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# THE POLAR MOMENT OF INERTIA OF THE ENVELOPING CURVE

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**Abstract.** The polar moment of inertia of a closed orbit curve of a point under one-parameter closed planar motions was studied by H. R. Müller, [1]. We study the polar moment of inertia of the enveloping curve of a line under one-parameter closed homothetic motions of planar kinematics.

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## 1. Introduction

Let $E$ and $E'$ be moving and fixed Euclidean planes and $\{O; \mathbf{e_1}, \mathbf{e_2}\}$ and $\{O'; \mathbf{e'_1}, \mathbf{e'_2}\}$ be their orthonormal frames, respectively; $\varphi$ be the rotation angle between the vectors $\mathbf{e_1}$ and $\mathbf{e'_1}$. If the homothetic scale $h$, $u_1$, $u_2$ and $\varphi$ are continuously differentiable functions of a real parameter $t$, then a *one-parameter planar homothetic motion* of $E$ with respect to $E'$ is defined (such a motion will be denoted by $H_1$) and represented in vector notation by

$$\mathbf{x}' = h\mathbf{x} - \mathbf{u}, \tag{1}$$

where $\mathbf{OO'} = \mathbf{u} = u_1\mathbf{e_1} + u_2\mathbf{e_2}$, $\mathbf{x}$ and $\mathbf{x}'$ are the position vectors of a point $X \in E$ with respect to the moving and fixed frames, respectively.

The motion $H_1$ is called *closed* if there exists $T > 0$ such that

$$h(t+T) = h(t), \quad \varphi(t+T) = \varphi(t) + 2\pi\nu, \quad u_i(t+T) = u_i(t), \quad i = 1, 2$$

for all $t$. The smallest number $T$ with this property is called the *period* of the closed motion, the integer $\nu$ is called the *rotation number* of the motion.

During $H_1$, the *moving* and *fixed polodes* are the curves obtained as locus of momentarily fixed points (centers of infinitesimal homotheties) in both, moving and fixed plane, respectively. Thus, for the pole point $P = (p_1, p_2) \in E$ we have

$$\dot{u}_1 = p_1\dot{h} - p_2 h\dot{\varphi} + u_2\dot{\varphi}, \quad \dot{u}_2 = p_2\dot{h} + p_1 h\dot{\varphi} - u_1\dot{\varphi}, \quad [2]. \tag{2}$$

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*The Steiner point* $S = (s_1, s_2)$, which is the center of gravity of the moving pole curve, is given by

(3) $$s_i = \frac{\oint h^2 p_i d\varphi}{\oint h^2 d\varphi} = \frac{\oint h^2 p_i d\varphi}{2h^2(t_0)\pi\nu}, \quad \text{for some } t_0 \in [0,T], \quad [2].$$

A point $X$ traces a closed curve $k_X$ in $E'$ during the closed $H_1$. The *polar moment of inertia* (PMI) of the curve $k_X$ covered with the mass elements $d\varphi$ is

(4) $$T_X = \oint \left((x')^2 + (y')^2\right) d\varphi,$$

where $x'$ and $y'$ are the coordinates of $X = (x, y) \in E$ with respect to the fixed coordinate system.

Let a line $g$ be given by

(5) $$g : x\cos\theta + y\sin\theta = k, \quad k, \theta = \text{constant}$$

in the moving coordinate system, where $k$ is the distance of the origin point $O$ to the line $g$ (Fig. 1).

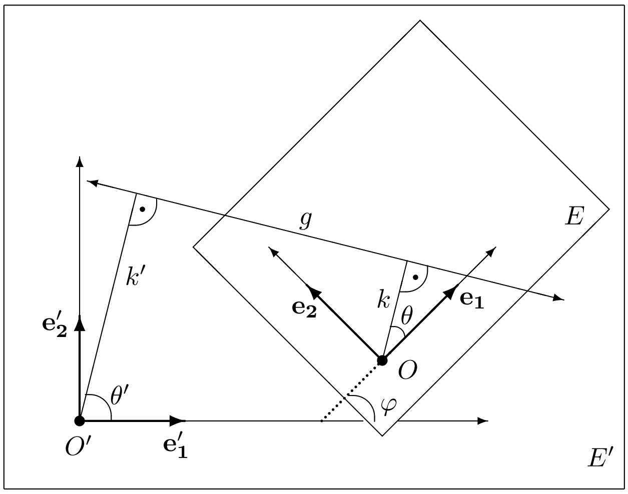


Fig. 1

The equation of the line $g$ with respect to the fixed coordinate system is

$$x'\cos\theta' + y'\sin\theta' = k', \quad \theta' = \theta + \varphi, \quad d\theta' = d\varphi,$$

where

(6) $$k'(\theta') = hk - u_1\cos\theta - u_2\sin\theta.$$

Then, the PMI of the enveloping curve of $g$ with respect to the origin point $O'$ is

(7) $$T_g = \oint \left((k')^2 + (\dot{k'})^2\right) d\theta', \qquad \text{“.” means } \frac{d}{d\theta'}.$$

