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UNIQUENESS THEOREMS FOR MEROMORPHIC FUNCTIONS THAT SHARE THREE SETS WITH WEIGHTS[1]

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Abstract. Using the notion of weighted sharing of sets we prove a uniqueness theorem which improves the result proved by Lin and Yi [6, 7].

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1. Introduction, definitions and results

Let f and g be two nonconstant meromrophic functions defined in the open complex plane C. We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \to \infty$, outside a possible exceptional set of finite linear measure.

If for some $a \in C\bigcup\{\infty\}$, f and g have the same set of a-points with same multiplicities then we say that f and g share the value a CM (counting multiplicities). If we do not take the multiplicities into account, f and g are said to share the value a IM (ignoring multiplicities).

Let S be a set of distinct elements $C\bigcup\{\infty\}$ and $E_f(S) = \bigcup_{a\in S}\{z : f(z)-a = 0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity the set $\bigcup_{a\in S}\{z : f(z) - a = 0\}$ is denoted by $\overline{E}_f(S)$.

If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand, if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM. Especially, let $S = \{a\}$, we say f and g share the value a CM. If $E_f(S) = E_g(S)$, and we say that f and g share the value a IM if $\overline{E}_f(S) = \overline{E}_g(S)$(see [3]).

In 1976, Gross [4] proved the following theorem:

Theorem A There exist three sets $S_j (j = 1, 2, 3)$ such that any two nonconstant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2, 3$ must be identical.

Gross posed the following question:

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Question A Can one find two finite sets $S_j(j = 1, 2)$ such that any two entire functions satisfying $E_f(S_j) = E_g(S_j)(j = 1, 2)$ must be identical?

Recently, H.X.Yi [10, 11, 12] gave the affirmative answer to the above questions completely. H.X.Yi [10] posed the following question:

Question B Can one find three finite sets $S_j(j = 1, 2, 3)$ such that any two meromorphic functions satisfying $E_f(S_j) = E_g(S_j)(j = 1, 2, 3)$ must be identical?

In 1994, H.X. Yi [10] proved the following result that answered Question B:

Theorem B Let $S_1 = \{a + b, a + b\omega, , \ldots, a + b\omega^{n-1}\}, S_2 = \{c_1, c_2\}$ and $S_3 = \{a\}$ or $S_3 = \{\infty\}$, where $n > 6, b \neq 0, c_1 \neq a, c_2 \neq a, (c_1 - a)^n \neq (c_2 - a)^n, (c_k - a)^n(c_j - a)^n \neq b^{2n}, (k, j = 1, 2)$. Suppose that f and g are nonconstant meromorphic functions satisfying $E_f(S_j) = E_g(S_j)(j = 1, 2, 3)$, then $f \equiv g$.

In 2003, W.C. Lin and H.X. Yi [6] proved the following result that is an improvement of Theorem B.

Theorem C Let $S_1 = \{0\}, S_2 = \{\infty\}$ and $S_3 = \{\omega|P(\omega) = a\omega^n - n(n - 1)\omega^2 +$
$2n(n - 2)b\omega - (n - 1)(n - 2)b^2 = 0\}$, where $n(> 4)$ is an integer, and a and b are two nonzero complex numbers satisfying $ab^{n-2} \neq 1, 2$. If f and g are two nonconstant meromorphic functions satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2, 3$, then $f \equiv g$.

Lin and Yi [6] remarked that the assumption $E_f(S_2) = E_g(S_2)$ in the above results can be relaxed to $\overline{E}_f(S_2) = \overline{E}_g(S_2)$.

Now based on the above theorems it is natural to ask the following question:

Question 1: Is it possible in any way to further relax the nature of sharing the set S_3 in Theorem C?

In the present paper we shall investigate this problem and obtain the following result which will improve the previous theorems mentioned earlier. To state our main result, we shall take the aid of weighted sharing of values and sets as introduced in [7, 8].

