 2004.

[14] Munteanu, G., Purcaru, M., About $\mathbb{R}$-complex Finsler Spaces. (submitted).

[15] Royden, H.L., Complex Finsler Metrics. Contemporary Math. 49 (1984), 119-124.

[16] Spiro, A., The Structure Equations of a Complex Finsler Manifold. Asian J. Math. 5 (2001), 291-326.

[17] Wong, P.-Mann, A survey of complex Finsler geometry. Advanced Studied in Pure Math. Vol.48 Math. Soc. of Japan (2007), 375-433.

[18] Sabău, V. S., Shimada, H., Remarkable classes of (α, β)-metric spaces. Rep. on Math. Phys. 47 No. 1 (2001), 31-48.

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