Novi Sad J. Math.
Vol. 38, No. 2, 2008, 115-121

ON NEARLY QUASI EINSTEIN MANIFOLDS

Uday Chand De[1], **Abul Kalam Gazi**

Abstract. The objective of the present paper is to study a type of non-flat Riemannian manifold called nearly quasi Einstein manifold. The existence of a nearly quasi Einstein manifold is also proved by a non-trivial concrete example.

AMS Mathematics Subject Classification (2000): 53C25

Key words and phrases: quasi Einstein manifolds, cyclic Ricci tensor, Ricci recurrent

1. Introduction

A Riemannian or a semi-Riemannian manifold (M^n, g), $n = dimM \geq 2$, is said to be an Einstein manifold if the following condition

$$S = \frac{r}{n} g \tag{1.1}$$

holds on M, where S and r denote the Ricci tensor and the scalar curvature of (M^n, g) respectively. According to ([1], p. 432), (1.1) is called the Einstein metric condition. Einstein manifolds play an important role in Riemannian Geometry, as well as in general theory of relativity. Also, Einstein manifolds form a natural subclass of various classes of Riemannian or semi-Riemannian manifolds by a curvature condition imposed on their Ricci tensor ([1], p. 432-433). For instance, every Einstein manifold belongs to the class of Riemannian manifolds (M^n, g) realizing the following relation :

$$S(X,Y) = ag(X,Y) + bA(X)A(Y), \tag{1.2}$$

where $a, b \in \mathbb{R}$ and A is a non-zero 1-form such that

$$g(X,U) = A(X), \tag{1.3}$$

for all vector fields X. Moreover, different structures on Einstein manifolds have also been studied by several authors. In 1993, Tamassy and Binh [13] studied weakly symmetric structures on Einstein manifolds.

A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is defined to be a quasi Einstein manifold [7] if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition (1.1). We shall call A the associated 1-form and U is called the generator of the manifold.

[1]Department of Mathematics, University of Kalyani, Kalyani 741235, West Bengal, India, email: uc_de@yahoo.com

Quasi Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. For instance, the Robertson-Walker spacetime are quasi Einstein manifolds [8]. Also quasi Einstein manifold can be taken as a model of the perfect fluid spacetime in general relativity [3]. So, quasi Einstein manifolds have some importance in the general theory of relativity. Considering this aspect we are motivated to generalize such a manifold.

It is to be noted that M. C. Chaki and R. K. Maity [2] also introduced the notion of quasi Einstein manifolds which is different from that of R. Deszcz [7]. They took a and b as scalars and the generator U of the manifold as a unit vector field.

In this paper the authors introduce a type of non-flat Riemannian manifold (M^n, g) $(n > 2)$ whose Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + bE(X, Y), \tag{1.4}$$

where a and b are non-zero scalars and E is a non-zero $(0, 2)$ tensor. Such a manifold shall be called a nearly quasi Einstein manifold.

It is known ([5], p.39) that the outer product of two covariant vectors is a covariant tensor of type (0,2) but the converse is not true, in general. Hence the manifolds which are quasi Einstein are also nearly quasi Einstein, but the converse is not true, in general. For this we choose the name nearly quasi Einstein.

An n–dimensional nearly quasi Einstein manifold will be denoted by $N(QE)_n$. We shall call E the associated tensor and a and b as associated scalars.

A Riemannian manifold (M^n, g), $n \geq 3$, is called semisymmetric [10] [11] [12] if

$$R.R = 0 \tag{1.5}$$

holds on M.

A Riemannian manifold (M^n, g), $n \geq 3$, is said to be Ricci-symmetric if

$$R.S = 0 \tag{1.6}$$

holds on M. The class of Ricci-semisymmetric manifolds includes the set of Ricci-symmetric manifolds $(\nabla S = 0)$ as a proper subset. Every semisymmetric manifold is Rici-symmetric. The converse statement is not true. Under some additional assumption (1.5) and (1.6) are equivalent for certain manifolds. A review of results related to this problem is given in [6].

A Riemannian manifold (M^n, g) is said to be Ricci-recurrent [9] if its Ricci tensor S of type $(0, 2)$ is non-zero and satisfies the condition

$$(\nabla_X S)(Y, Z) = \alpha(X)S(Y, Z), \tag{1.7}$$

where α is a non-zero 1–form called the 1–form of recurrence.

In 1995, U. C. De, N. Guha and D. Kamilya [4] introduced the notion of generalized Ricci-recurrent manifolds. A non-flat Riemannian manifold (M^n, g), ($n > 2$) is called a generalized Ricci-recurrent manifold if its Ricci tensor S of type (0, 2) satisfies the condition

$$(\nabla_X S)(Y,Z) = \alpha(X)S(Y,Z) + \beta(X)g(Y,Z), \tag{1.8}$$

where α and β are two 1-forms, β is non-zero. Such a manifold is denoted by GR_n. If $\beta = 0$, then the manifold reduces to a Ricci-recurrent manifold introduced by Patterson [9].

