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ATOMS AND A SAKS TYPE DECOMPOSITION IN EFFECT ALGEBRAS

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Abstract. The present paper deals with the study of the notion of an atom of a function m defined on an effect algebra L with values in $[0, \infty]$; a few examples of atoms for null-additive as well as for non-null-additive functions are also given. We have proved a Saks type decomposition theorem for an element a with $m(a) > 0$ (for a suitable m), which does not contain any atom of m, in a σ-complete effect algebra L. A characterization for a measure μ to be non-atomic (μ is defined on a σ-complete effect algebra with values in $[0, \infty]$) is established and a result for a non-atomic measure μ is proved, which has resemblance with the Intermediate Value Theorem for continuous functions.

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1. Introduction

In 1992, Kôpka [15] defined the D-posets of fuzzy sets, which are closed under the formations of differences of fuzzy sets, while studying the axiomatical systems of fuzzy sets. A generalization of such structures of fuzzy sets to an abstract partially ordered set, where the basic operation is the difference, yields a very general and, at the same time, a very simple structure called a D-poset [14]. The structure of a D-poset supports a non-commutative measure theory and allows the solution of some problems of non-commutative probability theory, including some problems of theory of quantum measurement. Almost at the same time, Bennett and Foulis [4] introduced so-called effect algebras with partial addition as a primary operation, which are essentially equivalent to D-posets, with the aim of modelling unsharp measurements in a quantum mechanical system. They are generalization of orthomodular lattices, MV-algebras, and also of many structures used in Quantum Physics [7], in Mathematical Economics [10, 11, 12, 25] and in Fuzzy Theory [8]. The categorical equivalence of D-posets and effect algebras is discussed in [9]. For a list of nice examples of effect algebras we refer to [5] and for some of its properties we refer also to [6] and [4].

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In the present paper, we have introduced the notion of an atom of a function m defined on an effect algebra L with values in $[0, \infty]$, which we have illustrated by means of several examples and investigated some properties of atoms of a null-additive function on L. Suzuki [21], for the first time introduced and investigated the concept of an atom of a fuzzy measure. Pap [17], further introduced and studied atoms of null-additive set functions. In 2000, Wu and Wu [24] pointed out the incorrectness of Lemma 1 and Theorem 1 of [17], by a counterexample and gave the correct proof, followed by some properties of atoms of a non-monotone function. Some important contributions to the study of the theory of atoms are done by several authors [16, 18, 22, 23]. Prerequisites and some basic results on effect algebras are collected in Section 2, which have been extensively used in the subsequent sections. In Section 3, we have studied some properties of atoms of a null-additive non-monotone function m and obtained that an element a in a σ-complete effect algebra L, with $m(a) > 0$, for a suitable m defined on L, can be written as a finite sum of elements $a_i \in L$, covered by a, with $m(a_i) \leqslant \varepsilon$, for every i and for any $\varepsilon > 0$. In Section 4, we have also considered a function μ with values in $[0, \infty]$ defined on a σ-complete effect algebra L. The notions of lower-semicontinuous (*lsc*), upper-semicontinuous (*usc*) and m-continuous functions on L are introduced, and it is proved that each atom of a null-additive function m is also an atom of an m-continuous null-additive function μ. In the rest of this section, we have concentrated on the study of a non-atomic measure, and the a characterization for a measure μ to be non-atomic is established.

Finally, we have proved a theorem for a non-atomic measure μ which has resemblance with the Intermediate Value Theorem for continuous functions. This result has an interesting history including contributions from Sierpinski, Fréchet and Hahn and the constructions in this proof are due to Newton [13].

2. Preliminaries and Basic Results

An *effect algebra* $(L; \oplus, 0, 1)$ is a structure consisting of a set L, two special elements 0 and 1, and a partially defined binary operation $\oplus$ on $L \times L$ satisfying the following conditions for every $a, b, c \in L$:

(1) If $a \oplus b$ is defined then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.

(2) If $b \oplus c$ and $a \oplus (b \oplus c)$ are defined, then $a \oplus b$ and $(a \oplus b) \oplus c$ are defined and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

(3) For every $a \in L$, there exists a unique $a^{\perp} \in L$ such that $a \oplus a^{\perp}$ is defined and $a \oplus a^{\perp} = 1$.

(4) If $a \oplus 1$ is defined, then $a = 0$.

Throughout the paper, $L = (L; \oplus, 0, 1)$ denotes, in general, an effect algebra. In every effect algebra L, a dual operation $\ominus$ to $\oplus$ can be defined as follows: $a \ominus c$ exists and equals b if and only if $b \oplus c$ exists and equals a. We say that two elements $a, b \in L$ are *orthogonal* and we write $a \perp b$, if $a \oplus b$ exists. For every $a \in L$, we have $a^{\perp} = 1 \ominus a$. We can define a binary relation on L by $a \leqslant b$ if and only if there exists $c \in L$ such that $c \oplus a = b$; $\leqslant$ is a partial ordering in L, with 0 as the smallest element. Also $a \perp b$ if and only if $a \leqslant b^{\perp}$ and $(a \oplus b)^{\perp} = a^{\perp} \ominus b = b^{\perp} \ominus a$. Moreover, $a^{\perp}$ satisfies naturally:

(i) $a \leqslant b \Rightarrow b^{\perp} \leqslant a^{\perp}$,

(ii) $(a^{\perp})^{\perp} = a$.

