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AN APPLICATION OF HIGHER ORDER FIXED POINTS OF NORMAL FUNCTIONS

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Abstract. We define higher order fixed points of normal functions, describe them and apply to obtain a constructive proof that, if κ is the least ordinal such that the ultrapower κ^I/F is non-trivial, then that ultrapower has at least κ^+ elements.

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Let F be a nonprincipal ultrafilter over a set I. By α^I/F we denote the ultrapower $\prod_{i\in I}\alpha/F$. The element of α^I/F which is the equivalence class of a function f will be denoted by f^F.

It is well known (see [1], page 134) that an ultraproduct of infinite ordinals modulo ultrafilter F is well-ordered iff F is σ-complete. Thus throughout this work we suppose that F is a fixed σ-complete ultrafilter over some I. The following proposition is easy to prove.

Proposition 1. If $\alpha<\beta$, then α^I/F is isomorphic to an initial segment of β^I/F.

Thus we can identify α^I/F with an initial segment of β^I/F. Therefore for $\alpha\in Ord$ we let, as in [4],

$$A_\alpha=(\alpha^I/F)\setminus\bigcup_{\beta<\alpha}A_\beta.$$

Obviously, $A_{\alpha+1}$ has exactly one element for $\alpha\in Ord$; it is f_α^F, where $f_\alpha(i)=\alpha$ for $i\in I$. Hence we will be interested only in the case when α is a limit ordinal. Also, the described element f_α^F is the image of α under the natural elementary embedding $d:\beta\to\beta^I/F$ (see [2]), and thus every β^I/F contains a copy of β. The main question we consider here is: how many more elements can β^I/F have, and on which A_α levels?

Definition 1. Let α be a limit ordinal. An ultrafilter F over I is α-descendingly incomplete if there is a sequence $\langle X_\beta:\beta<\alpha\rangle$ of elements of F such that $X_{\beta_1}\supseteq X_{\beta_2}$ for $\beta_1<\beta_2<\alpha$ and $\bigcap_{\beta<\alpha}X_\beta=\emptyset$; otherwise it is α-descendingly complete.

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Proposition 2. If α is a limit ordinal, then $A_\alpha = \emptyset$ iff F is $\mathrm{cf}(\alpha)$-descendingly complete.

But it is known (see [1], page 114) that the following proposition holds.

Proposition 3. The least ordinal α, such that F is α-descendingly incomplete, is equal to the least cardinal α such that F is α^+-incomplete, and it is a measurable cardinal.

So the first κ such that $A_\kappa \neq \emptyset$ must be an uncountable measurable cardinal (we will need only the fact that κ is regular). From this point on, κ will denote that cardinal. It is easy to prove that, if F is κ-descendingly incomplete, then $|A_\kappa| \geq \kappa$. One can prove (see [3], page 291) the following proposition.

Proposition 4. If F is a nonprincipal κ-complete ultrafilter over κ, then $|A_\kappa| \geq 2^\kappa$.

The result we are about to prove is weaker, but the proof gives a better insight into the structure of elements of A_κ. We begin with a lemma analogous to the Cantor Normal Form Theorem and emphasize that all operations that appear in its formulation are ordinal operations.

Lemma 1. Every nonzero ordinal less than κ^+ can be represented uniquely in the form

$$\kappa^{\alpha_0}\beta_0 + \kappa^{\alpha_1}\beta_1 + \cdots + \kappa^{\alpha_n}\beta_n,$$

where $\kappa^+ > \alpha_0 > \alpha_1 > \cdots > \alpha_n$ and $0 < \beta_k < \kappa$ for $0 \leq k \leq n$.

If $\xi = \kappa^{\alpha_0}\beta_0 + \kappa^{\alpha_1}\beta_1 + \cdots + \kappa^{\alpha_n}\beta_n$ is an ordinal represented as in the lemma above, with $\bar{\xi}$ we shall denote $\kappa^{\alpha_0}\beta_0 + \kappa^{\alpha_1}\beta_1 + \cdots + \kappa^{\alpha_{n-1}}\beta_{n-1}$. For every such ξ, we will call the set $\{\bar{\xi} + \kappa^{\alpha_n}\beta : 0 < \beta < \kappa\}$ a level (of course, ξ belongs to this set). Thus, if ξ and η are on the same level, then $\bar{\xi} = \bar{\eta}$ (but the opposite does not hold).

