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THE SUBSPACES OF HAMILTON SPACES OF HIGHER ORDER

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Abstract. To introduce the theory of subspaces in the Hamilton spaces of higher order, H, it was necessary to solve several difficulties, because the classical theory of subspaces could not be applied. In almost all theories the m-dimensional subspace in the n-dimensional space was given by the introduction of m-parameters and $n-m$ normal vectors N, but the transformation of their coordinates was always a problem.

Here, we introduce in H two complementary family of subspaces H_1 and H_2. In this way we obtain the complicated coordinate transformations expressed in elegant matrix form in H, H_1 and H_2, and determine their connections. This method allows us to obtain the transformations of the natural bases $\bar{B}$, $\bar{B}_1$ and $\bar{B}_2$ of $T(H)$, $T(H_1)$ and $T(H_2)$ further $\bar{B}^*$, $\bar{B}_1^*$ and $\bar{B}_2^*$ of $T^*(H)$, $T^*(H_1)$ and $T^*(H_2)$. As the elements of the natural bases are not transforming as tensors the adapted bases B, B_1, B_2 of $T(H)$, $T(H_1)$, and $T(H_2)$ are introduced using the matrices N, N_1 and N_2, respectively. For the dual spaces $T^*(H)$, $T^*(H_1)$ and $T^*(H_2)$ the adapted bases are B^*, B_1^* and B_2^* formed with the matrices M, M_1 and M_2, respectively.

It is proved that N and M, N_1 and M_1, N_2 and M_2 are inverse matrices to each other if B^* is dual to B, B_1^* is dual to B_1 and B_2^* is dual to B_2. The main result is the construction of adapted basis $B' = B_1 \cup B_2$ and $B^{*\prime} = B_1^* \cup B_2^*$ of $T(H)$ and $T^*(H)$ in such a way that the elements of B' and $B^{*\prime}$ are transforming as tensors and the tensor from space H can be decomposed as a sum of projections on H_1 and H_2. It is obtained by the determination of the relations between N, N_1 and N_2 further between M, M_1, and M_2. This very important result allows us to study the connections, torsion and curvature tensors, Jacobi fields, sprays and other invariants in the subspaces and surrounding space and determine their relations which will be done later on.

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1. The natural and adapted basis of $T(H)$ and $T^*(H)$

Let us denote by H the $(k+2)n$ dimensional manifold, where some point $p \in H$ in some local chart (U,φ) has the coordinates:

$$(x^a, p_{0a}, p_{1a}, \dots, p_{ka}) = (x, p_0, p_1, \dots, p_k) = (x^a, p_{Aa}),$$
$$a, b, c, d, \dots = \overline{1,n}, \quad A, B, C, D, \dots = \overline{0,k}.$$

If $(x^{a'}, p_{0a'}, \dots, p_{ka'})$ are the coordinates of the same point p in the coordinate chart (U', φ'), then the allowable coordinate transformation in H are given by

$$(1.1) \qquad x^{a'} = x^{a'}(x^a) \Leftrightarrow x^a = x^a(x^{a'})$$

$$p_{0a'} = {}^{(0)}B^a_{a'} p_{0a}, \quad {}^{(0)}B^a_{a'} = \frac{\partial x^a}{\partial x^{a'}} = \partial_{a'} x^a$$

$$p_{1a'} = \binom{1}{0} {}^{(1)}B^a_{a'} p_{0a} + \binom{1}{1} {}^{(0)}B^a_{a'} p_{1a}$$

$$p_{2a'} = \binom{2}{0} {}^{(2)}B^a_{a'} p_{0a} + \binom{2}{1} {}^{(1)}B^a_{a'} p_{1a} + \binom{2}{2} {}^{(0)}B^a_{a'} p_{2a}, \dots,$$

$$p_{ka'} = \binom{k}{0} {}^{(k)}B^a_{a'} p_{0a} + \binom{k}{1} {}^{(k-1)}B^a_{a'} p_{1a} + \dots + \binom{k}{k} {}^{(0)}B^a_{a'} p_{ka},$$

$$(1.2) \qquad {}^{(A)}B^a_{a'} = \frac{d^A \, {}^{(0)}B^a_{a'}}{dt^A}, \quad A = \overline{0,k}.$$

It is supposed that the C^∞ transformation $x^{a'} = x^{a'}(x^a)$ is $1-1$ and its inverse transformation $x^a = x^a(x^{a'})$, $a = \overline{1,n}$ is also C^∞. It can be proved:

Theorem 1.1. *The transformations of type (1.1) form a pseudo-group.*

A nice example of H can be obtained if we define

$$(1.3) \qquad p_{0a} = \frac{\partial}{\partial x^a}, \quad p_{1a} = \frac{d}{dt} p_{0a}, \dots, p_{ka} = \frac{d^k}{dt^k} p_{0a}.$$

Using the product rule for differentiation with respect to t, where $x^a = x^a(t)$, $p_{0a} = p_{0a}(t)$ are C^∞ functiones, we obtain all relations of (1.1).

