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A NOTE ON UNIFORMLY PRIMARY SUBMODULES

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Abstract. We introduce the concept of "uniformly primary submodules" of a module over a commutative ring R, which generalizes the concept of "uniformly primary ideals" of R, a concept that imposes a certain boundedness condition on the usual notion of "primary ideal". Several results on uniformly primary submodules are proved. Also, we characterize uniformly primary submodules of a multiplication module.

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1. Introduction

Uniformly primary ideals in a commutative ring with non-zero identity have been introduced and studied by J. A. Cox and A. J. Hetzel in [4]. In the present paper we introduce a new class of R-submodules of a module, called uniformly primary submodules (see Definition 2.1), and we study it in detail. In Section 2, we consider the relationship among the families of uniformly primary submodules, Noether strongly primary submodules and Mori strongly primary submodules (see Definition 2.2) of a module over a commutative ring R. Also, we provide an important characterization of uniformly primary submodules of an R-module in Theorem 2.4, and give results in Corollaries 3.4 and 3.5 that yield examples of uniformly primary submodules with a specific so-called order. In Section 3, we characterize uniformly primary submodules of a multiplication module. Also, we see that the notion of "uniformly primary submodule" of a multiplication module over a Noetherian ring is equivalent to the (equivalent) notions of Noether strongly primary submodule, Mori strongly primary submodule, and primary submodule.

For the sake of completeness, we state some definitions and notations used throughout. In this paper all rings are commutative with non-zero identity and all modules are unitary. If R is a ring and N is a submodule of an R-module M, the ideal $\{r \in R : rM \subseteq N\}$ is denoted by $(N : M)$. Then $(0 : M)$ is the annihilator of M. A proper submodule N of a module M over a ring R is said to be primary submodule if whenever $rm \in N$, for some $r \in R$, $m \in M$, then $m \in N$ or $r^k \in (N : M)$ for some positive integer k, so $\mathrm{rad}(0 : M/N) = P$ is a prime ideal of R, and N is said to be P-primary submodule. An R-module M

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is called a multiplication module if for each submodule N of M, $N = IM$ for some ideal I of R. In this case we can take $I = (N : M)$. We need the following definitions from [4].

Definition 1.1. *(i) A proper ideal Q of the commutative ring R is uniformly primary if there exists a positive integer n such that whenever $r, s \in R$ satisfy $rs \in Q$ and $r \notin Q$, then $s^n \in Q$. We say that a uniformly primary ideal Q has order k and write $\mathrm{ord}_R(Q) = k$, or simply $\mathrm{ord}(Q) = k$ if the ring R is understood, if k is the smallest positive integer for which the aforementioned property holds.*

(ii) Let R be a commutative ring, and let Q be a P-primary ideal of R.

(a) We call Q a Noether strongly primary ideal if $P^n \subseteq Q$ for some positive integer n. If Q is a Noether strongly primary ideal, then we call the smallest positive integer k for which $P^k \subseteq Q$ the exponent of Q and write $e(Q) = k$.

(b) We call Q a Mori strongly primary ideal if there exists $r \in R - Q$ such that $rP \subseteq Q$.

2. Properties of uniformly primary submodules

Our starting point is the following definition.

Definition 2.1. *Let M be a module over a commutative ring R. A proper submodule N of M is uniformly primary if there exists a positive integer n such that whenever $r \in$ and $m \in M$ satisfy $rm \in N$ and $m \notin N$, then $r^n M \subseteq N$. We say that a uniformly primary submodule N has order k and write $\mathrm{ord}(N) = k$ if k is the smallest positive integer for which the aforementioned property holds.*

It is easy to see that every prime submodule is a uniformly primary submodule of order 1.

Definition 2.2. *Let R be a commutative ring R, M an R-module and N a P-primary submodule of M.*

(a) We call N a Noether strongly primary submodule if $P^n M \subseteq N$ for some positive integer n. If N is a Noether strongly primary submodule, then we call the smallest positive integer k for which $P^k M \subseteq N$ the exponent of N and write $e(N) = k$.

(b) We call N is Mori strongly primary submodule if there exists $m \in M - N$ such that $Pm \subseteq N$.

In this section we list some basic properties concerning uniformly primary submodules of a module over a commutative ring R.

Proposition 2.3. *Let R be a commutative ring R, M an R-module and N a P-primary submodule of M. Then the following hold:*

(i) If N is Noether strongly primary, then N is Mori strongly primary.