The paper is organised as follows:

After preliminaries, in section 3 we have constructed a non-trivial concrete example of a nearly quasi Einstein manifold which is not a quasi Einstein manifold. In section 4 we deduce a necessary and suficient condition for a $N(QE)_n$ to be Ricci semi-symmetric. In section 5 we study conformally flat $N(QE)_n (n > 3)$. A necessary and suficient condition for an $N(QE)_n$ to satisfy Codazzi type of Riccci tensor has been deduced in section 6. Also, in this section we have proved that a Ricci-recurrent $N(QE)_n$ with associated scalar as constant is a generalised E-recurrent manifold.

2. Preliminaries

From (1.4) we get by interchanging X and Y

$$S(Y,X) = ag(Y,X) + bE(Y,X). \tag{2.1}$$

Now, since S and g are symmetric and b is non-zero, subtracting (2.1) from (1.4) we get

$$E(X,Y) - E(Y,X) = 0$$

which implies that the associated tensor E is symmetric.

Let L and Q be two symmetric endomorphisms of the tangent space at each point of the manifold corresponding to the Ricci tensor and the associated tensor E, respectively. Then

$$g(LX,Y) = S(X,Y), \qquad g(QX,Y) = E(X,Y). \tag{2.2}$$

Also, let $\tilde{E}$ be the scalar corresponding to E , that is,

$$\tilde{E} = E(e_i, e_i), \tag{2.3}$$

where $\{e_i\}$, $i = 1, 2, ..., n$, is an orthonormal basis of the tangent space at each point of the manifold.

Now, putting $X = Y = e_i$ in (1.4) we get

$$r = na + b\tilde{E}, \tag{2.4}$$

where r is the scalar curvature.

3. Example of a nearly quasi Einstein manifold

In this section we construct a non-trivial concrete example of a nearly quasi Einstein manifold which is not a quasi Einstein manifold.

Let us consider a Riemannian metric g on $\mathbb{R}^4$ by

$$ds^2 = g_{ij}dx^i dx^j = (x^4)^{\frac{4}{3}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2, \tag{3.1}$$

$(i, j = 1, 2, 3, 4)$. Then the only non-vanishing components of the Christoffel symbols, the curvature tensors and the Ricci tensors are

$$\Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = \frac{2}{3x^4}, \qquad \Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = -\frac{2}{3}(x^4)^{\frac{1}{3}},$$
$$R_{1441} = R_{2442} = R_{3443} = -\frac{2}{9(x^4)^{2/3}}$$

$$R_{11} = R_{22} = R_{33} = -\frac{2}{9(x^4)^{2/3}}, \qquad R_{44} = -\frac{2}{3(x^4)^2}. \tag{3.2}$$

It can be easily shown that the scalar curvature r of the resulting manifold $(\mathbb{R}^4, g)$ is $r = -\frac{4}{3(x^4)^2}$, which is non-vanishing and non-constant.

We shall now show that $\mathbb{R}^4$ is a $N(QE)_4$. Let us consider the associated scalar a, b and the associated tensor E as follows:

$$a = \frac{1}{(x^4)^2}, \qquad b = -\frac{1}{(x^4)^{5/3}} \tag{3.3}$$

and

$$E_{ij}(x) = \begin{cases} \frac{11}{9}x^4, & \text{for i=j=1,2,3,} \\ \frac{5}{3(x^4)1/3}, & \text{for i=j=4,} \\ 0, & \text{for } i \neq j, \end{cases} \tag{3.4}$$

at any point $x \in \mathbb{R}^4$. To verify the relation (1.4), it is sufficient to check the following:

$$R_{11} = ag_{11} + bE_{11}, \tag{3.5}$$
$$R_{22} = ag_{22} + bE_{22}, \tag{3.6}$$
$$R_{33} = ag_{33} + bE_{33}, \tag{3.7}$$
$$R_{44} = ag_{44} + bE_{44}, \tag{3.8}$$

since for the other cases (1.4) holds trivially. By (3.2), (3.3) and (3.4) we get

$$\begin{aligned} R.H.S.\ of\ (3.5) &= ag_{11} + bE_{11} \\ &= \frac{1}{(x^4)^2} \times (x^4)^{4/3} - \frac{1}{(x^4)^{5/3}} \times \frac{11}{9}(x^4) \\ &= \frac{1}{(x^4)^{2/3}} - \frac{11}{9(x^4)^{2/3}} = -\frac{2}{9(x^4)^{2/3}} \\ &= R_{11} = L.H.S.\ of\ (3.5). \end{aligned}$$

By similar argument it can be shown that (3.6), (3.7) and (3.8) are also true. Hence $\mathbb{R}^4$ equipped with the metric g, given in (3.1), is a $N(QE)_4$.

It is to be noted that (1.4) can be satisfied by a number of scalars a, b and tensor E, namely by those which fulfil (3.5), (3.6), (3.7) and (3.8).

We now show that $(\mathbb{R}^4, g)$ is not a quasi-Einstein manifold.