From (1.1)-(1.3) it follows that for this example

$$(1.4) \qquad {}^{(0)}B^a_{a'} = p_{0a'}(x^a), \quad {}^{(1)}B^a_{a'} = p_{1a'}(x^a), \dots, {}^{(k)}B^a_{a'} = p_{ka'}(x^a).$$

The new form of (1.1) is obtained if (1.4) is substituted in (1.1).

In the further examinations it will be supposed that $p_{Aa}(A = \overline{0,k})$ are arbitrary independent variables whose transformation law is prescribed by (1.1).

The natural basis of $T(H)$ is

$$\bar{B} = \{\partial_a, \partial^{0a}, \partial^{1a}, \dots, \partial^{ka}\}, \quad \partial_a = \frac{\partial}{\partial x^a}, \quad \partial^{Aa} = \frac{\partial}{\partial p_{Aa}}, A = \overline{0,k}. \tag{1.5}$$

Theorem 1.2. *The elements of the natural basis $\bar{B}$ of $T(H)$ transform in the following way*

$$\begin{aligned}
\partial_a &= {}^{(0)}B_a^{a'}\partial_{a'} + (\partial_a p_{0a'})\partial^{0a'} + (\partial_a p_{1a'})\partial^{1a'} + \dots + (\partial_a p_{ka'})\partial^{ka'} \\
\partial^{0a} &= \binom{0}{0}{}^{(0)}B_{a'}^{a}\partial^{0a'} + \binom{1}{0}{}^{(1)}B_{a'}^{a}\partial^{1a'} + \dots + \binom{k}{0}{}^{(k)}B_{a'}^{a}\partial^{ka'}, \\
\partial^{1a} &= \binom{1}{1}{}^{(0)}B_{a'}^{a}\partial^{1a'} + \binom{2}{1}{}^{(1)}B_{a'}^{a}\partial^{2a'} + \dots + \binom{k}{1}{}^{(k-1)}B_{a'}^{a}\partial^{ka'}, \\
\partial^{2a} &= \binom{2}{2}{}^{(0)}B_{a'}^{a}\partial^{2a'} + \binom{3}{2}{}^{(1)}B_{a'}^{a}\partial^{3a'} + \dots + \binom{k}{2}{}^{(k-2)}B_{a'}^{a}\partial^{ka'}, \dots, \\
\partial^{ka} &= \binom{k}{k}{}^{(0)}B_{a'}^{a}\partial^{ka'}.
\end{aligned} \tag{1.6}$$

If we introduce the notations:

$$[\partial^{(a)}] = [\partial_a \partial^{0a} \partial^{1a} \dots \partial^{ka}], \quad [\partial^{(a')}] = [\partial_{a'} \partial^{0a'} \partial^{1a'} \dots \partial^{ka'}], \tag{1.7}$$

$$[B_{(a')}^{(a)}] = \begin{bmatrix}
\partial_a x^{a'} & 0 & 0 & 0 & 0 \\
\partial_a p_{0a'} & \binom{0}{0}{}^{(0)}B_{a'}^{a} & 0 & 0 & 0 \\
\partial_a p_{1a'} & \binom{1}{0}{}^{(1)}B_{a'}^{a} & \binom{1}{1}{}^{(1)}B_{a'}^{a} & \binom{1}{1}{}^{(0)}B_{a'}^{a} & 0 \\
\dots & \dots & \dots & \dots & \dots \\
\partial_a p_{ka'} & \binom{k}{0}{}^{(k)}B_{a'}^{a} & \binom{k}{1}{}^{(k-1)}B_{a'}^{a} & \dots & \binom{k}{k}{}^{(0)}B_{a'}^{a}
\end{bmatrix} \tag{1.8}$$

then (1.6) can be written in the form

$$[\partial^{(a)}] = [\partial^{(a')}][B_{(a')}^{(a)}] \Rightarrow [\partial^{(a)}]^T = \left([\partial^{(a')}][B_{(a')}^{(a)}]\right)^T = [B_{(a')}^{(a)}]^T[\partial^{(a')}]^T. \tag{1.9}$$

Theorem 1.3. *The partial derivatives of the variables are connected by:*

$$\frac{\partial p_{0a'}}{\partial p_{0a}} = \frac{\partial p_{1a'}}{\partial p_{1a}} = \cdots = \frac{\partial p_{ka'}}{\partial p_{ka}} = {}^{(0)}B^a_{a'} = p_{0a'}(x^a) \tag{1.10}$$

$$\frac{\partial p_{1a'}}{\partial p_{0a}} = {}^{(1)}B^a_{a'} = p_{1a'}(x^a),$$

$$\frac{\partial p_{2a'}}{\partial p_{1a}} = \binom{2}{1}\frac{\partial p_{1a'}}{\partial p_{0a}} = \binom{2}{1}{}^{(1)}B^a_{a'} = \binom{2}{1}p_{1a'}(x^a), \ldots,$$

$$\frac{\partial p_{3a'}}{\partial p_{2a}} = \frac{3}{2}\frac{\partial p_{2a'}}{\partial p_{1a}} = \frac{3}{2}\cdot\frac{2}{1}\frac{\partial p_{1a'}}{\partial p_{0a}} = \binom{3}{2}{}^{(1)}B^a_{a'} = \binom{3}{2}p_{1a'}(x^2), \ldots,$$

$$\frac{\partial p_{(A+B)a'}}{\partial p_{Ba}} = \frac{A+B}{B}\frac{\partial p_{(A+B-1)a'}}{\partial p_{(B-1)a}} = \cdots$$

$$\cdots = \binom{A+B}{B}\frac{\partial p_{Aa'}}{\partial p_{0a}} = \binom{A+B}{B}{}^{(A)}B^a_{a'}.$$

