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ON ALMOST NEARLY CONTINUITY WITH REFERENCE TO MULTIFUNCTIONS IN BITOPOLOGICAL SPACES

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Abstract. An almost nearly continuity and almost nearly quasi-continuity have been investigated in a bitopological case. Several properties of almost upper (lower) nearly quasi-continuous and almost nearly quasi-continuous multifunctions have been obtained.

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1. Introduction and Preliminaries

A lot of forms of continuity has been investigated by many mathematicians. The term "nearly continuous" was used by Ptak in 1958 (see [7]) but an "almost continuity" term one can find in [3]. In 2004, the notion of almost nearly continuity of multifunctions was introduced in [2], while the nearly quasi-continuity of multifunctions was introduced by the author in [8].

The purpose of the present paper is to consider almost continuity and almost quasi-continuity for multifunction between topological space (which is domain) and bitopological space. In section 2 are compiled some basic facts connected with almost nearly continuity while almost nearly quasi-continuity has been investigated in section 3.

In what follows, $\mathrm{cl}_\tau(A)$ and $\mathrm{int}_\tau(A)$ will represent the closure and interior respectively of a subset A with respect to topology τ.

A set A in a topological space X will be termed semi-open (semi-closed) if $A \subset \mathrm{cl}(\mathrm{int}(A))$, $(A \subset \mathrm{int}(\mathrm{cl}(A))$. It is known that the arbitrary union of semi-open sets is a semi-open set. The notion of semi-open sets was introduced by N. Levine (see [4]).

A subset of a bitopological space (Y, τ_1, τ_2) is said to be $\tau_1\tau_2$-regularly open (closed) if it is the τ_1-interior of some τ_2-closed set (τ_1-closure of some τ_2-open set) or equivalently, if it is the τ_1-interior of its own τ_2-closure (the τ_1-closure of its own τ_2-interior) (see for instance [9] and [5]).

A bitopological space (Y, τ_1, τ_2) is said to be $\tau_1\tau_2$-nearly compact if for any τ_1-open cover $\mathcal{P}$ of Y there exists a finite subfamily $\mathcal{R} \subset \mathcal{P}$ such that $Y =$

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$\bigcup_{U \in \mathcal{R}} \mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(U))$ or, equivalently, if every cover of the space by $\tau_1\tau_2$-regularly open sets has a finite subcover (see [5]).

In order to localize the concept of nearly-compactness in bitopological space we can define $\tau_1\tau_2$-N-closed sets.

A subset A of a bitopological space (Y, τ_1, τ_2) is $\tau_1\tau_2$-*N-closed* if for any cover of A by τ_1-open sets there exists a finite subcolection the τ_1-interiors of the τ_2-closures of which cover A or equivalently if for any cover of A by $\tau_1\tau_2$-regularly open sets, there exists a finite subcover. Of course, a bitopological space Y is $\tau_1\tau_2$-nearly compact iff it is $\tau_1\tau_2$-N-closed.

A notion of N-closed sets (in usual topological space) was introduced by D. Carnahan (see [1]).

Lemma 1.1. *Let G be a τ_1-open subset of bitopological space (Y, τ_1, τ_2) such that $Y \setminus G$ is a $\tau_1\tau_2$-N-closed set. Then $\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G))$ is a $\tau_1\tau_2$-regularly open set having $\tau_1\tau_2$-N-closed complement.*

Proof. It is evident that $\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G))$ is a $\tau_1\tau_2$-regularly open set. Denote $K = Y \setminus \mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G))$. Of course, $K \subset Y \setminus G$. Let $\{G_t\}_{t \in T}$ be a τ_1-open cover of the set K. Then $\{G_t\}_{t \in T} \cup (Y \setminus K)$ is an open cover of the set $Y \setminus G$. Then there exist indexes $t_1, ..., t_k$ such that

$$Y \setminus G \subset \bigcup_{i=1}^{k} \mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_{t_i})) \cup \mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(Y \setminus K)).$$

As $\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(Y \setminus K)) = \mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G))) = \mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G))$ then $K \cap \mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(Y \setminus K)) = \emptyset$. It was shown that $K \subset \bigcup_{i=1}^{k} \mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_{t_i}))$. The proof of $\tau_1\tau_2$-N-closedness of the set K is finished. □

A multifunction $F : X \to (Y, \tau_1\tau_2)$ is said to be $\tau_1\tau_2$-lower (upper) almost nearly continuous at a point $x \in X$ if for any set $V \in \tau_1$ having $\tau_1\tau_2$-N-closed complement such that $x \in F^-(V)$ $(x \in F^+(V))$ there exists an open neighbourhood U of x such that $U \subset F^-(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(V)))$ $(U \subset F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(V))))$. A multifunction F is $\tau_1\tau_2$-lower (upper) almost nearly continuous if it is $\tau_1\tau_2$-lower (upper) almost nearly continuous at any point $x \in X$. A notion of lower (upper) almost nearly continuous multifunction with reference to topological spaces was introduced in [2], and was also investigated by the author in [8].