For $a_1, \dots, a_n \in L$, we inductively define $a_1 \oplus \dots \oplus a_n = (a_1 \oplus \dots \oplus a_{n-1}) \oplus a_n$ provided that the right-hand side exists. The definition is independent on permutations of the elements. We say that a finite subset $\{a_1, \dots, a_n\}$ of L is *orthogonal* if $a_1 \oplus \dots \oplus a_n$ exists. More generally, we say that $\{a_n\}$ is an *orthogonal family* if every finite subfamily is orthogonal. For a sequence $\{a_n\}$ in L, we say that it is orthogonal if, for every n, $\bigoplus_{i \leqslant n} a_i$ exists. If, moreover $\sup_n \bigoplus_{i \leqslant n} a_i$ exists, the *sum* $\bigoplus_{n \in \mathbb{N}} a_n$ of an orthogonal sequence $\{a_n\}$ in L is defined as $\sup_n \bigoplus_{i \leqslant n} a_i$; $\mathbb{N}$ denotes the set of natural numbers.

An effect algebra L is called a *σ-complete effect algebra* if every orthogonal sequence in L has its sum.

Let us recall the following results which we shall use in the sequel.

2.1 [1]. Let $a, b, c \in L$, such that $b \leqslant a$ and $c \leqslant (a \ominus b)$. Then $b \perp c$ and $b \oplus c \leqslant a$.

2.2 [2]. (i) Let $\{a_1, ..., a_n\} \subseteq L$ be orthogonal. If $1 \leq k \leq n$, then $\{a_1, ..., a_k\}$ and $\{a_{k+1}, ..., a_n\}$ are orthogonal and $\bigoplus_{i=1}^{n} a_i = \bigoplus_{i=1}^{k} a_i \oplus \bigoplus_{i=k+1}^{n} a_i$.

(ii) Let $\{a_n\}$ be an orthogonal sequence in L and $A, B \subseteq \mathbb{N}$ disjoint such that $a = \bigoplus_{n \in A} a_n$ and $b = \bigoplus_{n \in B} a_n$ exist. Then $a \perp b$ and $a \oplus b = \bigoplus_{n \in A \cup B} a_n$.

2.3 [1]. Let L be a σ-complete effect algebra. If $\{a_n\}$ is an increasing (respectively, decreasing) sequence, then $\sup_n a_n$ (respectively, $\inf_n a_n$) exists.

2.4. Assume that a, b, c are elements of an effect algebra L.

(i) If $a \leqslant b$, then $b = a \oplus (b \ominus a)$ [2].

(ii) If $a \leqslant b$, then $b \ominus a \leqslant b$ and $b \ominus (b \ominus a) = a$ [9].

(iii) If $a \leqslant b \leqslant c$, then $a \oplus (c \ominus b) = c \ominus (b \ominus a)$ and $(c \ominus b) \oplus (b \ominus a) = (c \ominus a)$ [1].

(iv) If $a \leqslant b \leqslant c$, then $(c \ominus b) \leqslant (c \ominus a)$ and $(c \ominus a) \ominus (c \ominus b) = (b \ominus a)$ [9].

(v) If $a \leqslant b \leqslant c$, then $(b \ominus a) \leqslant (c \ominus a)$ and $(c \ominus a) \ominus (b \ominus a) = (c \ominus b)$ [2].

(vi) If $a \leqslant b \leqslant c$, then $a \perp (c \ominus b)$ and $a \oplus (c \ominus b) = c \ominus (b \ominus a)$ [2].

(vii) If $a \leqslant b^{\perp} \leqslant c^{\perp}$, then $a \oplus (b \ominus c) = (a \oplus b) \ominus c$ [1].

(viii) If $a \perp b$, then $a \leqslant a \oplus b$ and $(a \oplus b) \ominus a = b$ [2].

(ix) If $a \perp b$ and $(a \oplus b) \leqslant c$, then $c \ominus (a \oplus b) = (c \ominus a) \ominus b = (c \ominus b) \ominus a$ [9].

3. Atoms in an Effect Algebra

Throughout this section m is a $[0, \infty]$-valued function defined on an effect algebra L. We have the following definitions.

Definition 3.1. *m is called monotone, if we have $m(a) \leqslant m(b)$, whenever $a, b \in L$ and $a \leqslant b$; m is called exhaustive, if we have $\lim_{n \to \infty} m(a_n) = 0$ for any orthogonal sequence $\{a_n\}$ of elements from L; m is called order continuous* (*at* 0) *if $\lim_{n \to \infty} m(a_n) = 0$ whenever $a_n \downarrow 0$.*

Example 3.1. Let $E = \{0, 1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{p}, ...\}$. Let us define: for each $\frac{1}{p}$, $0 \oplus \frac{1}{p} = \frac{1}{p}$, $\frac{1}{p} \oplus \frac{1}{p} = 1$, $0 \oplus 1 = 1$ and if $p \neq q$, $\frac{1}{p} \oplus \frac{1}{q}$ cannot be defined. Evidently, E is an effect algebra. Let us consider the function m defined on E by $m(x) = 1$ if $x = 1$ and otherwise $m(x) = 0$. Then one can observe that m is a monotone, exhaustive and order continuous function.