The natural basis $\bar{B}^*$ of $T^*(H)$ is

$$\bar{B}^* = \{dx^a, dp_{0a}, dp_{1a}, \ldots, dp_{ka}\}. \tag{1.11}$$

From the relation

$$x^{a'} = x^{a'}(x^a), p_{0a'} = p_{0a}(x^a, p_{0a}), \ldots, p_{ka'} = p_{ka}(x^a, p_{0a}, p_{1a}, \ldots, p_{ka})$$

we have

Theorem 1.4. *The elements of the natural basis $\bar{B}^*$ are transforming in the following way*

$$dx^{a'} = \frac{\partial x^{a'}}{\partial x^a}dx^a \tag{1.12}$$

$$dp_{0a'} = \frac{\partial p_{0a'}}{\partial x^a}dx^a + \frac{\partial p_{0a'}}{\partial p_{0a}}dp_{0a}$$

$$dp_{1a'} = \frac{\partial p_{1a'}}{\partial x^a}dx^a + \frac{\partial p_{1a'}}{\partial p_{0a}}dp_{0a} + \frac{\partial p_{1a'}}{\partial p_{1a}}dp_{1a}, \ldots$$

$$\vdots$$

$$dp_{ka'} = \frac{\partial p_{ka'}}{\partial x^a}dx^a + \frac{\partial p_{ka'}}{\partial p_{0a}}dp_{0a} + \frac{\partial p_{ka'}}{\partial p_{1a}}dp_{1a} + \cdots + \frac{\partial p_{ka'}}{\partial p_{ka}}dp_{ka}.$$

Using (1.10) and the notation

$$(1.13)\qquad [d_{(a')}] = \begin{bmatrix} dx^{a'} \\ dp_{0a'} \\ dp_{1a'} \\ \vdots \\ dp_{ka'} \end{bmatrix}, \quad [d_{(a)}] = \begin{bmatrix} dx^{a} \\ dp_{0a} \\ dp_{1a} \\ \vdots \\ dp_{ka} \end{bmatrix}$$

we have the shorter form of (1.12) as follows:

$$(1.14)\qquad [d_{(a')}] = [B^{(a)}_{(a')}][d_{(a)}].$$

Theorem 1.5. *If the bases* $\bar{B}^*$ *and* $\bar{B}$ *are dual to each other, then* $\bar{B}'^* = \{dx^{a'}, dp_{0a'}, dp_{1a'}, \dots, dp_{ka'}\}$ *and* $\bar{B}' = \{\partial_{a'}, \partial^{0a'}, \partial^{1a'}, \dots, \partial^{ka'}\}$ *are also dual to each other.*

Proof. From (1.9) it follows

$$(1.15)\qquad [\partial^{(b')}] = [\partial^{(c)}][B^{(b')}_{(c)}], \quad [B^{(a)}_{(a')}][B^{(b')}_{(a)}] = \delta^{b'}_{a'} I.$$

Using the assumption

$$[d_{(b)}][\partial^{(a)}] = \delta^a_b I,$$

(1.14) and (1.15) we get

$$[d_{(a')}][\partial^{(b')}] = [B^{(a)}_{a'}][d_{(a)}][\partial^{(c)}][B^{(b')}_{(c)}] =$$

$$[B^{(a)}_{(a')}]\delta^c_a I[B^{(b')}_{(c)}] = [B^{(a)}_{(a')}][B^{(b')}_{(a)}] = \delta^{b'}_{a'} I.$$

□

From (1.6) and (1.12) it is obvious that the elements of the natural bases $\bar{B}$ and $\bar{B}^*$ are not transforming as tensors. To obtain more convenient bases of $T(H)$ and $T^*(H)$ we construct the so-called adapted bases B and B^*.