We call a multifunction $F : X \to Y$ $\tau_1\tau_2$-*almost nearly continuous at a point* $x \in X$ if for any sets $V_1, V_2 \in \tau_1$ having $\tau_1\tau_2$-N-closed complement such that $x \in F^+(V_1)$ and $x \in F^-(V_2)$ there exists an open neighbourhood U of x such that $U \subset F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(V_1))) \cap F^-(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(V_2)))$. We define a $\tau_1\tau_2$-almost nearly continuous multifunction at any point $x \in X$ to be $\tau_1\tau_2$-*almost nearly continuous multifunction.*

Let us recall the notions of quasi-continuous (lower and upper) and quasi-continuous multifunctions.

F is called upper (lower) quasi-continuous at a point $x \in X$ if for any open subset V of Y such that $x \in F^+(V)$ $(x \in F^-(V))$ and for any open neighbourhood U of x there exists a nonempty open set $W \subset U$ such that $W \subset F^+(V)$ ($W \subset F^-(V)$) (see [6]). We call F the quasi-continuous at a point $x \in X$ if for any open subsets V_1 and V_2 of Y such that $x \in F^+(V_1) \cap F^-(V_2)$ and for any open neighbourhood U of x there exists a nonempty open set $W \subset U$ such that $U \subset F^+(V_1) \cap F^-(V_2)$ (see [6]). A multifunction F is said to be quasi-continuous (lower and upper) and quasi-continuous multifunctions if F has this property at any point of X.

Now we are ready to introduce the notions of a $\tau_1\tau_2$-almost nearly quasi-continuous and upper (lower) almost nearly quasi-continuous multifunction.

A multifunction F is said to be $\tau_1\tau_2$-*upper* (*lower*) *almost nearly quasi-continuous at a point* $x \in X$ if for any subset $V \in \tau_1$ of Y having $\tau_1\tau_2$-N-closed complement such that $x \in F^+(V)$ ($x \in F^-(V)$) and for any open neighbourhood U of x there exists a nonempty open set $W \subset U$ such that $W \subset F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(V)))$ ($W \subset F^-(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(V)))$). We call a multifunction F $\tau_1\tau_2$-*almost nearly quasi-continuous at a point* $x \in X$ if for any subsets $V_1, V_2 \in \tau_1$ of Y having $\tau_1\tau_2$-N-closed complement such that $x \in F^+(V_1) \cap F^-(V_2)$ and for any open neighbourhood U of x there exists a nonempty open set $W \subset U$ such that $W \subset F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(V_1))) \cap F^-(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(V_2)))$. A multifunction F is said to be a $\tau_1\tau_2$-*almost nearly quasi-continuous* ($\tau_1\tau_2$-*upper almost nearly quasi-continuous*, $\tau_1\tau_2$-*lower almost nearly quasi-continuous*) multifunction if F has this property at any point of X.

2. Almost nearly continuity

Theorem 2.1. *Let $F : X \to (Y, \tau_1, \tau_2)$ be a multifunction. The following statements are equivalent.*

(a) *F is $\tau_1\tau_2$-upper almost nearly continuous.*

(b) *For any $x \in X$ and for any $\tau_1\tau_2$-regularly open set G having $\tau_1\tau_2$-N-closed complement such that $F(x) \subset G$ there exists an open neighbourhood U of x such that $F(U) \subset G$.*

(c) *For any $x \in X$ and for any τ_1-closed $\tau_1\tau_2$- N-closed set K such that $x \in F^+(Y \setminus K)$ there exists a closed set $H \neq X$ such that $x \in X \setminus H$ and $F^-(\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K))) \subset H$.*

(d) *$F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(V)))$ is an open set for any τ_1-open set $V \subset Y$ having $\tau_1\tau_2$-N-closed complement.*