(ii) If N is Noether strongly primary, then N is a uniformly primary submodule of M. Moreover, $\operatorname{ord}(N) \leq e(N)$.

(iii) If R is Noetherian, then N is Noether strongly primary.

Proof. (i) Assume that $e(N) = k$ and let $m \in P^{k-1}M - N$. It follows that $Pm \subseteq P^k M \subseteq N$, and so N is a Mori strongly primary submodule of M.

(ii) Suppose that $r \in R$ and $m \in M$ such that $rm \in N$ and $m \notin N$. Then N primary gives $r \in P$; hence $r^{e(N)}M \subseteq P^{e(N)}M \subseteq N$. Therefore, N is a uniformly P-primary submodule of M with $\operatorname{ord}(N) \leq e(N)$.

(iii) Set $I = (0 : M/N)$. Then by assumption, $\operatorname{Rad}(I) = P$ and $P^t \subseteq I$ for some t, as needed. So the concepts of the primary and the Noether strongly primary submodules are equivalent over the class of Noetherian rings (see [4, Proposition 3]). □

Theorem 2.4. *Let M be an R-module. Then N is a uniformly P-primary submodule of M if and only if the following two conditions hold:*

(i) N is a P-primary submodule of M, and

(ii) there exists a positive integer n such that $P = \{r \in R : r^n M \subseteq N\}$. Moreover, $\operatorname{ord}(N) = k$ *if and only if k is the smallest positive integer for which condition (ii) holds.*

Proof. Let N be a uniformly P-primary submodule of M of order k. Clearly, the condition (i) is satisfied. Let $r \in P$. There are a positive integer s and $m \in M$ such that $r^s m \in N$, but $r^{s-1}m \notin N$; hence $r^k M \subseteq N$ since N has order k, and so condition (ii) is established. Conversely, suppose conditions (i) and (ii). Let $r \in R$ and $m \in M$ such that $rm \in N$ and $m \notin N$. By assumption, $r \in P$ and condition (ii) provides for positive integer t, independent of r, such that $r^t M \subseteq N$. Therefore, N is uniformly primary submodule of M. Finally, by the above consideration, $\operatorname{ord}(N) = k$ if and only if k is the smallest positive integer for which condition (ii) holds (see [4, Proposition 8]). □

Proposition 2.5. *Let $N_1 \subseteq N_2$ be uniformly P-primary submodules of an R-module M. Then* $\operatorname{ord}(N_2) \leq \operatorname{ord}(N_1)$.

Proof. Let $\operatorname{ord}(N_1) = m$ and $\operatorname{ord}(N_2) = n$. Then there are elements $r \in R$ and $x \in M$ such that $rx \in N_2$, $x \notin N_2$, $r^{n-1}M \nsubseteq N_2$, and $r^n M \subseteq N_2$; hence $r \in \operatorname{rad}(0 : M/N_1) = P$. Therefore, $r^m M \subseteq N_1 \subseteq N_2$; thus, $m > n - 1$, and so $m \geq n$ (see [4, Proposition 14]). □

Theorem 2.6. *Assume that M is an R-module and let $\{N_i\}_{i \in I}$ be a family of uniformly P-primary submodules of the module M such that* $\max_{i \in I}\{\operatorname{ord}(N_i)\} = k$*, where k is a positive integer. Then $N = \bigcap_{i \in I} N_i$ is a uniformly P-primary submodule of M of order k.*

Proof. It is clear that $\mathrm{rad}(N) = \bigcap_{i\in I} \mathrm{rad}(N_i) = P = \{r \in R : r^k \in N\}$ by Theorem 2.4. Now, let $r \in R$ and $m \in M$ such that $rm \in N$ and $m \notin N$. Then there is an element $j \in I$ such that $rm \in N_j$ and $m \notin N_j$, so $r \in P$, whence $r^k M \subseteq N$. Therefore, N is a uniformly P-primary of M of order at most k. Let N_t $(t \in I)$ be a uniformly P-primary submodule of M of order k. Then Theorem 2.4 gives that k is the smallest positive integer such that $P = \{r \in R : r^n M \subseteq N\}$. It follows that there exists $r \in P$ such that $r^{k-1}M \nsubseteq N_t$, whence $r^{k-1}M \nsubseteq N$. Therefore, N must have order k (see [4, Theorem 15]). □

Proposition 2.7. *Let R be any commutative ring, M and M' R-modules, and $\varphi : M \to M'$ an R-homomorphism. If N' is a uniformly P'-primary submodule of M' such that $\varphi(M) \nsubseteq N'$, then $\varphi^{-1}(N') = N$ is a uniformly P-primary submodule of M with* $\mathrm{ord}(N) \le \mathrm{ord}(N')$.