The adapted basis B of $T(H)$ will be denoted by

$$(1.16)\qquad B = \{\delta_a, \delta^{0a}, \delta^{1a}, \dots, \delta^{ka}\}.$$

We shall use the notations

$$(1.17)\qquad [\delta^{(a)}] = [\delta_a \delta^{0a} \delta^{1a} \dots \delta^{ka}]$$

$$
[N^{(a)}_{(b)}] = \begin{bmatrix} \delta^b_a & 0 & 0 & 0 & \cdots & 0 \\ -N_{a0b} & \delta^a_b & 0 & 0 & \cdots & 0 \\ -N_{a1b} & -N^{0a}_{1b} & \delta^a_b & 0 & \cdots & 0 \\ -N_{a2b} & -N^{0a}_{2b} & -N^{1a}_{2b} & \delta^a_b & \cdots & 0 \\ \vdots & & & & & \\ -N_{akb} & -N^{0a}_{kb} & -N^{1a}_{kb} & -N^{2a}_{kb} & \cdots & \delta^a_b \end{bmatrix}. \tag{1.18}
$$

Definition 1.1. . *The adapted basis B of $T(H)$ is defined by*

$$
[\delta^{(a)}] = [\partial^{(b)}][N^{(a)}_{(b)}] \ \textit{i.e.}\ [\delta^{(a)}]^T = [N^{(a)}_{(b)}]^T[\partial^{(b)}]^T. \tag{1.19}
$$

From this relation it is obvious that the elements of B are linear combination of the elements of $\bar{B}$, where the coefficients N are function of the coordinates of a point $p \in H$.

Theorem 1.6. *The necessary and sufficient conditions for elements of the basis B of $T(H)$ to transform as d-tensor, i.e.*

$$
\delta_a = {}^{(0)}B^{a'}_a \delta_{a'} \quad \delta^{Aa} = {}^{(0}B^a_{a'}\delta^{Aa'}, \quad A = \overline{0,k} \tag{1.20}
$$

is the following matrix equation

$$
[N^{(a')}_{(b')}][{}^{(0)}B^{(a)}_{(a')}] = [B^{(b)}_{(b')}][N^{(a)}_{(b)}], \tag{1.21}
$$

where

$$
[{}^{(0)}B^{(a)}_{(a')}] = \begin{bmatrix} {}^{(0)}B^{a'}_a & 0 & 0 & \cdots & 0 \\ 0 & {}^{(0)}B^a_{a'} & 0 & \cdots & 0 \\ 0 & 0 & {}^{(0)}B^a_{a'} & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & {}^{(0)}B^a_{a'} \end{bmatrix}. \tag{1.22}
$$

This matrix will appear frequently later on. It is important to remark, that in the above matrix the element in the place (1.1) differs from the other elements on the main diagonal.

Proof. Equations (1.20) can be written in the matrix form as follows

$$
[\delta^{(a)}] = [{}^{(0)}B^{(a)}_{(a')}][\delta^{(a')}]. \tag{1.23}
$$

Using Definition 1.1 or (1.19) we can write (1.23) as:

$$[\partial^{(b)}][N^{(a)}_{(b)}] = [{}^{(0)}B^{(a)}_{(a')}][\partial^{(b')}][N^{(a')}_{(b')}].$$

The substitution of (1.9) into the above equation and the fact that $[{}^{(0)}B^{(a)}_{(a')}]$ is a diagonal matrix result

$$[\partial^{(b')}][B^{(b)}_{(b')}][N^{(a)}_{(b)}] = [\partial^{(b')}][N^{(a')}_{(b')}][{}^{(0)}B^{(a)}_{(a')}].$$

The above equation is satisfied if

$$[B^{(b)}_{(b')}][N^{(a)}_{(b)}] = [N^{(a')}_{(b')}][{}^{(0)}B^{(a)}_{(a')}].$$

i.e. when (1.21) is valid. □

The elements of the adapted basis B^* of $T^*(H)$ will be denoted by

$$B^* = \{\delta x^a, \delta p_{0a}, \delta p_{1a}, \dots, \delta p_{ka}\}. \tag{1.24}$$

The following notations will be used:

$$[\delta_{(a)}] = \begin{bmatrix} \delta x^a \\ \delta p_{0a} \\ \delta p_{1a} \\ \delta p_{2a} \\ \vdots \\ \delta p_{ka} \end{bmatrix} \qquad [M^{(b)}_{(a)}] = \begin{bmatrix} \delta^a_b & 0 & 0 & 0 & \cdots & 0 \\ M_{a0b} & \delta^b_a & 0 & 0 & \cdots & 0 \\ M_{a1b} & M^{0b}_{1a} & \delta^b_a & 0 & \cdots & 0 \\ M_{a2b} & M^{0b}_{2a} & M^{1b}_{2a} & \delta^b_a & \cdots & 0 \\ \vdots & & & & & \\ M_{akb} & M^{0b}_{ka} & M^{1b}_{ka} & M^{2b}_{ka} & \cdots & \delta^b_a \end{bmatrix}. \tag{1.25}$$

Definition 1.2. . *The adapted basis B^* of $T^*(H)$ is defined by*

$$[\delta_{(a)}] = [M^{(b)}_{(a)}][d_{(b)}]. \tag{1.26}$$

Theorem 1.7. *The elements of B^* are transforming as d-tensors i.e.*

$$dx^{a'} = {}^{(0)}B^{a'}_{a}dx^a, \quad \delta p_{Aa'} = {}^{(0)}B^{a}_{a'}\delta p_{Aa}, \quad A = \overline{0,k} \tag{1.27}$$

if and only if the elements of the matrix M are transforming in the following way

$$[{}^{(0)}B^{(a)}_{(a')}][M^{(b)}_{(a)}] = [M^{(b')}_{(a')}][B^{(b)}_{(b')}]. \tag{1.28}$$