Definition 3.2. *m is called null-additive if we have $m(a \oplus b) = m(a)$, whenever $a, b \in L$, $a \perp b$ and $m(b) = 0$.*

Example 3.2. Let $E = \{0, a, b, c, 1\}$. Let us define : $a \oplus b = b \oplus a = c$, $b \oplus c = c \oplus b = a \oplus a = 1$ and let $x \oplus 0 = 0 \oplus x$ for all $x \in E$. Then one can easily see that E is an effect algebra. Consider two functions m_1 and m_2 defined on E as follows:
(*i*) $m_1(x) = 1$ if $x \in \{c, 1\}$ and otherwise $m_1(x) = 0$,
(*ii*) $m_2(x) = 1$ if $x \in \{0, a, c, 1\}$ and otherwise $m_2(x) = 0$.
Then m_1 is not a null-additive function while m_2 is a null-additive function.

Example 3.3. Let $E = \{0, a, b, c, d, e, 1\}$. Let us define: $a \oplus b = b \oplus a = c$, $b \oplus c = c \oplus b = a \oplus d = d \oplus a = e \oplus e = 1$ and let $x \oplus 0 = 0 \oplus x$ for all $x \in E$. Then one can easily see that E is an effect algebra. Consider two functions m_1 and m_2 defined on E as follows:
(i) $m_1(x) = 1$ if $x \in \{0, a, c\}$ and otherwise $m_1(x) = 0$,
(ii) $m_2(x) = 1$ if $x \in \{a, c, 1\}$ and otherwise $m_2(x) = 0$.
Then both m_1 and m_2 are not null-additive functions.

Definition 3.3. *An element $a \in L$ with $m(a) > 0$ is called an atom of m if for $a, b \in L$ with $b \leqslant a$,*
(*i*) *$m(b) = 0$ or*
(*ii*) *$m(a) = m(b)$ and $m(a \ominus b) = 0$.*

Remark 3.1. *For a null-additive function m, we may suppose in* (*ii*) *only $m(a \ominus b) = 0$. Since $b \oplus a^{\perp} \geqslant b$, we get $(b \oplus a^{\perp})^{\perp} \leqslant b^{\perp}$, i.e. $a \ominus b \leqslant b^{\perp}$. Hence $b \oplus (a \ominus b)$ exists and therefore from null-additivity of m we get $m(b \oplus (a \ominus b)) = m(b)$, i.e. $m(a) = m(b)$.*

Example 3.4. (*i*) In Example 3.2, the elements c and 1 of E are atoms of m_1, while the elements 0, a, c and 1 of E are not atoms of m_2.
(*ii*) In Example 3.3, the elements 0, a and c of E are not atoms of m_1, while elements a, c and 1 of E are atoms of m_2.

Definition 3.4. *An element $a \in L$ with $m(a) > 0$ is said to have an atom of m if there exists $b \in L$ with $b \leqslant a$, such that b is an atom of m.*

One can observe that if an element $a \in L$ with $m(a) > 0$ does not contain any atom of m, then a itself is not an atom of m and any element $b \leq a$, $b \in L$ with $m(b) > 0$ also does not contain any atom of m.

Proposition 3.1. *Let m be a null-additive function and let a_1 and a_2 be orthogonal atoms of m with $m(a_1 \wedge a_2) = 0$. Then $a_1 \ominus (a_1 \wedge a_2)$ and $a_2 \ominus (a_1 \wedge a_2)$ are orthogonal atoms of m and $m(a_1 \ominus (a_1 \wedge a_2)) = m(a_1)$, $m(a_2 \ominus (a_1 \wedge a_2)) = m(a_2)$.*

Proof. Let $b \leqslant a_1 \ominus (a_1 \wedge a_2)$, $b \in L$ and $m(b) > 0$. Since $b \leqslant a_1 \ominus (a_1 \wedge a_2) \leqslant a_1$, using 2.4(ii) and (iii) we get $((a_1 \ominus (a_1 \wedge a_2)) \ominus b) \oplus (a_1 \wedge a_2)$ exists and $((a_1 \ominus (a_1 \wedge a_2)) \ominus b) \oplus (a_1 \wedge a_2) = (a_1 \ominus b)$. Also, since m is null-additive and $m(a_1 \wedge a_2) = 0$, we get

$$\begin{aligned} m((a_1 \ominus (a_1 \wedge a_2)) \ominus b) &= m(((a_1 \ominus (a_1 \wedge a_2)) \ominus b) \oplus (a_1 \wedge a_2)) \\ &= m(a_1 \ominus b) \\ &= 0. \end{aligned}$$

Hence $a_1 \ominus (a_1 \wedge a_2)$ is an atom of m. Similarly, we can prove that $a_2 \ominus (a_1 \wedge a_2)$ is also an atom of m.

