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DIVERGENT LEGENDRE-SOBOLEV POLYNOMIAL SERIES

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Abstract. Let be introduced the Sobolev-type inner product

$$(f, g) = \frac{1}{2}\int_{-1}^{1} f(x)g(x)dx + M[f'(1)g'(1) + f'(-1)g'(-1)],$$

where $M \geq 0$. In this paper we will prove that for $1 \leq p \leq \frac{4}{3}$ there are functions $f \in L^p([-1, 1])$ whose Fourier expansion in terms of the orthonormal polynomials with respect to the above Sobolev inner product are divergent almost everywhere on $[-1, 1]$. We also show that, for some values of δ, there are functions whose Legendre-Sobolev expansions have almost everywhere divergent Cesàro means of order δ.

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1. Introduction

For f and g in $L^2([-1, 1])$, such that there exists the first derivative in 1 and -1, we can introduce the Sobolev-type inner product

$$(f, g) = \frac{1}{2}\int_{-1}^{1} f(x)g(x)dx + M[f'(1)g'(1) + f'(-1)g'(-1)], \tag{1.1}$$

where $M > 0$. We denote by $\hat{B}_n$ the orthonormal polynomials with respect to the inner product (1.1) (see [5]). We call them Legendre-Sobolev polynomials. For $M = 0$ we have classical Legendre polynomials.

For every function f such that $(f, \hat{B}_n)$ exists for $n = 0, 1, ...$ we introduce the Nth partial sum of the associated Fourier-Sobolev series

$$S_N(f) = \sum_{n=0}^{N} c_n(f)\hat{B}_n(x), \tag{1.2}$$

where

$$c_n(f) = (f, \hat{B}_n).$$

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The study of the convergence of standard Fourier-Legendre expansion has been discussed by many authors. We refer to ([13], [11], [10]) and the references therein. It was proved that $p \in (4/3, 4)$ if and only if

$$||S_N f||_{L^p([-1,1])} \leq C||f||_{L^p([-1,1])} \qquad \forall N \geq 0,\ \forall f \in L^p([-1,1]).$$

In 1972 Pollard [14] raised the following question: Is there an $f \in L^{4/3}([-1,1])$ whose Fourier-Legendre expansion diverges almost everywhere? This problem was solved by Meaney [8]. Furthermore, he proved that this is a special case of the divergence result for series of Jacobi polynomials.

In this paper we will prove that for $1 \leq p \leq \frac{4}{3}$ there are functions $f \in L^p([-1,1])$ whose expansions in terms of the polynomials associated to the Sobolev inner product

$$(f,g) = \frac{1}{2}\int_{-1}^{1} f(x)g(x)dx + M[f'(1)g'(1) + f'(-1)g'(-1)],$$

where $M > 0$, are divergent almost everywhere on $[-1,1]$.

Notice that the behaviour of the Fourier expansion in terms of the polynomials with respect to the Sobolev inner product

$$(f,g) = \int_{-1}^{1} f(x)g(x)dx + \sum_{k=1}^{K}\sum_{i=0}^{N_k} N_{k,i} f^{(i)}(a_k)g^{(i)}(a_k), \qquad N_{k,i} > 0$$

has been discussed in ([6],[15]) and for $i = 0$ in [4]. Also we refer to [12], where some interesting results about Fourier expansions with respect to Sobolev orthogonal polynomials are obtained.

2. Legendre-Sobolev polynomials

Some basic properties of $\hat{B}_n$ [5] (see also [1], [2]), we will needed in the sequel are given in below:

$$|\hat{B}_n(1)| \sim n^{1/2} \tag{2.1}$$

$$|(\hat{B}_n)'(1)| \sim n^{-7/2} \tag{2.2}$$

$$\hat{B}_n(-x) = (-1)^n \hat{B}_n(x) \tag{2.3}$$

$$|\hat{B}_n(cos\theta)| = \begin{cases} O(\theta^{-1/2}) & \text{if } c/n \leq \theta \leq \pi/2, \\ O(n^{1/2}) & \text{if } 0 \leq \theta \leq c/n \end{cases} \tag{2.4}$$

where $n \geq 1$ and c is a positive constant.

Asymptotic behaviour of the ultraspherical polynomials $\{P_n^{(\alpha)}\}_{n=0}^{\infty}$ is given in [16, (8.21.10)]

$$P_n^{(\alpha)}(cos\theta) = \frac{1}{\sqrt{\pi n}} \left(sin\frac{\theta}{2} cos\frac{\theta}{2} \right)^{-\alpha-1/2} cos(k_\alpha \theta + \gamma_\alpha) + O(n^{-3/2}),$$

where $k_\alpha = n + \alpha + 1/2$, $\gamma_\alpha = -(\alpha + 1/2)\pi/2$ and $\theta \in [\epsilon, \pi - \epsilon]$.