Proof. (1.27) can be written in the matrix form as

$$[\delta_{(a')}] = [{}^{(0)}B^{(a)}_{(a')}][\delta_{(a)}].$$

Using (1.14) and (1.26) the above equation gives

$$[M^{(b')}_{(a')}][d_{(b')}] = [M^{(b')}_{(a')}][B^{(b)}_{(b')}][d_{(b)}] = [{}^{(0)}B^{(a)}_{(a')}][M^{(b)}_{(a)}][d_{(b)}]$$

from which it follows (1.28). □

Theorem 1.8. *The adapted bases B^* and B are dual to each other when $\bar{B}^*$ and $\bar{B}$ are dual to each other and*

$$[M^{(c)}_{(a)}][N^{(b)}_{(c)}] = \delta^b_a I, \tag{1.29}$$

i.e. $[M^{(a)}_{(b)}]$ *is the inverse matrix of* $[N^{(a)}_{(b)}]$.

Proof. The duality of $\bar{B}^*$ and $\bar{B}$ is equivalent with:

$$< dx^a, \partial_b >= \delta^a_b \quad < dp_{Aa}, \partial^{Bb} >= \delta^B_A \delta^\delta_b$$

$$< dx^a, \partial^{Bb} >= 0 \quad < dp_{Aa}, \partial_b >= 0.$$

or shorter $[d_{(c)}][\partial^{(d)}] = \delta^d_c I$. Now we have

$$[\delta_{(a)}][\delta^{(b)}] = [M^{(c)}_{(a)}][d_{(c)}][\partial^{(d)}][N^{(b)}_{(d)}] =$$

$$[M^{(c)}_{(a)}]\delta^d_c I[N^{(b)}_{(d)}] = [M^{(c)}_{(a)}][N^{(b)}_{(c)}] = \delta^b_a I.$$

□

2. The subspaces in H

First we introduce the family of subspaces and complementary subspaces in the base manifold M. Let us consider the equations

$$x^a = x^a(u^1, \dots, u^m, v^{m+1}, \dots, v^n) = x^a(u^\alpha, v^{\hat{\alpha}}), \tag{2.1}$$

$$a = \overline{1,n}, \quad \alpha, \beta, \gamma, \delta, \varepsilon, \dots = \overline{1,m}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}, \hat{\varepsilon}, \dots = \overline{m+1,n}.$$

If the Jacobian matrix

$$J = \left[\frac{\partial(x^1, \dots, x^n)}{\partial(u^1, \dots, u^m, v^{m+1}, \dots, v^n)}\right] = \begin{bmatrix} \left(\frac{\partial x^a}{\partial u^\alpha}\right) \\ \left(\frac{\partial x^a}{\partial v^{\hat{\alpha}}}\right) \end{bmatrix} = \begin{bmatrix} [B^a_\alpha]_{m\times n} \\ [B^a_{\hat{\alpha}}]_{(n-m)\times n} \end{bmatrix} \tag{2.2}$$

has rank n, then we can express u^α and $v^{\hat\alpha}$ as functions of x^a, i.e.

(2.3) $$u^\alpha = u^\alpha(x^a), v^{\hat\alpha} = v^{\hat\alpha}(x^a)$$

(2.4) $$J^{-1} = \left[\frac{\partial(u^1,\dots,u^m,v^{m+1},\dots,v^n)}{\partial(x^1,\dots,x^n)}\right] = \left[[B_b^\beta]_{n\times m}[B_b^{\hat\beta}]_{n\times(n-m)}\right].$$

In the above the following notations were used:

$$B_\alpha^a = \frac{\partial x^a}{\partial u^\alpha}, \quad B_{\hat\alpha}^a = \frac{\partial x^a}{\partial v^{\hat\alpha}}, \quad B_b^\beta = \frac{\partial u^\beta}{\partial x^b}, \quad B_b^{\hat\beta} = \frac{\partial v^{\hat\beta}}{\partial x^b}.$$

From (2.2) and (2.4) it follows:

(2.5) $$[B_\alpha^a][B_a^\beta] = [\delta_\alpha^\beta]_{m\times m} \quad [B_\alpha^a][B_a^{\hat\alpha}] = 0_{m\times(n-m)}$$

(2.6) $$[B_{\hat\beta}^a][B_a^\beta] = 0_{(n-m)\times m} \quad [B_{\hat\alpha}^a][B_a^{\hat\beta}] = [\delta_{\hat\alpha}^{\hat\beta}]_{(n-m)\times(n-m)}$$

and

(2.7) $$JJ^{-1} = \begin{bmatrix} [\delta_\beta^\alpha] & 0 \\ 0 & [\delta_{\hat\beta}^{\hat\alpha}] \end{bmatrix} = [B_\alpha^a][B_b^\alpha] + [B_{\hat\alpha}^a][B_b^{\hat\alpha}] = [\delta_b^a]_{n\times n}.$$

We shall restrict our consideration on such special transformations for which $B_{\alpha\hat\beta}^{\ a} = 0$ for all indices, because on the subspaces M_1 and M_2, determined by (2.8) and (2.9), this relation is valid.