(e) *$F^-(\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K)))$ is a closed set for any τ_1-closed $\tau_1\tau_2$-N-closed set K.*

(f) *$F^+(G)$ is an open set for any $\tau_1\tau_2$-regularly open set G having $\tau_1\tau_2$-N-closed complement.*

(g) *$F^-(K)$ is a closed set for any $\tau_1\tau_2$-regularly closed $\tau_1\tau_2$-N-closed set K.*

Proof. (a) $\Rightarrow$ (b). Let $x \in X$ and G be any $\tau_1\tau_2$-regularly open set having $\tau_1\tau_2$-N-closed complement such that $F(x) \subset G$. Because F is $\tau_1\tau_2$-upper almost nearly continuous at x and G is $\tau_1\tau_2$-regularly open (and τ_1-open at the same time) set there exists an open neighbourhood U of x such that $U \subset F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G))) = F^+(G)$. This gives an inclusion $F(U) \subset G$.

(b) $\Rightarrow$ (a). Let $x \in X$ and V be a τ_1-open set having $\tau_1\tau_2$-N-closed complement such that $x \in F^+(V)$. By Lemma 1.1 $\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(V))$ is a $\tau_1\tau_2$-regularly open set having $\tau_1\tau_2$-N-closed complement. It is obvious that $F(x) \subset \mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(V))$. Under assumption there exists an open neighbourhood U of x such that $F(U) \subset \mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(V))$. This clearly forces $U \subset F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(V)))$.

(a) $\Rightarrow$ (c). Let $x \in X$ and K be a τ_1-closed $\tau_1\tau_2$- N-closed set such that $x \in F^+(Y \setminus K)$. By the above and assumption there exists an open neighbourhood U of x such that $U \subset F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(Y \setminus K))) = F^+(Y \setminus \mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K))) = X \setminus F^-(\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K)))$. Denoting $H = X \setminus U$ we have that $F^-(\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K))) \subset H$. It is evident that H is a closed proper subset of X.

(c) $\Rightarrow$ (a). Let $x \in X$ and V be a τ_1-open set having $\tau_1\tau_2$-N-closed complement such that $x \in F^+(V)$. Let us denote $K = Y \setminus V$. It is clear that K is a τ_1-closed $\tau_1\tau_2$- N-closed set such that $x \in F^+(Y \setminus K)$. By the assumption there exists a closed set $H \neq X$ such that $x \in X \setminus H$ and $F^-(\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K))) \subset H$. An analysis similar to that in the proof of $(a) \Rightarrow (c)$ shows that $U \subset F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(V)))$, where $U = X \setminus H$ is an open set containing x.

(a) $\Rightarrow$ (d). Let $V \subset Y$ be a τ_1-open set having $\tau_1\tau_2$-N-closed complement. Let $x \in F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(V)))$. Clearly, $\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(V)) \in \tau_1$ and has $\tau_1\tau_2$-N-closed complement. By the definition of $\tau_1\tau_2$-upper almost nearly continuity at a point x, there exists an open neighbourhood U of x such that $U \subset F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(V))))) = F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(V)))$. Since x was arbitrary chosen, the set $F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(V)))$ is an open set.

(d) $\Rightarrow$ (a). Let $x \in X$ and V be a τ_1-open set having $\tau_1\tau_2$-N-closed complement such that $x \in F^+(V)$. Under the assumption the set $U = F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(V)))$ is an open set containing x.

(d) $\Rightarrow$ (e). Let K be a τ_1-closed $\tau_1\tau_2$-N-closed set. Therefore $Y \setminus K$ is τ_1-open and has $\tau_1\tau_2$-N-closed complement. Let us observe that $U = F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(Y \setminus K))) = X \setminus F^-(\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K)))$. The above equality and openness of the set U imply that the set $F^-(\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K)))$ is closed.

(e) $\Rightarrow$ (d). The proof is similar to the above.

(d) $\Rightarrow$ (f). Let G be a $\tau_1\tau_2$-regularly open set having $\tau_1\tau_2$-N-closed complement. Then $F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G))) = F^+(G)$ is an open set.

(f) $\Rightarrow$ (d). Let V be a τ_1-open set having $\tau_1\tau_2$-N-closed complement. It is clear that it suffices to observe that the set $\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(V))$ is $\tau_1\tau_2$-regularly open set having $\tau_1\tau_2$-N-closed complement.