Definition 1.1. *Let k be a nonnegative integer or infinity. For $a \in C\bigcup\{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.*

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer $p, 0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2. *Let S be a set of distinct elements of $C\bigcup\{\infty\}$ and k be a nonnegative integer or ∞. We denote by $E_f(S, k)$ the set $\bigcup_{a\in S} E_k(a; f)$.*

Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.

We now state the following theorem which is the main result of this paper.

Theorem 1.3. *Let $S_1 = \{0\}, S_2 = \{\infty\}$ and $S_3 = \{\omega | P(\omega) = a\omega^n - n(n-1)\omega^2 + 2n(n-2)b\omega - (n-1)(n-2)b^2 = 0\}$, where $n(>4)$ is an integer, and a and b are two nonzero complex numbers satisfying $ab^{n-2} \neq 1, 2$. If f and g are two nonconstant meromorphic functions satisfying $E_f(S_1,\infty) = E_g(S_1,\infty), E_f(S_2,0) = E_g(S_2,0), E_f(S_3,4) = E_g(S_3,4)$, then $f \equiv g$.*

As for the standard definitions and notations of the value distribution theory we refer to [5], we now explain some notations that are used in this paper.

Definition 1.4. [9] *For $a \in C \bigcup \{\infty\}$ we denote by $N(r, a; f| = 1)$ the counting function of simple a points of f. For a positive integer m we denote by $N(r, a; f| \leq m)(N(r, a; f| \geq m))$ the counting function of those a points of f whose multiplicities are not greater(less) than m, where each a point is counted according to its multiplicity.*

$\overline{N}(r, a; f| \leq m), (\overline{N}(r, a; f| \geq m))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r, a; f| < m), N(r, a; f| > m), \overline{N}(r, a; f| < m)$ and $\overline{N}(r, a; f| > m)$ are defined analogously.

In addition, the following notions will be used in this paper: for $a \in C \bigcup \{\infty\}$ and an integer k, $\overline{N}(r, a; f| = k), \overline{N}_L(r, a; f), \overline{N}_E^{(k+1}(r, a; f), N_2(r, a; f), \overline{N}_*(r, a; f, g)$ (see [7, 8, 11]).

2. Lemmas

Lemma 2.1. [8] *Let F, G be two nonconstant meromorphic functions sharing $(1,1)$ and $H \not\equiv 0$, then*

$$N(r, 1; F| = 1) = N(r, 1; G| = 1) \leq N(r, H) + S(r, F) + S(r, G),$$

where $H = (\frac{F''}{F'} - \frac{2F'}{F-1}) - (\frac{G''}{G'} - \frac{2G'}{G-1})$.

Lemma 2.2. *If F, G share $(1,0), (\infty,0), (0,\infty)$ and $H \not\equiv 0$, then for any complex number $c \neq 1$*

$$\begin{aligned} N(r, H) \quad \leq \quad & \overline{N}_*(r, 1; F, G) + \overline{N}_*(r, \infty; F, G) + \overline{N}(r, c; F| \geq 2) \\ & + \overline{N}(r, c; G| \geq 2) + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G') \end{aligned}$$

where $\overline{N}_0(r, 0; F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)(F-c)$ and $\overline{N}_0(r, 0; G')$ is similarly defined.

Proof. We can easily verify that possible poles of H occur at

(i) multiple zeros of $F - c$ and $G - c$;

(ii) those poles of F and G whose multiplicities are distinct from the multiplicities of the corresponding poles of G and F respectively;

(iii) those 1-points of F and G whose multiplicities are distinct from the multiplicities of the corresponding 1-points of G and F respectively;

(iv) zeros of F' which are not the zeros of $F(F-1)(F-c)$;

(v) zeros of G' which are not the zeros of $G(G-1)(G-c)$.