Two complementary subspaces of the base manifold M are determined by the equations:

(2.8) $$x^a = x^a(u^1,u^2,\dots,u^m,C^{m+1},\dots,C^n),$$

(2.9) $$x^a = x^a(C^1,C^2,\dots,C^m,v^{m+1},\dots,v^n).$$

Equation (2.8) determines the family of m-dimensional subspaces M_1 of M and (2.9) the family of $(n-m)$ dimensional subspaces M_2 of M.

Here we shall consider some special case of general transformation (2.1), namely when (2.1) is valid, the new coordinates of the same point in the base manifold M are $(u^{1'},\dots,u^{m'},v^{(m+1)'},\dots,v^{n'})$, but

(2.10) $$u^{\alpha'} = u^{\alpha'}(u^1,\dots,u^m), \quad v^{\hat\alpha'} = v^{\hat\alpha'}(v^{m+1},\dots,v^n),$$
$$u^\alpha = u^\alpha(u^{1'},\dots,u^{m'}), \quad v^{\hat\alpha} = v^{\hat\alpha}(v^{(m+1)'},\dots,v^{n'})$$

and

$$x^{a'} = x^{a'}(u^{1'}, \dots, u^{m'}, v^{(m+1)'}, \dots, v^{n'}) = x^{a'}(u^{\alpha'}, v^{\hat{\alpha}'}). \tag{2.11}$$

If the above transformations are C^∞ and $1-1$, then there exist inverse transformations of the form (2.3), namely

$$u^{\alpha'} = u^{\alpha'}(x^{a'}), \quad v^{\hat{\alpha}'} = v^{\hat{\alpha}'}(x^{a'}). \tag{2.12}$$

Now we have

$$B_a^{a'} = B_{\alpha'}^{a'} B_\beta^{\alpha'} B_a^\beta + B_{\hat{\alpha}'}^{a'} B_{\hat{\beta}}^{\hat{\alpha}'} B_a^{\hat{\beta}} \tag{2.13}$$

$$B_{a'}^a = B_\alpha^a B_{\beta'}^\alpha B_{a'}^{\beta'} + B_{\hat{\alpha}}^a B_{\hat{\beta}'}^{\hat{\alpha}} B_{a'}^{\hat{\beta}'}.$$

For such special transformation of the base manifold M, the above equations have big influence on the second, third, ..., equations of (1.1).

From $p_{0a'} = {}^{(0)}B_{a'}^a p_{0a}$ it is clear that p_{0a} is transforming as a covariant vector field. As now the transformations on the base manifold M are determined by (2.1)-(2.13) we have:

$$\frac{\partial}{\partial x^a} = \frac{\partial u^\alpha}{\partial x^a}\frac{\partial}{\partial u^\alpha} + \frac{\partial v^{\hat{\alpha}}}{\partial x^a}\frac{\partial}{\partial v^{\hat{\alpha}}}, \tag{2.14}$$

$$\frac{\partial}{\partial u^\alpha} = \frac{\partial x^a}{\partial u^\alpha}\frac{\partial}{\partial x^a}, \quad \frac{\partial}{\partial v^{\hat{\alpha}}} = \frac{\partial x^a}{\partial v^{\hat{\alpha}}}\frac{\partial}{\partial x^a}. \tag{2.15}$$

As $p_{0a}, p_{0\alpha}, p_{0\hat{\alpha}}$ are transforming as covariant vector fields in T^*M, T^*M_1, T^*M_2 respectively from (2.14) and (2.15) it follows that p_{0a}, $p_{0\alpha}$ and $p_{0\hat{\alpha}}$ are transforming as $\frac{\partial}{\partial x^a}$, $\frac{\partial}{\partial u^\alpha}$ and $\frac{\partial}{\partial v^{\hat{\alpha}}}$ and from (2.14), (2.15) we get

$$p_{0\alpha} = B_\alpha^a p_{0a}, \quad p_{0\hat{\alpha}} = B_{\hat{\alpha}}^a p_{0a}, \quad p_{0a} = B_a^\alpha p_{0\alpha} + B_a^{\hat{\alpha}} p_{0\hat{\alpha}}. \tag{2.16}$$

From

$$\frac{\partial}{\partial x^{a'}} = \left(\frac{\partial x^a}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial u^{\alpha'}}\frac{\partial u^{\alpha'}}{\partial x^{a'}} + \frac{\partial x^a}{\partial v^{\hat{\alpha}}}\frac{\partial v^{\hat{\alpha}}}{\partial v^{\hat{\alpha}'}}\frac{\partial v^{\hat{\alpha}'}}{\partial x^{a'}}\right)\frac{\partial}{\partial x^a}$$

and the notations

$$B_{a'}^\alpha = B_{\alpha'}^\alpha B_{a'}^{\alpha'}, \quad B_{a'}^{\hat{\alpha}} = B_{\hat{\alpha}'}^{\hat{\alpha}} B_{a'}^{\hat{\alpha}'} \tag{2.17}$$

we can see that the relation

$$p_{0a'} = B_{a'}^\alpha p_{0\alpha} + B_{a'}^{\hat{\alpha}} p_{0\hat{\alpha}} \tag{2.18}$$

is satisfied.