Definition 2. Let $p : \kappa^+ \to \kappa^+$ be a normal function. We define the sequence of functions $\langle p_\xi : \xi < \kappa^+\rangle$ in this way:

1) $p_0 = p$;

2) $p_{\xi+1}(\alpha)$ is the α-th fixed point of p_ξ;

3) if ξ is a limit ordinal, $p_\xi(\alpha)$ is the αth ordinal that is fixed for all p_η for $\eta < \xi$.

It is easy to prove that all the functions p_ξ for $\xi < \kappa^+$ are normal and map κ^+ to κ^+. We need some more information, contained in the following lemmas:

Lemma 2. The function $p_{\xi+1}$ can be calculated in the following way:

1) $p_{\xi+1}(0) = \sup_{n<\omega} \theta_n$, where $\theta_0 = 1$ and $\theta_{n+1} = p_\xi(\theta_n)$ for $n < \omega$;

2) $p_{\xi+1}(\alpha + 1) = \sup_{n<\omega} \theta_n$, where $\theta_0 = p_{\xi+1}(\alpha) + 1$ and $\theta_{n+1} = p_\xi(\theta_n)$ for $n < \omega$;

3) if α is a limit ordinal, then $p_{\xi+1}(\alpha) = \sup_{\beta<\alpha} p_{\xi+1}(\beta)$.

Lemma 3. If δ is a limit ordinal, then the function p_δ can be calculated in the following way:

1) $p_\delta(0) = \sup_{\xi<\delta}\theta_\xi$, where $\theta_0 = 1$, for $\xi < \delta$ $\theta_{\xi+1} = p_\xi(\theta_\xi)$, and $\theta_\xi = \sup_{\eta<\xi}\theta_\eta$ for a limit cardinal ξ;

2) $p_\delta(\alpha+1) = \sup_{\xi<\delta}\theta_\xi$, where $\theta_0 = p_\delta(\alpha)+1$, for $\xi < \delta$ $\theta_{\xi+1} = p_\xi(\theta_\xi)$, and $\theta_\xi = \sup_{\eta<\xi}\theta_\eta$ for a limit cardinal $\xi < \delta$;

3) if α is a limit ordinal, then $p_\delta(\alpha) = \sup_{\beta<\alpha} p_\delta(\beta)$.

Before we begin with the proof of the main theorem, let us notice that if $p(0) > 0$ then for every $\zeta \in \operatorname{ran}(p)$ there is the greatest $\xi < \kappa^+$ such that $\zeta \in \operatorname{ran}(p_\xi)$. This is beacuse $p_{\xi+1}(0) > p_\xi(0)$ for all $\xi < \kappa^+$ (easy proof by induction), so the sequence $\langle p_\xi(0) : \xi < \kappa^+\rangle$ is cofinal in κ^+. Knowing this, let us call $\zeta < \kappa^+$ a fixed point of order δ if ζ belongs to $\operatorname{ran}(p_\delta) \setminus \operatorname{ran}(p_{\delta+1})$.

In the rest of this article, we will consider the function $p : \kappa^+ \to \kappa^+$ given by $p(\gamma) = \kappa^\gamma$. Obviously, it is normal and $p(0) > 0$, so all the preceding results apply to it. Let us also introduce the abbreviation $e_{\delta,\alpha} = p_\delta(\alpha)$, and note that $\kappa^{e_{\delta,\alpha}} = e_{\delta,\alpha}$ for $\delta > 0$.

Theorem 1. If κ is the least cardinal such that the ultrafilter F over I is κ^+-incomplete, then $|A_\kappa| \geq \kappa^+$.

Proof. We will construct, by recursion on ξ, a sequence $\langle g_\xi : \xi < \kappa^+\rangle$ of functions such that $Y_{\eta\xi} = \{i \in I : g_\eta(i) < g_\xi(i)\} \in F$ for $\eta < \xi < \kappa^+$, thus obtaining an ascending sequence $\langle g_\xi^F : \xi < \kappa^+\rangle$ of elements of A_κ. So let us define for every $\xi < \kappa^+$ an element $g_\xi \in A_\kappa$ such that for all $\eta < \xi$

$$(\mathrm{I}_{\eta,\xi}) \qquad g_\eta <^* g_\xi$$

holds. First, let g_0 be such that g_0^F is the minimal element of A_κ, and for $\xi > 0$:

1° If $\xi = \eta + 1$, we define $g_\xi(i) = g_\eta(i) + 1$.