Since H only has simple poles, the lemma follows from the above. □

Lemma 2.3. [1] *Let f and g be two nonconstant meromorphic functions sharing $(1,k)$, where $2 \leq k \leq \infty$. Then*

$$N(r,1;f|=2)+2\overline{N}(r,1;f|=3)+\cdots+(k-1)\overline{N}(r,1;f|=k)+k\overline{N}_L(r,1;f)$$
$$+(k+1)\overline{N}_L(r,1;g)+k\overline{N}_E^{(k+1}(r,1;f)\leq N(r,1;g)-\overline{N}(r,1;g).$$

Lemma 2.4. [13] *Let f be a nonconstant meromorphic function and let*

$$R(f)=\sum_{k=0}^{n}a_kf^k/\sum_{j=0}^{m}b_jf^j$$

be an irreducible rational function in f with coefficients $\{a_k\}$ and $\{b_j\}$, where $a_n\neq 0$ and $b_m\neq 0$. Then

$$T(r,R(f))=dT(r,f)+S(r,f),$$

where $d=max\{n,m\}$.

Lemma 2.5. *Let F,G share $(1,k),(\infty,0),(0,\infty)$ where $2\leq k<\infty$, then one of the following cases must occur:*

(i)

$$\begin{aligned}2T(r,F)+T(r,G) \leq\ & \overline{N}(r,0;F)+N_2(r,c;F)+\overline{N}(r,\infty;F)\\ &+\overline{N}_*(r,\infty;F,G)+\overline{N}(r,0;G)+N_2(r,c;G)\\ &+\overline{N}(r,\infty;G)-m(r,1;G)-\overline{N}(r,1;F|=3)-\cdots\\ &-(k-2)\overline{N}(r,1;F|=k)-(k-2)\overline{N}_L(r,1;F)\\ &-(k-1)\overline{N}_L(r,1;G)-(k-1)\overline{N}_E^{(k+1}(r,1;F)\\ &+S(r,F)+S(r,G);\end{aligned}$$

(ii) $F=\frac{(B+1)G+(A-B-1)}{BG+(A-B)}$, *where $A(\neq 0),B$ are two constants,*

where $c\neq 1$ is a complex number.

Proof. We consider the following two cases:

Case 1. Suppose that $H \not\equiv 0$. By the second fundamental theorem we get

(1)
$$\begin{aligned} 2T(r,F)+2T(r,G) \leq\ & \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,c;F)+\overline{N}(r,1;F) \\ & \overline{N}(r,0;G)+\overline{N}(r,\infty;G)+\overline{N}(r,c;G)+\overline{N}(r,1;G) \\ & -N_0(r,0;F')-N_0(r,0;G')+S(r,F)+S(r,G). \end{aligned}$$

Since F, G share $(1,k), (\infty,0), (0,\infty)$, by using Lemmas 2.1 - 2.2 and Lemma 2.3 we see that

(2)
$$\begin{aligned} \overline{N}(r,1;F)+\overline{N}(r,1;G) \leq\ & N(r.1;F|=1)+\overline{N}(r,1;F|=2)+\overline{N}(r,1;F|=3) \\ & +\cdots+\overline{N}(r,1;F|=k)+\overline{N}_E^{(k+1}(r,1;F) \\ & +\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}(r,1;G) \\ \leq\ & \overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_*(r,\infty;F,G) \\ & +\overline{N}(r,c;F|\geq 2)+\overline{N}(r,c;G|\geq 2) \\ & +\overline{N}(r,1;F|=2)+\cdots+\overline{N}(r,1;F|=k) \\ & +\overline{N}_L^{(k+1}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G) \\ & +T(r,G)-m(r,1;G)+O(1)-\overline{N}(r,1;F|=2) \\ & -2\overline{N}(r,1;F|=3)-\cdots-(k-1)\overline{N}(r,1;F|=k) \\ & -k\overline{N}_L^{(k+1}(r,1;F)-k\overline{N}_L(r,1;F) \\ & -(k+1)\overline{N}_L(r,1;G)+\overline{N}_0(r,0;F') \\ & +\overline{N}_0(r,0;G')+S(r,F)+S(r,G) \\ \leq\ & \overline{N}(r,c;F|\geq 2)+\overline{N}(r,c;G|\geq 2) \\ & +\overline{N}_*(r,\infty;F,G)+T(r,G)-m(r,1;G) \\ & -\overline{N}(r,1;F|=3)-\cdots-(k-2)\overline{N}(r,1;F|=k) \\ & -(k-1)\overline{N}_E^{(k+1}(r,1;F)+\overline{N}_0(r,0;F') \\ & +\overline{N}_0(r,0;G')+S(r,F)+S(r,G). \end{aligned}$$

From (1) and (2), then the relation (i) of this lemma follows.