We shall use the notations:

$$p_{Aa} = \frac{d^A p_{0a}}{dt^A}, \quad p_{A\alpha} = \frac{d^A p_{0\alpha}}{dt^A}, \quad p_{A\hat{\alpha}} = \frac{d^A p_{0\hat{\alpha}}}{dt^A}, \quad A = \overline{1,k}. \tag{2.19}$$

Theorem 2.1. *The transformations of the form (2.8) induce the* $(k+2)m$*-dimensional Hamilton space* H_1*, where the transformations of the point* $(u^\alpha = u^{0\alpha}, p_{0\alpha}, p_{1\alpha}, \dots, p_{k\alpha}) \in H_1$ *are given by*

$$
\begin{aligned}
(2.20)\qquad & u^{0\alpha'} = u^{0\alpha'}(u^{0\alpha}),\\
& p_{0\alpha'} = B^{\alpha}_{\alpha'} p_{0\alpha},\\
& p_{1\alpha'} = \binom{1}{0} {}^{(1)}B^{\alpha}_{\alpha'} p_{0\alpha} + \binom{1}{1} B^{\alpha}_{\alpha'} p_{1\alpha},\\
& p_{2\alpha'} = \binom{2}{0} {}^{(2)}B^{\alpha}_{\alpha'} p_{0\alpha} + \binom{2}{1} {}^{(1)}B^{\alpha}_{\alpha'} p_{1\alpha} + \binom{2}{2} B^{\alpha}_{\alpha'} p_{2\alpha}, \dots,\\
& p_{k\alpha'} = \binom{k}{0} {}^{(k)}B^{\alpha}_{\alpha'} p_{0\alpha} + \binom{k}{1} {}^{(k-1)}B^{\alpha}_{\alpha'} p_{1\alpha} + \dots + \binom{k}{k} B^{\alpha}_{\alpha'} p_{k\alpha},
\end{aligned}
$$

where

$$ {}^{(A)}B^{\alpha}_{\alpha'} = \frac{d^A}{dt^A} B^{\alpha}_{\alpha'}. $$

If in (2.20) we substitute everywhere α by $\hat\alpha$ obtain the transformation law of coordinates of point $(v^{\hat\alpha} = v^{0\hat\alpha}, p_{0\hat\alpha}, p_{\hat\alpha}, \dots, p_{k\hat\alpha}) \in H_2$, where the base manifold M_2 of H_2 is determined by (2.9) and $\dim H_2 = (k+2)(n-m)$.

Theorem 2.2. *The relations between two types of coordinates of the same point* $p \in H$*:*

$$ (x^a, p_{0a}, p_{1a}, \dots, p_{ka}) \text{ and } (u^\alpha, p_{0\alpha}, \dots, p_{k\alpha}, v^{\hat\alpha}, p_{0\hat\alpha}, p_{1\hat\alpha}, \dots, p_{k\hat\alpha}) $$

are given by:

$$
\begin{aligned}
(2.21)\qquad & x^a = x^a(u^1, \dots, u^m, v^{m+1}, \dots, v^n)\\
& p_{0a} = B^{\alpha}_{a} p_{0\alpha} + B^{\hat\alpha}_{a} p_{0\hat\alpha}\\
& p_{1a} = ({}^{(1)}B^{\alpha}_{a} p_{0\alpha} + {}^{(0)}B^{\alpha}_{a} p_{1\alpha}) + (\alpha/\hat\alpha)\\
& p_{2a} = ({}^{(2)}B^{\alpha}_{a} p_{0\alpha} + 2\,{}^{(1)}B^{\alpha}_{a} p_{1\alpha} + {}^{(2)}B^{\alpha}_{a} p_{2\alpha}) + (\alpha/\hat\alpha), \dots,\\
& p_{ka} = ({}^{(k)}B^{\alpha}_{a} p_{0\alpha} + \binom{k}{1} {}^{(k-1)}B^{\alpha}_{a} p_{1\alpha} + \dots + \binom{k}{k} {}^{(0)}B^{\alpha}_{a} p_{k\alpha}) + (\alpha/\hat\alpha),
\end{aligned}
$$

where in some equation $(\alpha/\hat\alpha)$ *means the expression in the former bracket in which* α *is substituted by* $\hat\alpha$*.*