Further, since a_1 and a_2 are orthogonal, we get

$$a_1 \ominus (a_1 \wedge a_2) \leqslant a_2^{\perp} \leqslant (a_2 \ominus (a_1 \wedge a_2))^{\perp},$$

i.e. $(a_1 \ominus (a_1 \wedge a_2)) \oplus (a_2 \ominus (a_1 \wedge a_2))$ exists. Hence $a_1 \ominus (a_1 \wedge a_2)$ and $a_2 \ominus (a_1 \wedge a_2)$ are orthogonal atoms of m.

Finally, since m is null-additive and a_1 and a_2 are atoms of m, we get $m(a_1 \ominus (a_1 \wedge a_2)) = m(a_1)$ and $m(a_2 \ominus (a_1 \wedge a_2)) = m(a_2)$. □

Proposition 3.2. *Let m be a null-additive function and let a_1 and a_2 be atoms of m with $m(a_1 \wedge a_2) > 0$. Then $a_1 \wedge a_2$ is an atom of m and $m(a_1 \ominus (a_1 \wedge a_2)) = m(a_2 \ominus (a_1 \wedge a_2)) = 0$ and $m(a_1 \wedge a_2) = m(a_1) = m(a_2)$.*

Proof. Since $a_1 \wedge a_2 \leqslant a_1$, $a_1 \wedge a_2 \leqslant a_2$ and a_1 and a_2 are atoms of m, we get by the definition of an atom of m, $m(a_1 \ominus (a_1 \wedge a_2)) = 0$ and $m(a_2 \ominus (a_1 \wedge a_2)) = 0$.

Again, let $b \leqslant a_1 \wedge a_2$, $b \in L$ and $m(b) > 0$. Since $(a_1 \wedge a_2) \ominus b \leqslant a_1 \wedge a_2 \leqslant a_1$. Using 2.4(ii) and (vi), we get $((a_1 \wedge a_2) \ominus b) \oplus (a_1 \ominus (a_1 \wedge a_2))$ exists and $((a_1 \wedge a_2) \ominus b) \oplus (a_1 \ominus (a_1 \wedge a_2)) = (a_1 \ominus b)$. Also since m is null-additive and a_1 is an atom of m, it follows that

$$\begin{aligned} m((a_1 \wedge a_2) \ominus b) &= m(((a_1 \wedge a_2) \ominus b) \oplus (a_1 \ominus (a_1 \wedge a_2))) \\ &= m(a_1 \ominus b) \\ &= 0. \end{aligned}$$

Hence $a_1 \wedge a_2$ is an atom of m.

Finally, since $a_1 \wedge a_2 \leqslant a_1$, $a_1 \wedge a_2 \leqslant a_2$ and m is null-additive, we get

$$m(a_1 \wedge a_2) = m(a_1) \text{ and } m(a_1 \wedge a_2) = m(a_2)$$

and therefore $m(a_1 \wedge a_2) = m(a_1) = m(a_2)$. □

Remark 3.2. *For two atoms a_1 and a_2 of m, only one of the relations*

$$m(a_1 \wedge a_2) = 0, \ m(a_1 \wedge a_2) = m(a_1) = m(a_2)$$

is possible.

Proposition 3.3. *Let m be a finite, order continuous and exhaustive function defined on a σ-complete effect algebra L. Let $\mathcal{A}$ be an orthogonal family of atoms of m. Then*

(i) $\mathcal{A}$ is at most countable;

(ii) $m(a_n) \geqslant m(a_{n+1})$, $n \in \mathbb{N}$;

(iii) $\lim_{n\to\infty} m(a_n) = 0$;

(iv) for any $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $m\left(\bigoplus_{n=k_0+1}^{\infty} a_n\right) \leqslant \varepsilon$.

Proof. (i) For $n \in \mathbb{N}$, let $F_n = \left\{a \in \mathcal{A} : m(a) \geqslant \frac{1}{n}\right\}$. Then F_n is at most finite. Otherwise, by exhaustivity of m, we get $\lim_{i\to\infty} m(a_i) = 0$ for an orthogonal sequence $\{a_i\}$ in F_n and so, for $\varepsilon = \frac{1}{n}$, $n \in \mathbb{N}$, we have $\nu \in \mathbb{N}$ with $m(a_i) < \frac{1}{n}$, $i \geqslant \nu$. Clearly, $\mathcal{A} = \bigcup_{n\in\mathbb{N}} F_n$. This yields that $\mathcal{A}$ is at most countable.

(ii) Clearly, we can suppose that a_n satisfies $m(a_n) \geqslant m(a_{n+1})$, $n \in \mathbb{N}$.

(iii) Follows by the exhaustivity of m.