Case 2. Suppose that $H \equiv 0$. We can obtain

(3)
$$\frac{2F'}{F-1}-\frac{F''}{F'} \equiv \frac{2G'}{G-1}-\frac{G''}{G'}.$$

By integration, we have from (3) that

(4)
$$\frac{1}{F-1} \equiv \frac{A}{G-1}+B,$$

where $A(\neq 0), B$ are constants. From (4) we get

$$F=\frac{(B+1)G+(A-B-1)}{BG+(A-B)}.$$

All these prove the lemma. □

Lemma 2.6. *Let*

$$F = \frac{af^n}{n(n-1)(f-\alpha_1)(f-\alpha_2)}, \qquad G = \frac{ag^n}{n(n-1)(g-\alpha_1)(g-\alpha_2)} \tag{5}$$

where f and g are nonconstant meromorphic functions,$n(>3)$ is an integer, and α_1, α_2 are distinct finite complex numbers. If F, G share $(1,k), (0,\infty)$ and f, g share $(\infty, 0)$ and $H \not\equiv 0$, then

$$\overline{N}(r,\infty;f) = \overline{N}(r,\infty;g) \le \frac{1}{n-3}\overline{N}_*(r,1;F,G) + S(r,F) + S(r,G). \tag{6}$$

Proof. We discuss the following two cases:

Case 1. Suppose that $\overline{E}(\{\infty\}, f) = \emptyset$, then (6) holds obviously.

Case 2. Suppose that $\overline{E}(\{\infty\}, f) \ne \emptyset$. Since $H \not\equiv 0$, we can get $\Phi \not\equiv 0$, where $\Phi = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}$. Noting that f and g share ∞ IM, we suppose that z_0 is a pole of f of order p, a pole of g of order q. From (5) we know that z_0 is a pole of F of order $(n-2)p$, a pole of G of order $(n-2)q$. In view of the definition of Φ we know that z_0 is zero of Φ of order at least $n-3$. Thus

$$(n-3)\overline{N}(r,\infty;f) \le N(r,0;\Phi) \le T(r,\Phi) + O(1), \tag{7}$$

and

$$m(r,\Phi) = S(r,F) + S(r,G). \tag{8}$$

Noting that F and G share $(1,k), (0,\infty)$, we obtain

$$N(r,\infty;\Phi) \le \overline{N}_*(r,1;F,G) + S(r,F) + S(r,G). \tag{9}$$

Noting $n > 3$, we get (6) from (7)-(9). So the proof of Lemma 2.6 comes to an end. □

Lemma 2.7. [2] *Let*

$$Q(w) = (n-1)^2(w^n - 1)(w^{n-2} - 1) - n(n-2)(w^{n-1} - 1)^2,$$

then

$$Q(w) = (w-1)^4(w-\beta_1)(w-\beta_2)\cdots(w-\beta_{2n-6}),$$

where $\beta_j \in C\backslash\{0,1\}(j = 1, 2, \ldots, 2n-6)$, which are distinct respectively.

3. Proof of Main Results

Proof. First, suppose that the polynomial P is defined by

$$P(\omega) = a\omega^n - n(n-1)\omega^2 + 2n(n-2)b\omega - (n-1)(n-2)b^2,$$

where $n(>4)$ is an integer, and a, b are two nonzero finite complex numbers such that $ab^{n-2} \ne 2$. Now we claim that the polynomial $P(\omega)$ has only simple zeros.