The proof of equivalence (e) $\Leftrightarrow$ (g) runs as the proof of (d) $\Leftrightarrow$ (f) $\square$

Theorem 2.2. *Let $F : X \to (Y, \tau_1, \tau_2)$ be a multifunction. The following statements are equivalent.*

(a) *F is $\tau_1\tau_2$-lower almost nearly continuous.*

(b) *For any $x \in X$ and for any $\tau_1\tau_2$-regularly open set G having $\tau_1\tau_2$-N-closed complement such that $F(x) \cap G \neq \emptyset$ there exists an open neighbourhood U of x such that $F(z) \cap G \neq \emptyset$ for any $z \in U$.*

(c) *For any $x \in X$ and for any τ_1-closed $\tau_1\tau_2$- N-closed set K such that $x \in F^-(Y \setminus K)$ there exists a closed set $H \neq X$ such that $x \in X \setminus H$ and*

$F^+(\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K))) \subset H$.

(d) *$F^-(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(V)))$ is an open set for any τ_1-open set $V \subset Y$ having $\tau_1\tau_2$-N-closed complement.*

(e) *$F^+(\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K)))$ is a closed set for any τ_1-closed $\tau_1\tau_2$-N-closed set K.*

(f) *$F^-(G)$ is an open set for any τ_1-open $\tau_1\tau_2$-regularly open set G having $\tau_1\tau_2$-N-closed complement.*

(g) *$F^+(K)$ is a closed set for any $\tau_1\tau_2$-regularly closed $\tau_1\tau_2$-N-closed set K.*

Theorem 2.3. *Let $F : X \to (Y, \tau_1, \tau_2)$ be a multifunction. The following statements are equivalent.*

(a) *F is $\tau_1\tau_2$-almost nearly continuous.*

(b) *For any $x \in X$ and for any $\tau_1\tau_2$-regularly open sets G_1, G_2 having $\tau_1\tau_2$-N-closed complement such that $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \emptyset$ there exists an open neighbourhood U of x such that $F(z) \subset G_1$ and $F(z) \cap G_2 \neq \emptyset$ for any $z \in U$.*

(c) *For any $x \in X$ and for any τ_1-closed $\tau_1\tau_2$- N-closed sets K_1, K_2 such that $x \in F^+(Y \setminus K_1) \cap F^-(Y \setminus K_2)$ there exists a closed set $H \neq X$ such that $x \in X \setminus H$ and $F^-(\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K_1))) \cup F^+(\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K_2))) \subset H$.*

(d) *$F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(V_1))) \cap F^-(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(V_2)))$ is an open set for any τ_1-open sets $V_1, V_2 \subset Y$ having $\tau_1\tau_2$-N-closed complement.*

(e) *$F^-(\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K_1))) \cup F^+(\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K_2)))$ is a closed set for any τ_1-closed $\tau_1\tau_2$-N-closed sets K_1, K_2.*

(f) *$F^+(G_1) \cup F^-(G_2)$ is an open set for any $\tau_1\tau_2$-regularly open sets G_1, G_2 having closed $\tau_1\tau_2$-N-closed complement.*

(g) *$F^-(K_1) \cap F^+(K_2)$ is a closed set for any $\tau_1\tau_2$-regularly closed $\tau_1\tau_2$-N-closed sets K_1, K_2.*

Proof. (a) $\Rightarrow$ (b). Let $x \in X$ and G_1, G_2 be $\tau_1\tau_2$-regularly open subsets of Y having $\tau_1\tau_2$-N-closed complement such that $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \emptyset$. By $\tau_1\tau_2$-almost nearly continuity there exists an open neighbourhood U of x such that $U \subset F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_1))) \cap F^-(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_2))) = F^+(G_1) \cap F^-(G_2)$.

(b) $\Rightarrow$ (a). It is sufficient to observe that for any $\tau_1\tau_2$-regularly open sets G_1, G_2 having $\tau_1\tau_2$-N-closed complement the sets $\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_1)), \mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_2))$ are $\tau_1\tau_2$-regularly open having $\tau_1\tau_2$-N-closed complement (see Lemma 1.1).