(iv) Since $\bigoplus_{i=k+1}^{\infty} a_i \downarrow 0$ and m is order continuous, we have the assertion.
□

Theorem 3.1. *Let m be a bounded, null-additive and exhaustive function, defined on a σ-complete effect algebra L. If each $a \in L$ with $m(a) > 0$ contains an atom of m, then there exist at most a countable number of pairwise orthogonal atoms a_i from L with $a_i \leqslant a$ $(i \in I)$, such that $m(a \ominus \bigoplus_{i\in I} a_i) = 0$, where each element $a \in L$ contains at most a countable number of distinct atoms of m.*

Proof. Let $a \in L$ with $m(a) > 0$. If a itself is an atom of m, then the theorem is proved. Otherwise, we take $a_1 \in L$, $a_1 \leqslant a$, which is an atom of m such that $m(a_1) > \frac{1}{2}\sup\{m(c) : c \leqslant a, \ c \text{ is an atom of } m\}$. Then either $m(a \ominus a_1) = 0$ and then the theorem is proved, or $m(a \ominus a_1) > 0$ and in this case we have $a_2 \in L$, $a_2 \leqslant a \ominus a_1$, which is an atom of m and such that

$$m(a_2) > \frac{1}{2}\sup\{m(c) : c \leqslant a \ominus a_1, \ c \text{ is an atom of } m\}.$$

Now let us consider $(a \ominus a_1) \ominus a_2 \leqslant a$. Then either $m((a \ominus a_1) \ominus a_2) = 0$, i.e. with the aid of 2.1 and 2.4(ix), $m((a \ominus a_1) \ominus a_2) = m(a \ominus (a_1 \oplus a_2)) = 0$, and then the theorem is proved or $m(a \ominus (a_1 \oplus a_2)) > 0$ and in this case we have $a_3 \in L$, $a_3 \leqslant a \ominus (a_1 \oplus a_2)$, which is an atom of m and such that

$$m(a_3) > \frac{1}{2}\sup\{m(c) : c \leqslant a \ominus (a_1 \oplus a_2), \ c \text{ is an atom of } m\}.$$

By the successive use of 2.1 and 2.4(ix), inductively at j^{th} stage, either we have $m\left(a \ominus \bigoplus_{i=1}^{j-1} a_i\right) = 0$ and then the theorem is proved or $m\left(a \ominus \bigoplus_{i=1}^{j-1} a_i\right) > 0$

and in this case, continuing the above procedure, we obtain a sequence $\{a_i\}$ of pairwise orthogonal atoms of m from L such that $a_j \in L$, $a_j \leqslant a \ominus \bigoplus_{i=1}^{j-1} a_i$, for $j \in \mathbb{N}$ and

$$(1) \qquad m(a_j) > \frac{1}{2} \sup \left\{ m(c) : c \leqslant \left(a \ominus \bigoplus_{i=1}^{j-1} a_i \right), \ c \text{ is an atom of } m \right\}.$$

We claim that $m\left(a \ominus \bigoplus_{i=1}^{\infty} a_i\right) = 0$. If we suppose on contrary that $m\left(a \ominus \bigoplus_{i=1}^{\infty} a_i\right) > 0$, then there exists $a^1 \in L$, $a^1 \leqslant \left(a \ominus \bigoplus_{i=1}^{\infty} a_i\right)$, and a^1 is an atom of m. Now using 2.2 and 2.4(iv), we get $a^1 \leqslant \left(a \ominus \bigoplus_{i=1}^{\infty} a_i\right) \leqslant \left(a \ominus \bigoplus_{i=1}^{j-1} a_i\right)$ and therefore (1) yields

$$2m(a_j) > m(a^1) \quad (j \in \mathbb{N}).$$

Finally, the exhaustivity of m yields that $m(a^1) = 0$, which contradicts the fact that a^1 is an atom of m. □

Lemma 3.1. *Let m be a null-additive and exhaustive function. If a given element $a \in L$ with $m(a) > 0$ does not contain any atom of m, then for every $\varepsilon > 0$, there exists $b \in L$, $b \leqslant a$, such that $0 < m(b) \leqslant \varepsilon$.*

Proof. Since a is not an atom of m, there exists $a_1 \in L$, $a_1 \leqslant a$, such that $m(a_1) > 0$ and $m(a \ominus a_1) > 0$. If $m(a_1) \leqslant \varepsilon$ or $m(a \ominus a_1) \leqslant \varepsilon$, then the conclusion holds. Otherwise, we have $m(a_1) > \varepsilon$ and $m(a \ominus a_1) > \varepsilon$. Since $(a \ominus a_1)$ is an atom of m, there exists $a_2 \in L$, $a_2 \leqslant (a \ominus a_1)$ such that $m(a_2) > 0$ and $m((a \ominus a_1) \ominus a_2) > 0$, i.e. with the aid of 2.1 and 2.4 (ix), $m((a \ominus a_1) \ominus a_2) = m(a \ominus (a_1 \oplus a_2)) > 0$. If $m(a_2) \leqslant \varepsilon$ or $m(a \ominus (a_1 \oplus a_2)) \leqslant \varepsilon$ holds, then the conclusion holds. Otherwise we have $m(a_2) > \varepsilon$ and $m(a \ominus (a_1 \oplus a_2)) > \varepsilon$. By the successive applications of 2.1 and 2.4(ix) and proceeding inductively in this manner, we obtain that if the conclusion does not hold, a sequence $\{a_n\}$ of pairwise orthogonal elements, $a_n \leqslant a$, from L such that $m(a_n) > \varepsilon$ $(n \in \mathbb{N})$, a contradiction by the exhaustivity of m. □

Theorem 3.2. (Saks Type Decomposition Theorem)
Let m be a null-additive, exhaustive and order continuous function defined on a σ-complete effect algebra L. Let a given element $a \in L$ with $m(a) > 0$ contain no atom of m. Then for every $\varepsilon > 0$, there exists at most a finite number of pairwise orthogonal elements $a_0, a_1, a_2, \ldots, a_k$ with $a_i \leqslant a$, $a_i \in L$, $(i = 0, 1, 2, \ldots, k)$ such that $a = \bigoplus_{i=0}^{k} a_i$ and $m(a_i) \leqslant \varepsilon$ $(i = 0, 1, 2, \ldots, k)$.

