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A REMARK ON REGULAR STURM-LIOUVILLE SYSTEM

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Abstract. A self-contained proof of a classical (text-book) oscillation theorem for a regular Sturm-Liouville problem is presented.

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The (text-book) oscillation theorem states that the regular Sturm-Liouville system

$$(P(x)y')' + Q(x,\lambda)y = 0, \quad x \in [a,b], \qquad \lambda \in \mathbb{R}$$

$$hy(a) + h'y'(a) = ky(b) + k'y'(b) = 0,$$

where h, h' and k, k' are given constants not simultaneously equal to zero, possesses increasing sequence of eigenvalues $\{\lambda_n\}$, tending to infinity with n, and a sequence of corresponding eigenfunctions $y_n(x)$ having n zeros in the interval (a,b).

For the proof of that fact the following result is crucial:

Theorem 1. *Let $P(x)$ be continuous and positive for $x \in [a,b]$, $Q(x,\lambda)$ continuous on $[a,b] \times \mathbb{R}$ and such that*

$$Q(x,\lambda) \to \infty, \ \text{as } \lambda \to \infty \tag{1}$$

uniformly in $x \in [a,b]$. Then for the solution $\theta(x) = \theta(x,\lambda)$ of the initial value problem

$$\begin{aligned} &a) \quad \theta'(x) = Q(x,\lambda)\sin^2\theta(x) + \tfrac{1}{P(x)}\cos^2\theta(x) \\ &b) \quad \theta(a,\lambda) = \gamma, \ 0 \le \gamma < \pi, \ \text{for each } \lambda \in \mathbb{R} \end{aligned} \tag{2}$$

there holds

$$\lim_{\lambda\to\infty} \theta(x,\lambda) = \infty \text{ for each } x \in (a,b].$$

The aim of this paper is to present a short and self-contained proof of that result at variance to the ones known to us (see, for example [1]).

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Proof. Suppose to the contrary that for at least one $x^* \in (a, b]$ and for at least one sequence $\{\mu_\nu\}$ such that $\mu_\nu \to \infty$, as $\nu \to \infty$, there exists an $M \in (0, \infty)$ such that

$$\lim_{\nu\to\infty} \theta(x^*, \mu_\nu) = M. \tag{3}$$

First observe that, due to (1), for any constant $M_1 > 0$ there exists $m = m(M_1)$ such that for all $x \in [a, b]$ and $\lambda > m$

$$Q(x, \lambda) > M_1, \tag{4}$$

which, by (2a) imply $\theta'(x, \lambda) > 0$ for $x \in [a, b]$ and $\lambda > m$, so that the solution $\theta(x, \lambda)$ is strictly increasing in x for all $\lambda > m$.

Put $I := (a, x^*]$, $k_0 = \left[\frac{M}{\pi}\right]$ and choose δ such that $\delta \in (0, \frac{\pi}{2})$. For $k = 0, 1, \ldots, k_0 + 1$, one can define the following closed intervals

$$I_k(\mu_\nu) := \{x \in I : |\theta(x, \lambda) - k\pi| \le \delta\} = [x_k, x'_k], \tag{5}$$

where the end points x_k, x'_k depend on μ_ν. Notice that the intervals $I_0(\mu_\nu)$ and $I_{k_0+1}(\mu_\nu)$ are empty for $\gamma \ge \delta$ and $(k_0 + 1)\pi - M \ge \delta$ but the others are never such due to (2b) and the monotonicity of $\theta(x)$.

Further put

$$I^1(\mu_\nu) = \bigcup_{k=0}^{k_0+1} I_k(\mu_\nu), \quad I^2(\mu_\nu) = I \setminus I^1(\mu_\nu).$$

Then, in view of (2a) and (4), the following estimates hold for $\mu_\nu > m$:

$$\begin{aligned} \theta'(x, \mu_\nu) &\ge \tfrac{\cos^2\delta}{P(x)} \ge M_2 > 0 \quad &&\text{for } x \in I^1(\mu_\nu) \\ \theta'(x, \mu_\nu) &\ge M_1 \cdot \sin^2\delta \quad &&\text{for } x \in I^2(\mu_\nu). \end{aligned} \tag{6}$$

By applying the mean value theorem over each of the intervals $I_k(\mu_\nu)$ and their complements, one obtains

$$\theta(x^*, \mu_\nu) - \gamma = \sum_{k=0}^{k_0+1} \theta'(\xi_k, \mu_\nu)(x'_k - x_k) + \sum_{k=0}^{k_0} \theta'(\eta_k, \mu_\nu)(x_{k+1} - x'_k) \tag{7}$$

where ξ_k, η_k belong to the corresponding (open) intervals. Denote the sum of the lengths of intervals $I_k(\mu_\nu)$ by $d(I^1(\mu_\nu))$, so that

$$d(I^2(\mu_\nu)) = x^* - a - d(I^1(\mu_\nu)). \tag{8}$$

Then, equality (7) and estimate (6) yield for $\mu_\nu > m$

$$\theta(x^*, \mu_\nu) \ge M_2 d(I^1(\mu_\nu)) + M_1 d(I^2(\mu_\nu)) \sin^2\delta. \tag{9}$$

Since M_1 is arbitrary, the above inequality will lead to a contradiction provided that $d(I^2(\mu_\nu))$ is bounded below.

But, by applying the mean value theorem over each of intervals $I_k(\mu_\nu)$, and due to (6), one obtains

$$d(I^1(\mu_\nu)) = \sum_{k=0}^{k_0+1} d(I_k(\mu_\nu)) \leq 2\delta \sum_{k=0}^{k_0+1} \frac{1}{\theta'(\xi_k)} \leq 2\delta \frac{k_0+2}{M_2}.$$

Whence, for conveniently chosen δ, (8) implies

$$d(I^2(\mu_\nu)) \geq x^* - a - 2\delta \frac{k_0+2}{M_2} \geq M_3 > 0.$$

Therefore, one can choose M_1 (sufficiently large) such that, in virtue of (9)

$$\theta(x^*, \mu_\nu) > M \text{ for all } \mu_\nu > m$$

contradicting (3).

It is worthwhile to add that for $Q(x,\lambda) = \lambda r(x) + q(x)$ the hypothesis (1) is fulfilled if $r(x)$ and $q(x)$ are continuous and $r(x) > 0$ for $x \in [a,b]$, which, therefore, are the sole hypotheses needed in this special case important in applications.

References

[1] Birkhoff, G., Rota, G. C., Ordinary differential equations. New York, 1978.

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