(a) $\Rightarrow$ (c). Let $x \in X$ and K_1, K_2 be τ_1-closed $\tau_1\tau_2$-N-closed sets such that $x \in F^+(Y \setminus K_1) \cap F^-(Y \setminus K_2)$. Therefore $Y \setminus K_1, Y \setminus K_2$ are τ_1-open sets having $\tau_1\tau_2$-N-closed complement. Under assumptions there exists an open neighbourhood U of x such that $U \subset F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(Y \setminus K_1))) \cap F^-(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(Y \setminus K_2))) = X \setminus [F^+(\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K_1))) \cup F^-(\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K_2)))]$. It is clear that $H = X \setminus U$ is closed subset of X and the inclusion $F^+(\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K_1))) \cup F^-(\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K_2))) \subset H$ is satisfied.

(c) $\Rightarrow$ (a). The proof is similar to the proof (a) $\Rightarrow$ (c).

$(a) \Rightarrow (d)$. The statement is a result of Theorems 2.1d) and 2.2d).

(d) $\Rightarrow$ (a). The proof is clear.

(d) $\Rightarrow$ (e). It is enough to observe that the complement of the set

$$F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(A))) \cap F^-(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(B)))$$

is equal to $F^+(\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(A))) \cup F^-(\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(B)))$ for any sets $A, B \subset Y$.

(d) $\Rightarrow$ (f). It is easily seen that the set

$$F^+(G_1) \cap F^-(G_2) = F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_1))) \cap F^-(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_2)))$$

for any $\tau_1\tau_2$-regularly open sets G_1, G_2.

(f) $\Rightarrow$ (d). The proof is a consequence of Lemma 1.1.

(f) $\Leftrightarrow$ (d). The proof is similar to the proof (d) $\Leftrightarrow$ (e). □

3. Almost nearly quasi-continuity

Theorem 3.1. *Let $F : X \to (Y, \tau_1, \tau_2)$ be a multifunction. The following statements are equivalent.*

(a) *F is $\tau_1\tau_2$-upper almost nearly quasi-continuous.*

(b) *For any $x \in X$ and for any $\tau_1\tau_2$-regularly open set $G \subset Y$ having $\tau_1\tau_2$-N-closed complement such that $F(x) \subset G$ and for any open neighbourhood U of x there exists a nonempty open set $W \subset U$ such that $F(z) \subset G$ for any $z \in W$.*

(c) *For any $x \in X$ and for any τ_1-closed $\tau_1\tau_2$-N-closed set $K \subset Y$ such that $x \in F^+(Y \setminus K)$ and for any closed set H such that $x \in X \setminus H$ there exists a closed set M such that $H \subset M, M \neq X$ and $F^-(\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K))) \subset M$.*

(d) *For any $x \in X$ and for any τ_1-open set $G \subset Y$ having $\tau_1\tau_2$-N-closed complement such that $F(x) \subset G$ there exists a semi-open set A such that $x \in A$ and $A \subset F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G)))$.*

(e) *A set $F^+(G)$ is semi-open for any $\tau_1\tau_2$-regularly open set $G \subset Y$ having $\tau_1\tau_2$-N-closed complement.*

(f) *A set $F^-(K)$ is semi-closed for any $\tau_1\tau_2$-regularly closed $\tau_1\tau_2$-N-closed set $K \subset Y$.*

Proof. (a) $\Rightarrow$ (b). Let $x \in X$ and G be a $\tau_1\tau_2$-regularly open subset of Y having $\tau_1\tau_2$-N-closed complement such that $F(x) \subset G$ and let U be an open subset of X and $x \in U$. Under the assumptions (F is $\tau_1\tau_2$-upper almost nearly quasi-continuous) there exits an open nonempty set $W \subset U$ such that $W \subset F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G)))$. As G is $\tau_1\tau_2$-regularly open we have $G = \mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G))$ and, consequently $W \subset F^+(G)$.

(b) $\Rightarrow$ (a). Let now $x \in X$ and G be a τ_1-open set having $\tau_1\tau_2$-N-closed complement such that $F(x) \subset G$ and let U be an open subset of X such that $x \in U$. By Lemma 1.1 we know that the set $\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G))$ is a $\tau_1\tau_2$-regularly open and $Y \setminus \mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G))$ is $\tau_1\tau_2$-N-closed. Because $F(x) \subset \mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G))$ then there exists an open nonempty set $W \subset U$ such that $W \subset F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G)))$.