Proof. If the element $a \in L$ with $m(a) > 0$ does not contain any atom of m, then by Lemma 3.1 there exists $a_1 \in L$, $a_1 \leqslant a$, such that $0 < m(a_1) \leqslant \varepsilon$. If $m(a \ominus a_1) = 0$, the conclusion holds. Otherwise $m(a \ominus a_1) > 0$, and so by

Lemma 3.1, there exists $a_2 \in L$, $a_2 \leqslant a \ominus a_1$, such that $0 < m(a_2) \leqslant \varepsilon$ and we may suppose that

$$m(a_2) \geqslant \frac{1}{2} \sup\{m(c) : c \in L, c \leqslant a \ominus a_1, 0 < m(c) \leqslant \varepsilon\}.$$

If $m((a \ominus a_1) \ominus a_2) = 0$, i.e. with the aid of 2.1 and 2.4(ix), $m((a \ominus a_1) \ominus a_2) = m(a \ominus (a_1 \oplus a_2)) = 0$, then the conclusion holds. Otherwise, $m(a \ominus (a_1 \oplus a_2)) > 0$, then by Lemma 3.1 there exists $a_3 \in L$, $a_3 \leqslant a \ominus (a_1 \oplus a_2)$, such that $0 < m(a_3) \leqslant \varepsilon$ and we may suppose that

$$m(a_3) \geqslant \frac{1}{2} \sup\{m(c) : c \in L, c \leqslant a \ominus (a_1 \oplus a_2), 0 < m(c) \leqslant \varepsilon\}.$$

Continuing on in this manner and by the successive use of 2.1 and 2.4(ix), we obtain that there exist $a_j \in L$, $a_j \leqslant a \ominus \bigoplus_{i=1}^{j-1} a_i$, such that $0 < m(a_j) \leqslant \varepsilon$ and we may suppose that

$$m(a_j) \geqslant \frac{1}{2} \sup\left\{m(c) : c \in L, c \leqslant a \ominus \bigoplus_{i=1}^{j-1} a_i, 0 < m(c) \leqslant \varepsilon\right\}. \tag{2}$$

If $m\left(a \ominus \bigoplus_{i=1}^{j} a_i\right) = 0$, then the conclusion holds. Otherwise, there exists $a_j \in L$, $a_j \leqslant a$,$(j \in \mathbb{N})$, which are pairwise orthogonal and satisfy (2).

We claim that $m\left(a \ominus \bigoplus_{i=1}^{\infty} a_i\right) = 0$. If we suppose on the contrary that $m\left(a \ominus \bigoplus_{i=1}^{\infty} a_i\right) > 0$, there exists $a^1 \in L$, $a^1 \leqslant \left(a \ominus \bigoplus_{i=1}^{\infty} a_i\right)$, such that $0 < m(a^1) \leqslant \varepsilon$. Now using 2.2 and 2.4(iv), we get $a^1 \leqslant \left(a \ominus \bigoplus_{i=1}^{\infty} a_i\right) \leqslant \left(a \ominus \bigoplus_{i=1}^{j-1} a_i\right)$ and (2) yields that $2m(a_j) \geqslant m(a^1)$, $\quad j \in \mathbb{N}$. Consequently, $\lim_{j\to\infty} m(a_j) \geqslant m(a^1) > 0$, a contradiction by the exhaustivity of m.

Since $\bigoplus_{i=j+1}^{\infty} a_i \downarrow 0$, by order continuity of m, we have

$$\lim_{j\to\infty} m(\bigoplus_{i=j+1}^{\infty} a_i) = 0$$

and hence there is $k \in \mathbb{N}$ such that $m\left(\bigoplus_{i=k+1}^{\infty} a_i\right) \leqslant \varepsilon$. Taking $a_0 = a \ominus \bigoplus_{i=1}^{k} a_i$, by 2.2 and 2.4(i), (ix), we deduce that

$$a_0 = \left(\bigoplus_{i=k+1}^{\infty} a_i\right) \bigoplus \left(a_0 \ominus \bigoplus_{i=k+1}^{\infty} a_i\right) \text{ and } m\left(a_0 \ominus \bigoplus_{i=k+1}^{\infty} a_i\right) = 0$$

and we get by null-additivity of m,

$$\begin{aligned} m(a_0) &= m\left(\left(a_0 \ominus \bigoplus_{i=k+1}^{\infty} a_i\right) \bigoplus \left(\bigoplus_{i=k+1}^{\infty} a_i\right)\right) \\ &= m\left(\bigoplus_{i=k+1}^{\infty} a_i\right) \leqslant \varepsilon. \end{aligned}$$

Hence $a = \bigoplus_{i=0}^{k} a_i$ and $m(a_i) \leqslant \varepsilon \quad (i = 0, 1, 2, \ldots, k)$. □

4. Non-atomic Measure on a σ-Complete Effect Algebra

Let μ be a $[0,\infty]$-valued function defined on an effect algebra L. Then μ is called a measure [3], if we have $\mu(a \oplus b) = \mu(a) + \mu(b)$, whenever $a, b \in L$ and $a \perp b$. It is easy to see that μ is a measure iff $a \leqslant b$ implies $\mu(b \ominus a) = \mu(b) - \mu(a)$. Using 2.4(iv), one can observe that if $a \in L$ is an atom of a measure μ then any element $b \leq a, b \in L$ with $\mu(b) > 0$ is also an atom of the measure μ.