Combining this with [5, Lemma 1] we obtain the strong inner asymptotics of $\hat{B}_n$ for $\theta \in [\epsilon, \pi - \epsilon]$ and $\epsilon > 0$

$$\hat{B}_n(cos\theta) = u_n \left(sin\frac{\theta}{2} cos\frac{\theta}{2} \right)^{-1/2} cos(k_0 \theta + \gamma_0) + O(n^{-1}), \tag{2.5}$$

where $k_0 = n + 1/2$, $\gamma_0 = -\pi/4$ and $\lim_{n\to\infty} u_n = \frac{1}{2\sqrt{\pi}}$.

For every function f such that $(f, \hat{B}_n)$ exists for $n = 0, 1, ...$ the Fourier-Sobolev coefficients of the series (1.2) can be written as

$$c_n(f) = (f, \hat{B}_n) = c'_n(f) + M[f'(1) \left(\hat{B}_n\right)'(1) + f'(-1) \left(\hat{B}_n\right)'(-1)], \tag{2.6}$$

where

$$c'_n(f) = \frac{1}{2} \int_{-1}^{1} f(x) \hat{B}_n(x) dx.$$

Now we will estimate the Lebesgue norm

$$||\hat{B}_n||_q^q = \int_{-1}^{1} |\hat{B}_n(x)|^q dx$$

where $1 \leq q < \infty$. For $M = 0$ the calculation of this norm appears in [16, p. 391. Exercise 91] (see also [7]).

Theorem 2.1. *Let $M \geq 0$. Then*

$$\int_0^1 |\hat{B}_n(x)|^q dx \sim \begin{cases} c & if\ q < 4, \\ log\ n & if\ q = 4, \\ n^{q/2-2} & if\ q > 4. \end{cases}$$

Proof. From (2.4), for $q \neq 4$, we have

$$\begin{aligned} \int_0^1 |\hat{B}_n(x)|^q dx &\sim \int_0^{\pi/2} \theta\ |\hat{B}_n(cos\theta)|^q d\theta \\ &= O(1) \int_0^{n^{-1}} \theta\ n^{q/2} d\theta + O(1) \int_{n^{-1}}^{\pi/2} \theta\ \theta^{-q/2} d\theta \\ &= O(n^{q/2-2}) + O(1), \end{aligned}$$

and for $q = 4$ we have

$$\int_0^1 |\hat{B}_n(x)|^q dx = O(log\ n).$$

Now we will prove the lower bounds for integrals involving Legendre-Sobolev polynomials. Taking into account the continuity of the polynomials $\hat{B}_n(\cos\theta)$, there exists $\delta > 0$ such that $2|\hat{B}_n(cos\theta)| \geq |\hat{B}_n(1)|$ for all θ with $0 \leq \theta < \delta$. Hence, from (2.1) and [16, Theorem 7.32.2], for $0 \leq \theta < \delta$ we have

$$2|\hat{B}_n(cos\theta)| \geq cn^{1/2} \geq c_1|p_n(cos\theta)|,$$

where p_n are Legendre orthonormal polynomials (see [16, Chapter IV]).

On the other hand, from (2.5) and [16, Theorem 8.21.8], we have

$$\hat{B}_n(cos\theta) = c_2 p_n(cos\theta) + O(n^{-1}),$$

where $\theta \in [\delta, \pi/2]$. Therefore, according to the Lebesgue norms of Legendre polynomials (see [16, p. 391. Exercise 91], [7]), we have

$$\int_0^{\pi/2} \theta\ |\hat{B}_n(cos\theta)|^q d\theta \geq c_3 \int_0^{\pi/2} \theta\ |p_n(cos\theta)|^q d\theta \sim \begin{cases} c_4 & \text{if } q < 4, \\ log\ n & \text{if } q = 4, \\ n^{q/2-2} & \text{if } q > 4. \end{cases}$$

The proof of Theorem 2.1 is complete. □

3. Divergent Legendre-Sobolev polynomial series

From Egorov's theorem follows that if the series (1.2) converges on a set of positive measure in $[-1, 1]$ then there is a subset of positive measure E on which

$$|c_n(f)\hat{B}_n(x)| \to 0, \quad as\ n \to \infty,$$

uniformly for $x \in E$. Hence, from (2.5), we have

$$|c_n(f)\left(cos(k_0\theta + \gamma_0) + O(n^{-1})\right)| \to 0, \quad as\ n \to \infty,$$

uniformly for $cos\theta \in E$. Using the Cantor-Lebesgue Theorem, as described in [9, Subsection 1.5](see also [17, p.316]), we obtain

$$|c_n(f)| \to 0, \quad as\ n \to \infty. \tag{3.1}$$

From Theorem 2.1, for $1 \leq q < \infty$, we have

$$||\hat{B}_n||_q > \left(\int_0^1 |\hat{B}_n(x)|^q dx\right)^{1/q} \sim \begin{cases} (log\ n)^{1/4} & \text{if } p = \frac{4}{3} \\ n^{1/2-2/q} & \text{if } p < \frac{4}{3} \end{cases} \tag{3.2}$$

where p is a conjugate of q i.e. $1/p + 1/q = 1$.