2° If $\xi = \bar{\xi} + \kappa^\alpha\beta$, $\alpha = \alpha_1 + 1$ and $\beta = \beta_1 + 1$, we define $\eta_\mu = \bar{\xi} + \kappa^\alpha\beta_1 + \kappa^{\alpha_1}\mu$ for $\mu < \kappa$. By Proposition 3 F is κ-descendingly incomplete, so there is a sequence $\langle X_\zeta : \zeta < \kappa\rangle$ of elements of F such that $X_0 = I$, $X_{\zeta_1} \supseteq X_{\zeta_2}$ for $\zeta_1 < \zeta_2 < \kappa$ and $\bigcap_{\zeta<\kappa} X_\zeta = \emptyset$. Now let us define $g_\xi(i) = g_{\eta_\mu}(i)$, where $\mu = \min\{\zeta < \kappa : i \notin X_\zeta\}$.

3° If $\xi = \bar{\xi} + \kappa^\alpha\beta$ and β is a limit ordinal, then we define $\eta_\mu = \bar{\xi} + \kappa^\alpha\mu$ for $\mu < \beta$ and $g_\xi(i) = \sup_{\mu<\beta} g_{\eta_\mu}(i)$. Since $\beta < \kappa$ and κ is regular, we have $g_\xi(i) < \kappa$.

4° If $\xi = \bar{\xi} + \kappa^\alpha\beta$, α is a limit ordinal not fixed for p and $\beta = \beta_1 + 1$, then we define $\eta_\mu = \bar{\xi} + \kappa^\alpha\beta_1 + \kappa^\mu$ for $\mu < \alpha$, and let g_ξ be defined by $\langle g_{\eta_\mu} : \mu < \alpha\rangle$ in the same way g_α was defined by $\langle g_\mu : \mu < \alpha\rangle$. (This means that we look up which of the rules $2° - 7°$ was used for constructing g_α and use it to construct g_ξ too; this depends of the ordinal α. It does not mean that we have to use all elements of the sequence $\langle g_{\eta_\mu} : \mu < \alpha\rangle$.)

5° If $\xi = \bar{\xi} + e_{\delta,\nu}\beta$, $e_{\delta,\nu}$ is a fixed point of order δ, where $\delta = \delta_1 + 1$, $\beta = \beta_1 + 1$ and $\nu = \nu_1 + 1$, then g_ξ is defined in the following way: we set $\eta_n = \bar{\xi} + e_{\delta,\nu}\beta_1 + \theta_n$, where $\theta_0 = e_{\delta,\nu_1} + 1$ and $\theta_{n+1} = p_{\delta_1}(\theta_n)$ for $n < \omega$; finally, let $g_\xi(i) = \sup_{n<\omega} g_{\eta_n}(i)$.

6° If $\xi = \bar{\xi} + e_{\delta,\nu}\beta$, $e_{\delta,\nu}$ is a fixed point of order δ, where δ is a limit ordinal, $\beta = \beta_1 + 1$ and $\nu = \nu_1 + 1$, g_ξ is defined in the following way: we set $\eta_\mu = \bar{\xi} + e_{\delta,\nu}\beta_1 + \theta_\mu$, where $\theta_0 = e_{\delta,\nu_1} + 1$, $\theta_{\mu+1} = p_\mu(\theta_\mu)$ for $\mu < \delta$ and, if $\mu < \delta$ is a limit ordinal, then $\theta_\mu = \sup_{\zeta<\mu} \theta_\zeta$. Finally, let g_ξ be defined by the sequence $\langle g_{\eta_\mu} : \mu < \delta \rangle$ in the same way we defined g_δ by the sequence $\langle g_\mu : \mu < \delta \rangle$. Since e_{δ,ν_1} is a fixed point of order at least δ, it follows that $\xi \geq e_{\delta,\nu} > e_{\delta,\nu_1} \geq \delta$.