Definition 4.1. *μ is called lower-semicontinuous (lsc)(respectively, upper-semi -continuous (usc)), if $a_n \in L$, $a_n \leqslant a_{n+1}$, $n = 1, 2, \ldots \Rightarrow \mu\left(\bigvee_{n=1}^{\infty} a_n\right) = \lim_{n\to\infty} \mu(a_n)$, provided $\bigvee_{n=1}^{\infty} a_n$ exists (respectively, if $a_n \in L$, $a_n \geqslant a_{n+1}$, $n = 1, 2, \ldots$ and $\mu(a_1) < \infty \Rightarrow \mu\left(\bigwedge_{n=1}^{\infty} a_n\right) = \lim_{n\to\infty} \mu(a_n)$, provided $\bigwedge_{n=1}^{\infty} a_n$ exists); μ is called semicontinuous, if it is both lower-semicontinuous and upper-semicontinuous.*

Definition 4.2. *Let m be a $[0,\infty)$-valued function defined on L. We say that μ is absolutely continuous with respect to m (or in brief m-continuous), if for any $a \in L$, $m(a) = 0$, implies $\mu(a) = 0$.*

Proposition 4.1. *If μ is an m-continuous and null-additive function on L, then each atom $a \in L$ of a null-additive function m is also an atom of μ, provided $\mu(a) > 0$.*

Proof. Let a be an atom of m, and let $b \leqslant a$. Then by the definition of an atom, when $m(b) > 0$, and by the null-additivity of m, we get $m(a \ominus b) = 0$. Also, since μ is m-continuous, we have $\mu(a \ominus b) = 0$. Therefore a is also an atom of μ, when $\mu(a) > 0$.

Definition 4.3. *In the case there is no atom of μ in L, μ is called non-atomic on L.*

Example 4.1. In Example 3.2, m_2 is a non-atomic, while in Example 3.3, m_1 is a non-atomic function.

From now onwards, $L = (L, \mu)$ denotes a finite m-continuous and semicontinuous measure space; L is a σ-complete effect algebra and m is a *usc* measure defined on L.

Lemma 4.1. *If μ is non-atomic on L, then for every $a \in L$ with $\mu(a) > 0$, there exists $b \in L$ such that $b \leqslant a$ and $0 < \mu(b) < \mu(a)$.*

Proof. Let us suppose on the contrary that

$$b \leqslant a,\ \mu(b) > 0 \Rightarrow \mu(b) = \mu(a). \tag{3}$$

Take an element $a_1 \in L$, $a_1 \leqslant a$, with $\mu(a_1) > 0$ satisfying $\mu(a_1) = \mu(a)$ and $\mu(a \ominus a_1) = \mu(a)$. Such an a_1 surely exists. Indeed, if for every $a_1 \leqslant a$, with $\mu(a_1) > 0$, $\mu(a \ominus a_1) = 0$, then a must be an atom of μ in view of (3). Thus, it follows that $\mu(a \ominus a_1) > 0$, for some $a_1 \leqslant a$ with $\mu(a_1) > 0$, which actually implies $\mu(a \ominus a_1) = \mu(a)$, together with (3). Since m is a measure defined on L, we may assume that $m(a_1) \leqslant \frac{1}{2}m(a)$. Applying the same argument to a_1 instead of a, we obtain $a_2 \in L$, $a_2 \leqslant a_1$, such that $\mu(a_2) = \mu(a_1)$ and $m(a_2) \leqslant \frac{1}{2}m(a_1)$. Continue this construction inductively to obtain a decreasing sequence $\{a_n\}$ from L such that

$$\mu(a_n) = \mu(a) \text{ and } m(a_n) < \left(\frac{1}{2}\right)^n m(a), \ n = 1, 2, \ldots.$$

Now, the upper-semicontinuity of m and μ and 2.3 gives, $m\left(\bigwedge_{n=1}^{\infty} a_n\right) = 0$ and $\mu\left(\bigwedge_{n=1}^{\infty} a_n\right) = \mu(a)$ respectively. Also, since μ is m-continuous, therefore $\mu\left(\bigwedge_{n=1}^{\infty} a_n\right) = 0$ and, consequently, we get $\mu(a) = 0$, which contradicts the fact that $\mu(a) > 0$. □

Theorem 4.1. *μ is non-atomic on L if and only if for a given element $a \in L$ with $\mu(a) > 0$ and $\varepsilon > 0$, there exists $b \in L$, $b \leqslant a$, such that $0 < \mu(b) < \varepsilon$.*

Proof. The *if* part: Suppose on the contrary that $c \in L$ be an atom of μ. Then $\mu(c) > 0$ and therefore there exists $b \in L$, $b \leqslant c$ such that $0 < \mu(b) < \varepsilon$, for any $\varepsilon > 0$. Also, $\mu(b) = \mu(c)$ and $\mu(c \ominus b) = 0$, which yields $\mu(c) = 0$, a contradiction.