In fact, we consider the rational function

$$R(\omega) = \frac{a\omega^n}{n(n-1)(\omega-\alpha_1)(\omega-\alpha_2)}, \tag{10}$$

where α_1 and α_2 are two distinct roots of the equation $n(n-1)\omega^2 - 2n(n-2)b\omega + (n-1)(n-2)b^2 = 0$. From (14), we have

$$R'(\omega) = \frac{(n-2)a\omega^{n-1}(\omega-b)^2}{n(n-1)(\omega-\alpha_1)^2(\omega-\alpha_2)^2}, \tag{11}$$

so $\omega = 0$ is one root with multiplicity n of the equation $R(\omega) = 0$ and $\omega = b$ is one root with multiplicity 3 of the equation $R(\omega) - c = 0$, where $c = ab^{n-2}/2(\neq 1)$.

Thus

$$R(\omega) - c = \frac{a(\omega-b)^3 Q_{n-3}(\omega)}{n(n-1)(\omega-\alpha_1)(\omega-\alpha_2)}, \tag{12}$$

where $Q_{n-3}(\omega)$ is a polynomial of degree $n-3$. Moreover, we have

$$R(\omega) - 1 = \frac{P(\omega)}{n(n-1)(\omega-\alpha_1)(\omega-\alpha_2)} \tag{13}$$

We obtain from (11) and (13) that $P(\omega) = a\omega^n - n(n-1)\omega^2 + 2n(n-2)b\omega - (n-1)(n-2)b^2$ has only simple zeros.

Now let F and G be defined by

$$F = R(f), \qquad G = R(g). \tag{14}$$

From (10),(13),(14) and in view of the condition of Theorem 1.3, we conclude that F and G share $(0,\infty)$, $(1,4)$ and $(\infty,0)$, and by Lemma 2.4 we have

$$T(r,f) = \frac{1}{n}T(r,F) + S(r,F), \qquad T(r,g) = \frac{1}{n}T(r,G) + S(r,G). \tag{15}$$

Let H be mentioned in Section 2, then we consider the following two cases:

Case 1. By Lemma 2.5 we have

$$\begin{aligned} 2T(r,F) + T(r,G) \leq\ & \overline{N}(r,0;F) + N_2(r,c;F) + \overline{N}(r,\infty;F) \\ & + \overline{N}_*(r,\infty;F,G) + \overline{N}(r,0;G) + N_2(r,c;G) \\ & + \overline{N}(r,\infty;G) - 2\overline{N}_L(r,1;F) - 2\overline{N}_L(r,1;G) \\ & + S(r,F) + S(r,G). \end{aligned} \tag{16}$$

In view of the definition of F, G we have

$$\overline{N}(r,\infty;F) = \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f) + \overline{N}(r,\infty;f), \tag{17}$$

$$\overline{N}(r,\infty;G) = \overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g) + \overline{N}(r,\infty;g), \tag{18}$$

$$\overline{N}(r,0;F) \le \overline{N}(r,0;f), \qquad \overline{N}(r,0;G) \le \overline{N}(r,0;g), \tag{19}$$

$$\overline{N}_2(r,c;F) \le 2\overline{N}(r,b;f) + N(r,0;Q_{n-3}(f)), \tag{20}$$

$$\overline{N}_2(r,c;G) \le 2\overline{N}(r,b;g) + N(r,0;Q_{n-3}(g)). \tag{21}$$

On the other hand, by Lemma 2.6 we have

$$\overline{N}(r,\infty;f) = \overline{N}(r,\infty;g) \le \frac{1}{n-3}\overline{N}_*(r,1;F,G) + S(r,F) + S(r,G). \tag{22}$$

From (15)-(22), we can get

(23)

$$\begin{aligned} 2nT(r,f) + nT(r,g) \quad \le \quad & \overline{N}(r,0;f) + 2\overline{N}(r,b;f) + N(r,0;Q_{n-3}(f)) \\ & + N(r,\alpha_1;f) + N(r,\alpha_2;f) + \overline{N}(r,0;g) \\ & + 2\overline{N}(r,b;g) + N(r,0;Q_{n-3}(g)) + N(r,\alpha_1;g) \\ & + N(r,\alpha_2;g) + S(r,f) + S(r,g) \\ \le \quad & (n+2)\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g). \end{aligned}$$