(a) $\Rightarrow$ (c). Let $x \in X$ and K be a τ_1-closed $\tau_1\tau_2$-N-closed subset of Y such that $x \in F^+(Y \setminus K)$. It is clear that $Y \setminus K$ is an τ_1-open subset of Y having $\tau_1\tau_2$-N-closed complement. Let H be a closed subset of X such that $x \in X \setminus H$. Then $X \setminus H$ is an open set. According to the definition of $\tau_1\tau_2$-upper almost nearly continuity, there exists an open nonempty set $W \subset X \setminus H$ such that $W \subset F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(Y \setminus K)))$. Let us observe that $\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(Y \setminus K)) = \mathrm{int}_{\tau_1}(Y \setminus \mathrm{int}_{\tau_2}(K)) = Y \setminus \mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K))$. It follows that $W \subset F^+(Y \setminus (\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K))) =$

$X \setminus F^-(\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K)))$. Let $M = X \setminus W$, then $X \setminus M \subset X \setminus F^-(\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K)))$ since $F^-(\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K))) \subset M$. It is evident that M is a closed set and $M \neq X$.

(c) $\Rightarrow$ (a). Let $x \in X$, G be an τ_1-open subset of Y having $\tau_1\tau_2$-N-closed complement such that $F(x) \subset G$. Therefore $K = Y \setminus G$ is τ_1-closed $\tau_1\tau_2$-N-closed subset of Y such that $x \in F^+(Y \setminus K)$. Let U be an open neighbourhood of x. Then $H = X \setminus U$ is a closed set such that $x \in X \setminus H$. Under the assumptions there exists a closed set M such that $H \subset M, M \neq X$ and $F^-(\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K)) \subset M$. The last inclusion follows that $X \setminus F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G))) \subset M = X \setminus W$, where $W = X \setminus M$ is an open nonempty set. It was shown that $W \subset F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G)))$. It is easy to see that $W \subset U$.

(a) $\Rightarrow$ (d). Let $x \in X$ and$G$ be an τ_1-open subset of Y having $\tau_1\tau_2$-N-closed complement such that $F(x) \subset G$. We know that for any open neighbourhood U of the point x there exists an open nonempty set $W_U \subset U$ such that $W_U \subset F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G)))$. Let $A = \{x\} \cup \bigcup\{W_U : U \text{ is an open neighbourhood of } x\}$. Hence $A \subset \mathrm{cl}(\mathrm{int}(A))$ and consequently A is a semi-open set and $x \in A$. Additionally $A \subset F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G)))$.

(d) $\Rightarrow$ (a). Let $x \in X$ and G be an τ_1-open subset of Y having $\tau_1\tau_2$-N-closed complement such that $F(x) \subset G$. Let U be an open neighbourhood of x. Under the assumptions there exists a semi-open set A such that $x \in A$ and $A \subset F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G)))$. Let $W = U \cap \mathrm{int}(A)$. Because $U \cap A \neq \emptyset$ then $W \neq \emptyset$. It is easy to check that $W \subset U$ and $W \subset A$. Therefore $W \subset F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G)))$.

(d) $\Rightarrow$ (e). Let G be a $\tau_1\tau_2$-regularly open subset of Y having $\tau_1\tau_2$-N-closed complement and let $x \in F^+(G)$. Then $F(x) \subset G$. Under the assumptions there exists a semi-open set A_x such that $x \in A_x$ and $A_x \subset F^+(G) = F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G)))$. It is easily seen that the set $A = \bigcup\{A_x : x \in F^+(G)\}$ is semi-open and is equal to the set $F^+(G)$.

(e) $\Rightarrow$ (d). Let $x \in X$ and G be an τ_1-open subset of Y having $\tau_1\tau_2$-N-closed complement such that $F(x) \subset G$. Then by Lemma 1.1 $\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G))$ is a $\tau_1\tau_2$-regularly open set having $\tau_1\tau_2$-N-closed complement. Therefore $F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G)))$ is semi-open. Of course $x \in F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G)))$.

(e) $\Rightarrow$ (f). Let K be a $\tau_1\tau_2$-regularly closed $\tau_1\tau_2$-N-closed subset of Y. Then $Y \setminus K$ is a $\tau_1\tau_2$-regularly open having $\tau_1\tau_2$-N-closed complement subset of Y. Under the assumptions $F^+(Y \setminus K)$ is a semi-open set. From this we see that the set $X \setminus F^+(Y \setminus K) = F^-(K)$ is semi-closed.