If possible, let $(\mathbb{R}^4, g)$ be a quasi-Einstein manifold. Then we have to show that

$$R_{ij} = ag_{ij} + bA_iA_j, \tag{3.9}$$

$(i, j = 1, 2, 3, 4)$. Now for $i = j$, we get from (3.9)

$$R_{ii} = ag_{ii} + bA_iA_i, \tag{3.10}$$

for all $i = 1, 2, 3, 4$. Since $R_{ii} \neq 0$ and $g_{ii} \neq 0$, we a can chose $a \neq 0$, $b \neq 0$ and $A_i \neq 0$ for all $i = 1, 2, 3, 4$ such that (3.10) holds. But for these values of $a, \quad b \quad and \quad A_i$ the equation

$$R_{ij} = ag_{ij} + bA_iA_j$$

for $i \neq j$, can not be satisfied because for $i \neq j$, $R_{ij} = g_{ij} = 0$ but $A_i \neq 0$.

So $(\mathbb{R}^4, g)$ is not a quasi Einstein manifold. This shows that an $N(QE)_n$ is not necessarily a quasi Einstein manifold. Thus we can state the following:

Theorem 3.1. *Let $(\mathbb{R}^4, g)$ be a Riemannian manifold endowed with the metric given by*

$$ds^2 = g_{ij}dx^idx^j = (x^4)^{\frac{4}{3}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2,$$

$(i, j = 1, 2, 3, 4)$*. Then $(\mathbb{R}^4, g)$ is an $N(QE)_4$ with non-zero and non-constant scalar curvature which is not a quasi Einstein manifold.*

4. Ricci semi-symmetric nearly quasi Einstein manifold

In this section we deduce a necessary and suficient condition for an $N(QE)_n$ to be Ricci semi-symmetric.

We first assume that an $N(QE)_n$ is a Ricci semi-symmetric space. Then from (1.6) we get

$$(R(X, Y) \cdot S)(Z, W) = -S(R(X, Y)Z, W) - S(Z, R(X, Y)W) = 0. \tag{4.1}$$

Now, since the curvature tensor $\tilde{R}$ of type (0,4), defined by

$$g(R(X, Y)Z, W) = \tilde{R}(X, Y, Z, W)$$

is skew-symmetric where R is the curvature tensor of type (1,3), we get from (1.4) and (4.1) by taking account of the fact that $b \neq 0$

$$E(R(X, Y)Z, W) + E(Z, R(X, Y)W) = 0, \tag{4.2}$$

which implies that

$$(R(X,Y)\cdot E)(Z,W)=0. \tag{4.3}$$

So the manifold is E semi-symmetric.

Again, let us suppose that the manifold is E semi-symmetric, that is, $R\cdot E = 0$ holds in a $N(QE)_n$. Then (4.2) holds. Now using (1.4), (4.2) and the skew-symmetric properties of $\tilde{R}$ we get after some calculations that $R\cdot S = 0$, which implies that the manifold is Ricci semi-symmetric. Hence the following theorem holds:

Theorem 4.1. *An $N(QE)_n$ is Ricci semi-symmetric if and only if it is E semi-symmetric.*

5. $N(QE)_n$ with constant associated scalars

In this section we assume that the associated scalars of an $N(QE)_n$ are constants, that is, a and b are constants.

Now, if an $N(QE)_n$ satisfies a Codazzi type of Ricci tensor then its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies

$$(\nabla_X S)(Y,Z)-(\nabla_Z S)(Y,X)=0. \tag{5.1}$$

Taking covariant differentiation to the both sides of (1.4) we get

$$(\nabla_X S)(Y,Z)=b(\nabla_X E)(Y,Z). \tag{5.2}$$

Since $b\neq 0$, the equations (5.1) and (5.2) together imply that the associated tensor E is of Codazzi type. This leads to the following theorem:

Theorem 5.1. *An $N(QE)_n$ with associated scalars as constants satisfies Codazzi type of Ricci tensor if and only if its associated tensor is of Codazzi type.*

Next we consider an $N(QE)_n$ that is Ricci-recurrent. Then its Ricci tensor S satisfies (1.7). Now, from (2.4) we have

$$dr(X)=b\; d\tilde{E}(X). \tag{5.3}$$

Again, contracting (1.7) we get

$$dr(X)=\alpha(X)r. \tag{5.4}$$

Equations (5.3) and (5.4) give

$$\alpha(X)=\frac{b}{r}d\tilde{E}(X). \tag{5.5}$$

From (1.7), (5.2), (1.4) and (5.5) we see that

$$(\nabla_X E)(Y,Z)=\frac{a\, d\tilde{E}(X)}{r}g(Y,Z)+\frac{b\, d\tilde{E}(X)}{r}E(Y,Z),$$

which implies that the manifold is generalized E-recurrent. Thus we can state the following:

Theorem 5.2. *A Ricci-recurrent $N(QE)_n$ with associated scalars as constants is a generalized E-recurrent manifold.*

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Received by the editors June 28, 2008