## 2. The PMI of the enveloping curve

Let us consider two non-parallel fixed lines (through $O$)

$$\begin{cases} g_1 : x\cos\theta_1 + y\sin\theta_1 = 0 \\ g_2 : x\cos\theta_2 + y\sin\theta_2 = 0 \end{cases} \tag{8}$$

which have the same closed enveloping curve under closed $H_1$ in $E'$ ($g_2$ can be taken as a second tangent to the envelope of $g_1$). Then, in the fixed coordinate system $\{O'; \mathbf{e}'_1, \mathbf{e}'_2\}$ we have

$$\begin{cases} k'_1(\theta'_1) = -u_1\cos\theta_1 - u_2\sin\theta_1, & \theta'_1 = \theta_1 + \varphi, & d\theta'_1 = d\varphi \\ k'_2(\theta'_2) = -u_1\cos\theta_2 - u_2\sin\theta_2, & \theta'_2 = \theta_2 + \varphi, & d\theta'_2 = d\varphi. \end{cases} \tag{9}$$

In this case, since $g_1$ and $g_2$ have the same closed enveloping curve, we obtain

$$\begin{aligned} T_{g_1} &= \cos^2\theta_1 \oint \left(u_1^2 + \dot{u}_1^2\right) d\theta'_1 + \sin^2\theta_1 \oint \left(u_2^2 + \dot{u}_2^2\right) d\theta'_1 + \\ &\quad 2\sin\theta_1\cos\theta_1 \oint (u_1u_2 + \dot{u}_1\dot{u}_2)\, d\theta'_1 \\ = T_{g_2} &= \cos^2\theta_2 \oint \left(u_1^2 + \dot{u}_1^2\right) d\theta'_2 + \sin^2\theta_2 \oint \left(u_2^2 + \dot{u}_2^2\right) d\theta'_2 + \\ &\quad 2\sin\theta_2\cos\theta_2 \oint (u_1u_2 + \dot{u}_1\dot{u}_2)\, d\theta'_2. \end{aligned} \tag{10}$$

We now choose the moving coordinate frame such that $\mathbf{e_1}$ is a bisector of the lines $g_1$ and $g_2$. In this case, we have $\theta_1 = -\theta_2$. If we substitute $\theta_1 = -\theta_2$ into (10), we find

$$\oint (u_1u_2 + \dot{u}_1\dot{u}_2)\, d\varphi = 0. \tag{11}$$

Now, we want to express the PMI of the enveloping curve of an arbitrary fixed line $g$ (given by (5) and $k \neq 0$). Let $g_1$ denote the line parallel to $g$ through $O$. Hence, if we substitute (6) into (7) and use (11), we get

$$\begin{aligned} T_g &= T_{g_1} + k^2 \oint \left(h^2 + \dot{h}^2\right) d\theta' - 2k\cos\theta \oint \left(hu_1 + \dot{h}\dot{u}_1\right) d\theta' \\ &\quad -2k\sin\theta \oint \left(hu_2 + \dot{h}\dot{u}_2\right) d\theta'. \end{aligned}$$

If we take $u_1 \equiv u_2 \equiv 0$, we obtain the closed homothetic motion $\overline{H}_1$ with the constant origin $O$. Then, the PMI of the enveloping curve of $g$ under $\overline{H}_1$ is

$$\overline{T}_g = k^2 \oint \left(h^2 + \dot{h}^2\right) d\theta'.$$

Thus, if we take $\varphi$ as the parameter of the motion and use (2) and (3), we get

$$T_g = T_{g_1} + \overline{T}_g - 4h^2(t_0)\pi\nu k(s_1\cos\theta + s_2\sin\theta) - 2k(A\cos\theta + B\sin\theta), \tag{12}$$

where

$$A = \oint \left( p_2 h\dot{h} - h\dot{u}_2 + \dot{h}\dot{u}_1 \right) d\varphi, \; B = \oint \left( -p_1 h\dot{h} + h\dot{u}_1 + \dot{h}\dot{u}_2 \right) d\varphi. \tag{13}$$

It is easy to see that (12) can be written as

$$T_g = T_{g_1} + \overline{T}_g - 4h^2(t_0)\pi\nu k(d + k) - 2k(D + k),$$

where $d$ and $D$ are the distances of the points $(s_1, s_2)$ and $(A, B)$ from the line $g$, respectively.

So, we may give the following theorem:

**Theorem:** *Let $g$ be an arbitrary fixed line on the moving plane and $g_1$ be the line through $O$ parallel to $g$. Under a one-parameter closed homothetic motion, the PMI of the enveloping curve of $g$ with respect to $O'$ is given by (12). It depends on*

- *the PMI of the enveloping curve of $g_1$,*
- *the PMI of the enveloping curve of $g$ under the homothetic motion obtained by omitting the translational components,*
- *the distance of the motion's Steiner point $(s_1, s_2)$ to the line $g$,*
- *the distance of $g$ to a certain point $(A, B)$ (given by the integral formula (13)) that depends only on the one-parameter homothetic motion.*

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## References

[1] Müller, H. R., Über Trägheitsmomente bei Steinerscher Massenbelegung. Abh. Braunschw. Wiss. Ges. 29 (1978), 115–119.

[2] Tutar, A., Kuruoğlu, N., The Steiner formula and the Holditch theorem for the homothetic motions on the planar kinematics. Mech. Mach. Theory 34 (1) (1999), 1–6.

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