Proof. First, we show that N is a primary submodule of M. Let $r \in R$ and $x \in M$ such that $rx \in N$ and $x \notin N$. Then $r\varphi(x) \in N'$ with $\varphi(x) \notin N'$; hence $r^n M' \subseteq N'$ for some positive integer n. It suffices to show that $r^n M \subseteq N$. If $r^n y \in r^n M$ for some $y \in M$, then $r^n\varphi(y) \in N'$, so $r^n y \in N$; hence $r^n M \subseteq N$. Therefore, N is primary. Next, assume that $\mathrm{ord}(N') = k$. Then Theorem 2.4 asserts that $P' = \{r \in R : r^k M' \subseteq N'\}$. Let $r \in R$ and $m \in M$ such that $rm \in N$ and $m \notin N$. Then $r\varphi(m) \in N'$ with $\varphi(m) \notin N'$, so $r^k M' \subseteq N'$; hence $r^k M \subseteq N$. It follows that N is a uniformly P-primary submodule of M of order at most k (see [4, Proposition 16]). □

Theorem 2.8. *Let N and L be submodules of an R-module M with $L \subseteq N$. Then N is a uniformly P-primary submodule of M of order k if and only if N/L is a uniformly P-primary of M/L of order k.*

Proof. Suppose that N/L is a uniformly P-primary submodule of M/L of order k. By Proposition 2.7 (using the canonical mapping $M \to M/L$), N is a uniformly P-primary submodule of M of order at most k. By Theorem 2.4, there exists $r \in P$ such that $r^{k-1}(M/L) \nsubseteq N/L$. Thus, $r \in P$ such that $r^{k-1}M \nsubseteq N$, and so the order of N must be k. Conversely, suppose that N is a uniformly P-primary submodule of M of order k. It is easy to see that N/L is a uniformly P-primary submodule of M/L. Moreover, by the proof of the "if" statement above, it must be the case that $\mathrm{ord}(N) = \mathrm{ord}(N/L) = k$ (see [4, Corollary 18]). □

3. Multiplication modules

In this section we characterize uniformly primary submodules of a multiplication module over a commutative ring R.

Proposition 3.1. *Let M be a faithful multiplication R-module and J a uniformly P-primary ideal of R of order k such that $JM \neq M$. Then JM is a uniformly primary submodule of M of order at most k.*

Proof. Suppose $r \in R$ and $x \in M$ such that $rx \in JM$ and $x \notin JM$. Then by [3, Theorem 2] and [4, Propositin 8], $r \in \mathrm{rad}(I) = P = \{s \in R : s^k \in J\}$; hence $r^k \in J \subseteq (JM : M))$. Therefore, JM is uniformly primary submodule of M of order at most k. □

Lemma 3.2. *Let M be an R-module, N a proper R-submodule of M and I an ideal of R such that $I \subseteq \mathrm{Ann}(M)$. Then N is a uniformly primary submodule of M if and only if N is uniformly primary submodule as an R/I-module.*

Proof. The proof is straightforward. □

Theorem 3.3. *The following statements are equivalent for a proper submodule N of a multiplication R-module M.*

(i) N is uniformly P-primary submodule of M.

(ii) $(N : M)$ is uniformly P-primary ideal of R.

(iii) $N = JM$ for some uniformly P-primary ideal J with $\mathrm{Ann}(M) \subseteq J$.

Proof. $(i) \to (ii)$. Assume that N is a uniformly P-primary submodule of M of order k and let $r, s \in R$ such that $rs \in (N : M)$ and $r \notin (N : M)$. Then there is an element $m \in M - N$ such that $rm \notin N$ and $rsm \in N$, so by (i), $s^k \in (N : M)$; hence $(N : M)$ is a uniformly P-primary ideal of R of order k. $(ii) \to (iii)$ is clear.