(f) $\Rightarrow$ (e). The proof is similar to the above. $\square$

Theorem 3.2. *Let $F : X \to (Y, \tau_1, \tau_2)$ be a multifunction. The following statements are equivalent.*

(a) *F is $\tau_1\tau_2$-lower almost nearly quasi-continuous.*

(b) *For any $x \in X$ and for any $\tau_1\tau_2$-regularly open set $G \subset Y$ having $\tau_1\tau_2$-N-closed complement such that $F(x) \cap G \neq \emptyset$ and for any open neighbourhood U of x there exists a nonempty open set $W \subset U$ such that $F(z) \cap G \neq \emptyset$ for any $z \in W$.*

(c) *For any $x \in X$ and for any τ_1-closed $\tau_1\tau_2$-N-closed set $K \subset Y$ such that $x \in F^-(Y \setminus K)$ and for any closed set H such that $x \in X \setminus H$ there exists a* *closed set M such that $H \subset M, M \neq X$ and $F^+(\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K))) \subset M$.*

(d) *For any $x \in X$ and for any τ_1-open set $G \subset Y$ having $\tau_1\tau_2$-N-closed complement such that $F(x) \cap G \neq \emptyset$ there exists a semi-open set A such that $x \in A$ and $A \subset F^-(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G)))$.*

(e) *A set $F^-(G)$ is semi-open for any $\tau_1\tau_2$-regularly open set $G \subset Y$ having $\tau_1\tau_2$-N-closed complement.*

(f) *A set $F^+(K)$ is semi-closed for any $\tau_1\tau_2$-regularly closed $\tau_1\tau_2$-N-closed set $K \subset Y$.*

Theorem 3.3. *Let $F : X \to (Y, \tau_1, \tau_2)$ be a multifunction. The following statements are equivalent.*

(a) *F is $\tau_1\tau_2$-almost nearly quasi-continuous.*

(b) *For any $x \in X$ and for any $\tau_1\tau_2$-regularly open sets $G_1, G_2 \subset Y$ having $\tau_1\tau_2$-N-closed complement such that $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \emptyset$ and for any open neighbourhood U of x there exists a nonempty open set $W \subset U$ such that $F(z) \subset G_1$ and $F(z) \cap G_2 \neq \emptyset$ for any $z \in W$.*

(c) *For any $x \in X$ and for any τ_1-closed $\tau_1\tau_2$-N-closed sets $K_1, K_2 \subset Y$ such that $x \in F^+(Y \setminus K_1) \cap F^-(Y \setminus K_2)$ and for any closed set H such that $x \in X \setminus H$ there exists a closed set M such that $H \subset M, M \neq X$ and $F^-(\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K_1))) \cup F^+(\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K_2))) \subset M$.*

(d) *For any $x \in X$ and for any τ_1-open sets $G_1, G_2 \subset Y$ having $\tau_1\tau_2$-N-closed complement such that $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \emptyset$ there exists a semi-open set A such that $x \in A$ and $A \subset F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_1))) \cap F^-(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_2)))$.*

(e) *A set $F^+(G_1) \cap F^-(G_2)$ is semi-open for any $\tau_1\tau_2$-regularly open sets $G_1, G_2 \subset Y$ having $\tau_1\tau_2$-N-closed complement.*

(f) *A set $F^-(K_1) \cup F^+(K_2)$ is semi-closed for any $\tau_1\tau_2$-regularly closed $\tau_1\tau_2$-N-closed set $K_1, K_2 \subset Y$.*

Proof. (a) $\Rightarrow$ (b). Let $x \in X$ and G_1, G_2 be two $\tau_1\tau_2$-regularly open subsets of Y having $\tau_1\tau_2$-N-closed complement such that $x \in F^+(G_1) \cap F^-(G_2)$. Let U be an open subset of X containing x. Under assumption there exists an open nonempty set $W \subset U$ such that $W \subset F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_1))) \cap F^-(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_2)))$. We have $W \subset F^+(G_1) \cap F^-(G_2)$ because G_1, G_2 are $\tau_1\tau_2$-regularly open sets.

(b) $\Rightarrow$ (a). Let $x \in X$ and G_1, G_2 be two τ_1-open sets having $\tau_1\tau_2$-N-closed complement such that $x \in F^+(G_1) \cap F^-(G_2)$. By Lemma 1.1, $\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_1)))$, $\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_2)))$ are $\tau_1\tau_2$-regularly open sets having $\tau_1\tau_2$-N-closed complement and it is clear that $x \in F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_1))) \cap F^-(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_2)))$. So for any open neighbourhood U of x there exists an open nonempty set $W \subset U$ such that $W \subset F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_1))) \cap F^-(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_2)))$.