Similarly, we can get

(24)

$$\begin{aligned} 2nT(r,g) + nT(r,f) \quad \le \quad & \overline{N}(r,0;f) + 2\overline{N}(r,b;f) + N(r,0;Q_{n-3}(f)) \\ & + N(r,\alpha_1;f) + N(r,\alpha_2;f) + \overline{N}(r,0;g) \\ & + 2\overline{N}(r,b;g) + N(r,0;Q_{n-3}(g)) + N(r,\alpha_1;g) \\ & + N(r,\alpha_2;g) + S(r,f) + S(r,g) \\ \le \quad & (n+2)\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g). \end{aligned}$$

Adding (23) and (24), we have

$$3n\{T(r,f) + T(r,g)\} \le (2n+4)\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g).$$

So we get $n \le 4$ which contradicts to the condition of Theorem 1.3.

Case 2. By Lemma 2.5 we have

$$F = \frac{(B+1)G + (A-B-1)}{BG + (A-B)}, \tag{25}$$

where $A(\neq 0), B$ are two constants, and

$$T(r,F) = T(r,G) + O(1) \tag{26}$$

Now we consider three subcases:

Subcase 2.1 $B \neq 0, 1$, (25) and the condition of Theorem 1.3 imply that ∞ is Picard exceptional value of f and g, then

$$\overline{N}(r,\infty;F) = \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f), \tag{27}$$

$$\overline{N}(r,\infty;G) = \overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g). \tag{28}$$

Suppose that $A-B-1\neq 0$. We have from (25) that

$$\overline{N}(r, \frac{A-B-1}{B+1}; G) = \overline{N}(r, 0; F).$$

By the second fundamental theorem, we have

$$T(r,G) \leq \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + \overline{N}(r, \frac{A-B-1}{B+1}; G) + S(r,G).$$

From (15),(25)-(28), we have

$$\begin{aligned} nT(r,g) &\leq \overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g) + \overline{N}(r,0;g) + \overline{N}(r,0;f) + S(r,g) \\ &\leq 4T(r,g) + S(r,g), \end{aligned}$$

which contradicts our assumption that $n \leq 4$. Therefore, we obtain $A-B-1 = 0$, and from (25) we have

$$F = \frac{(B+1)G}{BF+1}. \tag{29}$$

Hence we get

$$\overline{N}(r, 1/B; G) = \overline{N}(r,\infty;F) = \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f). \tag{30}$$

If $c \neq -1/B$, from (12) (14) and Lemma 2.4 we have

$$\overline{N}(r,c;G) \leq (n-2)T(r,g) + S(r,g). \tag{31}$$

By the second fundamental theorem we have

$$2T(r,G) \leq \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + \overline{N}(r,-1/B;G) + \overline{N}(r,c;G) + S(r,G),$$

i.e.,

$$\begin{aligned} 2nT(r,g) &\leq \overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g) + \overline{N}(r,0;g) + \overline{N}(r,\alpha_1;f) \\ &\quad + \overline{N}(r,\alpha_2;f) + (n-2)T(r,g) + S(r,g) \\ &\leq (n+3)T(r,g) + S(r,g). \end{aligned}$$

Then we get a contradiction since $n > 4$.
If $c = -1/B$, we can write (29) as

$$F = \frac{(1-c)G}{G-c},$$

i.e.,

$$G = \frac{cF}{F-(1-c)}. \tag{32}$$

Then we have

$$\overline{N}(r,1-c;F) = \overline{N}(r,\infty;G) = \overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g). \tag{33}$$