7° If $\xi = \bar{\xi} + e_{\delta,\nu}\beta$, $e_{\delta,\nu}$ is a fixed point of order δ, where ν is a limit ordinal and $\beta = \beta_1 + 1$, let $\eta_\mu = \bar{\xi} + e_{\delta,\nu}\beta_1 + e_{\delta,\mu}$ for $\mu < \nu$, and let g_ξ be defined by $\langle g_{\eta_\mu} : \mu < \nu \rangle$ in the same way as g_ν by $\langle g_\mu : \mu < \nu \rangle$. Note that $\xi \geq e_{\delta,\nu} > \nu$, because otherwise $p_\delta(\nu) = e_{\delta,\nu} = \nu$ would imply that $e_{\delta,\nu}$ is a fixed point of order at least $\delta + 1$.

Let us show by induction on $\xi > 0$ that the conditions $(\mathrm{I}_{\eta,\xi})$ are satisfied for all $\eta < \xi$. Let us assume $(\mathrm{I}_{\zeta,\eta})$ holds for $\zeta < \eta < \xi$. We will prove $Y_{\eta\xi} \in F$ for each of the cases in the definition:

1° Obvious.

2° Let us first prove that $Y_{\eta_\mu\xi} \in F$ for $\mu < \kappa$. If $Y = \bigcap_{\nu<\mu} Y_{\eta_\nu\eta_\mu}$, since $(\mathrm{I}_{\eta_\nu,\eta_\mu})$ holds for $\nu < \mu$ and F is κ-complete, we have $Y \in F$. But $g_{\eta_\nu}(i) < g_{\eta_\mu}(i)$ for all $i \in Y$ and all $\nu < \mu$. It follows from the definition of g_ξ that $Y_{\eta_\mu\xi} \supseteq X_\mu \cap Y$, hence $Y_{\eta_\mu\xi} \in F$. But $\xi = \sup_{\mu<\kappa} \eta_\mu$, so for every $\eta < \xi$ there is $\eta_\mu > \eta$, and by $(\mathrm{I}_{\eta,\eta_\mu})$ we have $Y_{\eta\eta_\mu} \in F$ and therefore, since $Y_{\eta\xi} \supseteq Y_{\eta\eta_\mu} \cap Y_{\eta_\mu\xi}$, we conclude $Y_{\eta\xi} \in F$.

3° For every $\mu < \xi$ we have $(I_{\eta_\mu,\eta_{\mu+1}})$ and, since $g_{\eta_{\mu+1}}(i) \leq g_\xi(i)$ for $i \in I$, it follows that $Y_{\eta_\mu\xi} \in F$ as well. Now, as in case 2°, for every $\eta < \xi$ we can find η_μ such that $\eta < \eta_\mu$, thus $Y_{\eta\eta_\mu} \in F$. This implies $Y_{\eta\xi} \in F$.

4° We can prove $Y_{\eta_\mu\xi} \in F$ in the same way we proved $Y_{\mu\alpha} \in F$, depending of which of the cases of the construction was used in defining g_α by g_μ $(\mu < \alpha)$, and proceed as in 2°.

5° By Lemma 2 $\xi = \sup_{n<\omega} \eta_n$, so the proof is analogous to case 3°.

6° By Lemma 3 $\xi = \sup_{\mu<\delta} \eta_\mu$, so the proof is analogous to case 4°.

7° Analogous to 4°, using Lemmas 2 and 3. □

It is easy to see that a similar construction can be done in any A_β for β such that $\mathrm{cf}(\beta) = \kappa$:

Corollary. If κ is the least ordinal such that the ultrafilter F is κ^+-incomplete and $\mathrm{cf}(\beta) = \kappa$, then $|A_\beta| \geq \kappa^+$.

Adding to results from [5], we get another direct corollary:

Corollary. Let κ be the least ordinal such that the ultrafilter F is κ^+-incomplete, $\mathrm{cf}(\beta) = \kappa$, and $B = \{t_\xi : \xi < \alpha\}$ be a branch of a tree T. Then there are at least κ^+ elements in T^I/F greater than all t_ξ for $\xi < \beta$ and (if $\beta < \alpha$) less than t_β.

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