The *only if* part: Suppose the contrary and choose an element $a \in L$ with $\mu(a) > 0$ and $t_0 > 0$, for which $\mu(b) \geqslant t_0$ holds if $b \leqslant a$, $b \in L$ and $\mu(b) > 0$. Define $t_1 = \inf\{\mu(b) : b \in L, b \leqslant a, \mu(b) > 0\}$. Then obviously $0 < t_0 \leqslant t_1$. Take $a_1 \leqslant a$, $a_1 \in L$ with $t_1 \leqslant \mu(a_1) < t_1 + 1$ and setting $t_2 = \inf\{\mu(b) : b \in L, b \leqslant a_1, \mu(b) > 0\}$. Choose $a_2 \leqslant a_1$ with $t_2 \leqslant \mu(a_2) < t_2 + \frac{1}{2}$. Continuing the process in the same manner, we obtain sequences $\{t_n\}$ and $\{a_n\}$ such that $t_0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant \mu(a)$ and $a \geqslant a_1 \geqslant a_2 \geqslant \ldots$ with $t_n \leqslant \mu(a_n) < t_n + \frac{1}{2^n}$, for all n. Using 2.3, put $a_0 = \bigwedge_{n=1}^{\infty} a_n$. Clearly, $\mu(a_0) = \mu\left(\bigwedge_{n=1}^{\infty} a_n\right) = \lim_{n\to\infty} \mu(a_n) = \lim_{n\to\infty} t_n > 0$. Let $b \leqslant a_0$ with $\mu(b) > 0$. Then $\mu(a_0) \geqslant \mu(b) \geqslant t_n$, for any n and hence $\mu(b) = \mu(a_0)$. This contradicts Lemma 4.1. □

Theorem 4.2. *If μ is non-atomic on L, then μ takes every value between 0 and $\mu(1)$.*

Proof. Let $0 < t < \mu(1)$. According to Theorem 4.1, there are elements $c \in L$ such that $0 < \mu(c) < t$. Let

$$s_1 = \sup\{\mu(c) : c \in L, \mu(c) \leqslant t\}.$$

(Obviously $0 < s_1 \leqslant t$). Then there exists an element $c_1 \in L$ such that $\frac{s_1}{2} < \mu(c_1) \leqslant s_1$. Let

$$s_2 = \sup\{\mu(c) : c \in L, c_1 \leqslant c, \mu(c) \leqslant t\}.$$

Then there exists an element $c_2 \in L$ such that $c_2 \geqslant c_1$ and $s_2 - \frac{s_1}{2^2} < \mu(c_2) \leqslant s_2$. Continue this construction inductively to obtain

$$s_n = \sup\{\mu(c) : c \in L, c_{n-1} \leqslant c, \mu(c) \leqslant t\},$$

and then there exists $c_n \geqslant c_{n-1}$, $c_n \in L$ such that $s_n - \frac{s_1}{2^n} < \mu(c_n) \leqslant s_n$. It is clear that $\{s_n\}$ is a decreasing sequence and $\{c_n\}$ is an increasing sequence of elements in L such that $d = \bigvee_{n=1}^{\infty} c_n \in L$ (using 2.3) and therefore, since μ is *lsc*, we get $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \mu(c_n) = \mu\left(\bigvee_{n=1}^{\infty} c_n\right) = \mu(d)$. Therefore $\mu(d) = \lim_{n\to\infty} s_n = s$ (let). Clearly $s \leqslant t$. Now we claim that $s = t$. For, otherwise, let us suppose that $s < t$. Since $0 < t < \mu(1)$, we get $\mu(1 \ominus d) > 0$, $d \in L$ and therefore, by Theorem 4.1, we obtain an element b of L such that $b \leqslant (1 \ominus d)$ and $s < \mu(d \oplus b) < t$. But then $d \oplus b \geqslant c_{n-1}$, for all $n > 1$, which yields $\mu(d \oplus b) \leqslant s_n$, for all n. This will further imply that $\mu(d \oplus b) \leqslant s$, a contradiction. Thus $\mu(d) = t$, as required. □

Concluding Remark

Several authors [9, 19, 20] have made their contributions to the theory of atoms in an effect algebra L where the definition of an atom involves only the structure of L, while in this paper we have coined the concept of an atom of a function on L and illustrated it by means of a few examples. We have established a decomposition theorem in the context of atoms of a null-additive function. The concept of atoms of a function in an effect algebra is complex and interesting, and null-additivity is found to be an effective tool for its study. We have also studied non-atomic measures on an effect algebra which becomes useful in proving some results leading to a theorem having resemblance with the Intermediate Value Theorem for continuous functions. We intend to address these concepts in the theory of effect algebras in our further research papers.

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