Similarly, from (25) we have

$$\overline{N}(r,c;F) \le (n-2)T(r,f) + S(r,f). \tag{34}$$

Since $c = ab^{n-2}/2 \ne 1/2$, we get that $1-c \ne c$. By the second fundamental theorem, we get

$$2T(r,F) \le \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1-c;F) + \overline{N}(r,c;F) + S(r,F),$$

i.e.,

$$\begin{aligned} 2nT(r,f) &\le \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f) + \overline{N}(r,0;f) + \overline{N}(r,\alpha_1;g) \\ &\quad + \overline{N}(r,\alpha_2;g) + (n-2)T(r,f) + S(r,f) \\ &\le (n+3)T(r,f) + S(r,f). \end{aligned}$$

Then we also get a contradiction since $n > 4$.

Subcase 2.2 $B = 0$. We can write (26) as

$$F = \frac{G + (A-1)}{A} \tag{35}$$

Suppose that $A - 1 \ne 0$. Since F, G share $(0,\infty)$, from (35) we know $G \ne 0, 1-A$, and by the second fundamental theorem we have

$$\begin{aligned} nT(r,g) &\le \overline{N}(r,\infty;G) + S(r,g) \\ &\le \overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g) + \overline{N}(r,\infty;g) + S(r,g) \\ &\le 3T(r,g) + S(r,g). \end{aligned}$$

This is impossible. Then $A = 1$, *i.e.* $F \equiv G$, in view of the definition of $R(\omega), F$ and G, we have

$$n(n-1)f^2g^2(f^{n-2}-g^{n-2}) - 2n(n-2)bfg(f^{n-1}-g^{n-1}) + (n-1)(n-2)b^2(f^n-g^n) = 0. \tag{36}$$

Let $h = f/g$, *i.e.* $f = hg$; substituting it in (36) we obtain

$$n(n-1)h^2g^2(h^{n-2}-1) - 2n(n-2)bhg(h^{n-1}-1) + (n-1)(n-2)b^2(h^n-1) = 0. \tag{37}$$

If h is nonconstant, using Lemma 2.7 and (37), we have

$$\{n(n-1)h(h^{n-2}-1)g - n(n-2)b(h^{n-1}-1)\}^2 = gn(n-2)b^2Q(h), \tag{38}$$

where $Q(h) = (h-1)^4(h-\beta_1)(h-\beta_2)\cdots(h-\beta_{2n-6})$, $\beta_j \in C\backslash\{0,1\}(j = 1,2,\ldots,2n-6)$, which are distinct respectively.

From (38) we know that every zero of $h - \beta_j (j = 1,2,\ldots,2n-6)$ is of order at least 2. By the second fundamental theorem we have $n \le 4$, which is a contradiction. Hence h is a constant. From (38), we have that $h^{n-2} = 1$ and $h^{n-1} = 1$, which imply $h \equiv 1$, *i.e.* $f \equiv g$.

Subcase 2.3 $B = -1$. We write (25) as

$$F = \frac{A}{-G + A + 1}. \tag{39}$$

If $A+1 \neq 0$, noting that F, G share $(0, \infty)$, from (39) we know $F \neq 0, A/(A+1)$. By using the second fundamental theorem, we have

$$\begin{aligned} nT(r,f) &\leq \overline{N}(r,\infty;F) + S(r,f) \\ &\leq \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f) + \overline{N}(r,\infty;f) + S(r,f) \\ &\leq 3T(r,f) + S(r,f). \end{aligned}$$

Then we get a contradiction. Hence we have $A+1=0$, *i.e.* $F \equiv G$. In view of the definition of F, G we obtain

$$\frac{f^n}{(f-\alpha_1)(f-\alpha_2)} \frac{g^n}{(g-\alpha_1)(g-\alpha_2)} = \frac{n^2(n-1)^2}{a^2}. \tag{40}$$

Since f, g share $(0, \infty), (\infty, 0)$, from (40) we know that $0, \alpha_1, \alpha_2, \infty$ are Picard exceptional values of g. This is also impossible.

Then, the proof Theorem 1.3 comes to an end. □

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