For $q = \infty$ we have

$$||\hat{B}_n||_\infty = cn^{1/2}. \tag{3.3}$$

Now we are in position to prove our first main result.

Theorem 3.1. *There is an $f \in L^p([-1,1])$, $1 \le p \le 4/3$, such that there exists the first derivative in* 1, *supported in* $[0,1]$, *whose Legendre-Sobolev series diverges almost everywhere on* $[-1,1]$.

Proof. The uniform boundedness principles, (3.2) and (3.3) imply that there are the functions $f \in L^p([-1,1])$, supported on $[0,1]$, for which the linear functional $c'_n(f)$ satisfies

$$\frac{c'_n(f)}{(log\ n)^{1/8}} \to \infty, \qquad as\ n \to \infty.$$

Hence, from (2.2), (2.3) and (2.6), we obtain

$$\frac{c_n(f)}{(log\ n)^{1/8}} \to \infty, \qquad as\ n \to \infty.$$

Since this result is contrary to (3.1) it follows that for this f the Fourier-Sobolev series diverges almost everywhere on $[-1,1]$. □

4. Divergent Cesàro means of Legendre-Sobolev expansions

The Cesàro means of order δ of the expansion (1.2) is defined by

$$\sigma_N^\delta f(x) = \sum_{n=0}^{N} \frac{A_{N-n}^\delta}{A_N^\delta} c_n(f) \hat{B}_n(x),$$

where $A_k^\delta = \binom{k+\delta}{k}$. In [17, Theorem 3.1.22] (see also [9, Lemma 1.1]) is proved

Lemma 4.1. *Suppose that* $\lim_{N\to\infty} \sigma_N^\delta f(x)$ *exists for some* $x \in [-1,1]$ *and* $\delta > -1$. *Then*

$$|c_N(f)\hat{B}_N(x)| \le O(N^\delta), \qquad \forall N \ge 1.$$

From Egorov's theorem and Lemma 4.1 it follows that if the series (1.2) is Cesàro summable of order δ on a set of positive measure in $[-1,1]$ then there is a subset E of positive measure where

$$|n^{-\delta} c_n(f) \hat{B}_n(x)| \le A$$

uniformly for $x \in E$. Hence, from (2.5), we have

$$|n^{-\delta} c_n(f) \left(cos(k\theta + \gamma) + O(n^{-1})\right)| \le A$$

uniformly for $cos\theta \in E$. Using again the Cantor-Lebesgue Theorem we obtain

$$|\frac{c_n(f)}{n^\delta}| \le A, \qquad \forall n \ge 1. \tag{4.1}$$

Theorem 4.1. *Let p and δ be real numbers such that*

$$1 \le p < \frac{4}{3};$$

$$0 \le \delta < \frac{2}{p} - \frac{3}{2}.$$

There is an $f \in L^p([-1,1])$ such that there exists the first derivative in 1, supported in $[0,1]$, whose Cesàro means $\sigma_N^\delta f(x)$ is divergent almost everywhere on $[-1,1]$.

Proof. Suppose that

$$0 \le \delta < \frac{2}{p} - \frac{3}{2}.$$

For q conjugate of p

$$\delta < \frac{1}{2} - \frac{2}{q}.$$

From the argument given in [9, Subsection 1.4], (3.2) and (3.3), for the linear functional $c_n'(f) = \frac{1}{2}\int_{-1}^{1} f(x)\hat{B}_n(x)dx$, it follows that there is an $f \in L^p([-1,1])$, supported on $[0,1]$, such that

$$\frac{c_n'(f)}{n^\delta} \to \infty, \qquad \textit{as } n \to \infty.$$

So, from (2.2), (2.3) and (2.6), we obtain

$$\frac{c_n(f)}{n^\delta} \to \infty, \qquad \textit{as } n \to \infty.$$

Combining the above results with (4.1) it follows that for this f, the $\sigma_N^\delta f(x)$ diverges almost everywhere. □

Remark 4.1. *Using formulae in [3], which relate the Riesz and Cesàro means of order $\delta \ge 0$, we conclude that Theorem 4.1 also holds for the Riesz means.*

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