(a) $\Rightarrow$ (c). Let $x \in X$ and $K_1, K_2 \subset Y$ be two τ_1-closed $\tau_1\tau_2$-N-closed sets such that $x \in F^+(Y \setminus K_1) \cap F^-(Y \setminus K_2)$. Let H be a closed subset of X such that $x \in U = X \setminus H$. Under the assumptions there exists an open nonempty set $W \subset U$ such that $W \subset F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(Y \setminus K_1))) \cap F^-(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(Y \setminus K_2)))$.

Let us denote $M = X \setminus W$. Then M is a closed set other than X and

$$\begin{aligned} & X \setminus [F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(Y \setminus K_1))) \cap F^-(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(Y \setminus K_2)))] \\ = \ & [X \setminus F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(Y \setminus K_1)))] \cup [X \setminus F^-(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(Y \setminus K_2)))] \\ = \ & F^-(\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K_1))) \cup F^+(\mathrm{cl}_{\tau_1}(\mathrm{int}_{\tau_2}(K_2))) \\ \subset \ & M. \end{aligned}$$

(c) $\Rightarrow$ (a). The proof is similar to the above.

(a) $\Rightarrow$ (d). Let $x \in X$ and G_1, G_2 be two τ_1-open subsets of Y having $\tau_1\tau_2$-N-closed complement such that $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \emptyset$. We know that for any open neighbourhood U of x there exists an open nonempty set $W_U \subset U$ such that $W \subset F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_1))) \cap F^-(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_2)))$. Let $A = \{x\} \cup \bigcup\{W_U :$ U is an open neighbourhood of $x\}$. It is clear that $A \subset \mathrm{cl}(\mathrm{int}(A))$ and hence semi-open. Of course, $x \in A$ and $A \subset F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_1))) \cap F^-(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_2)))$.

(d) $\Rightarrow$ (a). Let $x \in X$ and G_1, G_2 be two τ_1-open subsets of Y having $\tau_1\tau_2$-N-closed complement such that $x \in F^+(G_1) \cap F^-(G_2)$. Let U be an open neighbourhood of x. We know that there exists a semi open set $A \subset F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_1))) \cap F^-(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_2)))$ such that $x \in A$. Let W denote the set $U \cap \mathrm{int}(A)$. Because $U \cap A \neq \emptyset$ then $W \neq \emptyset$. Of course $W \subset U$ and $W \subset A$. Therefore $A \subset F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_1))) \cap F^-(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_2)))$.

(d) $\Rightarrow$ (e). Let G_1, G_2 be two $\tau_1\tau_2$-regularly open subsets of Y having $\tau_1\tau_2$-N-closed complement such that $x \in F^+(G_1) \cap F^-(G_2)$. Let us denote by A_x a semi-open set such that $x \in A_x \subset F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_1))) \cap F^-(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_2))) = F^+(G_1) \cap F^-(G_2)$. Then the set $A = \bigcup_{x \in F^+(G_1) \cap F^-(G_2)} A_x$ is semi-open and equal to the set $F^+(G_1) \cap F^-(G_2)$.

(e) $\Rightarrow$ (d). Let $x \in X$ and G_1, G_2 be two τ_1-open subsets of Y having $\tau_1\tau_2$-N-closed complement such that $F(x) \in G_1$ and $F(x) \cap G_2 \neq \emptyset$. Then by Lemma 1.1 $\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_1))$ and $\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_2))$ are $\tau_1\tau_2$-regularly open sets having $\tau_1\tau_2$-N-closed complement and $F(x) \subset \mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_1))$ and $F(x) \cap \mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_2)) \neq \emptyset$. Under assumption $F^+(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_1))) \cap F^-(\mathrm{int}_{\tau_1}(\mathrm{cl}_{\tau_2}(G_2)))$ is a semi-open set. According to the above remark, the proof is finished.

(e) $\Rightarrow$ (f). Let K_1, K_2 be two $\tau_1\tau_2$-regularly closed $\tau_1\tau_2$-N-closed subsets of Y. Then $F^+(Y \setminus K_1) \cap F^-(Y \setminus K_2)$ is a a semi-open set. The complement of this set is a semi-closed set and it is equal to the set $X \setminus [F^+(Y \setminus K_1) \cap F^-(Y \setminus K_2)] = F^-(K_1) \cup F^+(K_2)$.

(f) $\Rightarrow$ (e). The proof is similar to the above. $\square$

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