$(iii) \to (i)$. Since $N = JM \neq M$ and as $R/\mathrm{Ann}(M)$-module, N is uniformly P-primary by Proposition 3.1, so is uniformly P-primary as an R-module of M by Lemma 3.2. □

Corollary 3.4. *Let M be a faithful multiplication R-module and P a prime ideal of R such that P^n is a primary ideal with $P^n M \neq M$, where n is a positive integer. Then $P^n M$ is a uniformly primary submodule of M of order at most n. In particular, if P is a maximal ideal of R, then for each positive integer n, the submodule $P^n M$ is a uniformly primary submodule M of order at most n.*

Proof. Apply Propositin 3.1, [4, Corollary 4] and [1, Proposition 4.2]. □

Corollary 3.5. *Let M be a faithful multiplication module over a ring R with no non-zero nilpotent elements, and let $X = \{X_i\}_{i \in I}$ be a set of indeterminates over R indexed by the set I. Let P be a non-zero prime ideal of $R[X]$ that is generated by polynomials each of which has zero constant term. If n is a positive integer such that P^n is primary ideal of $R[X]$, then $P^n M$ is uniformly primary submodule of M of order n.*

Proof. Apply Proposition 3.1 and [4, Corollary 5]. □

Lemma 3.6. *Let R be a commutative Noetherian ring and M a multiplication R-module. A P-primary submodule N of M is Noether primary if and only if $(N : M)$ is Noether primary ideal of R.*

Proof. Since R is Noetherian, we must have M is Noetherian. Therefore, by Theorem 3.3, $P^{e(N)} \subseteq (N : M)$ if and only $P^{e(N)}M \subseteq (N : M)M = N$ by [2, Theorem 3.1]. □

Proposition 3.7. *Let R be a commutative Noetherian ring, M a multiplication R-module, and N a P-primary R-submodule of M. Then N is a Noether primary submodule of M and, hence, a uniformly primary submodule of M.*

Proof. By Theorem 3.3 and [4, Proposition 21], $(N : M)$ is a Noether primary ideal of R. Now the assertion follows from Lemma 3.6. □

Theorem 3.8. *Let R be a commutative Noetherian ring, M a multiplication R-module, and N an R-submodule of M. Then the following are equivalent:*

(i) N is a uniformly primary submodule of M;

(ii) N is a Noether strongly primary submodule of M;

(iii) N is a Mori strongly primary submodule of M;

(iv) N is a primary submodule of M.

Proof. Apply Propositions 3.7, Proposition 2.3 and [4, Corollary 22]. □

Definition 3.9. *Let M be a module over a commutative ring R.*

(a) R is called a U-ring if and only if for every proper ideal I of R, there are uniformly P_1-primary ideal Q_1, ..., and uniformly P_n-primary ideal Q_n such that $I = Q_1Q_2...Q_n$.

(b) M is called a U-module if every submodule N of M such that $N \neq M$ either is uniformly P-primary or has a uniform factorization $N = Q_1Q_2...Q_nN'$, where $Q_1, ..., Q_n$ are uniformly P_i-primary ideals of R and N' is a uniformly P'-primary submodule.

Theorem 3.10. *Let M be a faithful multiplication module over a U-ring R. Then M is a U-module.*

Proof. Let N be a proper submodule of M. Then $N = IM$ for some ideal I of R. Since R is a U-ring, $I = Q_1Q_2...Q_n$ and so $N = Q_1Q_2...Q_nM$, where Q_i is uniformly P_i-primary ideal of R $(1 \leq i \leq n)$. Since $N \neq M$, there exists i $(1 \leq i \leq n)$ such that $Q_iM \neq M$; hence either N is a uniformly P_i-primary submodule of M or has a uniformly primary factorization by Proposition 3.1. □

Recall that a ring R is ZPI-ring if every ideal of R can be written as a product of prime ideals of R. So every ZPI-ring is a U-ring and then we have the following result:

Corollary 3.11. *If R is a ZPI-ring (resp. Dedekind domain) and M a faithful multiplication R-module, then M is a U-module.*

Theorem 3.12. *Let M be a finitely generated faithful multiplication R-module. If M is a U-module, then R is a U-ring.*

Proof. Let I be an ideal of R with $I \neq R$. Then $IM \neq M$ by [2, Theorem 3.1]. Since M is a U-module, we must have $IM = N_1N_2...N_nN'$, where $N_1, ..., N_n$ are uniformly P_i-primary ideals of R and N' is a uniformly P-primary submodule of M. As M is multiplication, we can write $N' = (N' : M)M$, where $(N' : M)$ is a uniformly P-primary ideal of R by Theorem 3.3. Then $IM = Q_1Q_2...Q_n(N' : M)M$; hence $I = Q_1Q_2...Q_n(N' : M)$ by [2, Theorem 3.1]. Thus R is a U-ring. □

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