Theorem 2.3. *The coordinates in the subspaces are expressed as the functions of coordinates in the surrounding place in the following way:*

$$ u^\alpha = u^\alpha(x^1, \dots, x^n), \quad v^{\hat\alpha} = v^{\hat\alpha}(x^1, \dots, x^n) $$

$$ x^a = x^a(u^1, u^2, \dots, u^m), \quad x^a = x^a(v^{m+1}, \dots, v^n) $$

(2.22)
$$p_{0\alpha} = B^{a'}_{\alpha} p_{0a},$$
$$p_{1\alpha} = {}^{(1)}B^a_\alpha p_{0a} + B^a_\alpha p_{1a},$$
$$p_{2\alpha} = {}^{(2)}B^a_\alpha p_{0a} + 2{}^{(1)}B^a_\alpha p_{1a} + B^a_\alpha p_{2a}, \dots,$$
$$p_{k\alpha} = {}^{(k)}B^a_\alpha p_{0a} + \binom{k}{1}{}^{(k-1)}B^a_\alpha p_{1a} + \dots + \binom{k}{k} B^a_\alpha p_{ka}.$$

The formulae from (2.22) are valid if u and α are substituted by v and $\hat\alpha$ respectively.

Theorem 2.4. *Equations (2.21) and (2.22) are equivalent.*

Proof. First we prove that (2.22) $\Rightarrow$ (2.21).
From

$$p_{0\alpha} = B^a_\alpha p_{0a}, \quad p_{0\hat\alpha} = B^a_{\hat\alpha} p_{0a} \Rightarrow$$
$$p_{0\alpha} B^\alpha_b + p_{0\hat\alpha} B^{\hat\alpha}_b = (B^a_\alpha B^\alpha_b + B^a_{\hat\alpha} B^{\hat\alpha}_b) p_{0a} = \delta^a_b p_{0a} = p_{0b},$$

which is the first equation from (2.21). Further, from

$$p_{1\alpha} = {}^{(1)}B^a_\alpha p_{0a} + B^a_\alpha p_{1a}, \quad p_{1\hat\alpha} = {}^{(1)}B^a_{\hat\alpha} p_{0a} + B^a_{\hat\alpha} p_{1a} \Rightarrow$$
$$B^\alpha_b p_{1\alpha} + B^{\hat\alpha}_b p_{1\hat\alpha} = (B^{\alpha}_b {}^{(1)}B^a_\alpha + B^{\hat\alpha}_b {}^{(1)}B^a_{\hat\alpha}) p_{0a} + (B^a_\alpha B^\alpha_b + B^a_{\hat\alpha} B^{\hat\alpha}_b) p_{1a}.$$

The substitution of $p_{0a} = p_{0\beta} B^\beta_a + p_{0\hat\beta} B^{\hat\beta}_a$ gives

$$(B^\alpha_b B^a_\alpha + B^{\hat\alpha}_b B^a_{\hat\alpha})'_t = (\delta^a_b)'_t = 0 \Rightarrow$$
$$B^{\alpha}_b {}^{(1)}B^a_\alpha + B^{\hat\alpha}_b {}^{(1)}B^a_{\hat\alpha} = -({}^{(1)}B^\alpha_b B^a_\alpha + {}^{(1)}B^{\hat\alpha}_b B^a_{\hat\alpha}).$$

(2.5) and (2.6) result:

$$B^\alpha_b p_{1\alpha} + B^{\hat\alpha}_b p_{1\hat\alpha} = -({}^{(1)}B^\alpha_b B^a_\alpha + {}^{(1)}B^{\hat\alpha}_b B^a_{\hat\alpha})(B^\beta_a p_{0\beta} + B^{\hat\beta}_a p_{0\hat\beta}) + \delta^a_b p_{1a} \Rightarrow$$
$$B^\alpha_b p_{1\alpha} + B^{\hat\alpha}_b p_{1\hat\alpha} + {}^{(1)}B^\alpha_b p_{0\alpha} + {}^{(1)}B^{\hat\alpha}_b p_{0\hat\alpha} = p_{1a},$$

which is the second equation of (2.21). The other equations of (2.21) can be proved in the similar manner. The proof in the opposite direction is similar. □

To introduce the natural and adapted bases in tangent and cotangent spaces of the subspaces H_1 and H_2 of H it is convenient to use the matrix representation of coordinate transformations obtained in 1 and 2.

Let us introduce the notations

$$(2.23)\qquad [p_a]=\begin{bmatrix} p_{0a}\\ p_{1a}\\ \vdots\\ p_{ka}\end{bmatrix},\quad [p_\alpha]=\begin{bmatrix} p_{0\alpha}\\ p_{1\alpha}\\ \vdots\\ p_{k\alpha}\end{bmatrix},\quad [p_{\hat\alpha}]=\begin{bmatrix} p_{0\hat\alpha}\\ p_{1\hat\alpha}\\ \vdots\\ p_{k\hat\alpha}\end{bmatrix}$$

$$(2.24)\quad [B^{(a)}_{(a')}]_{(s)}=\begin{bmatrix} {}^{(0)}B^a_{a'} & 0 & 0 & \cdots & 0\\ \binom{1}{0}{}^{(1)}B^a_{a'} & \binom{1}{1}{}^{(0)}B^a_{a'} & 0 & \cdots & 0\\ \binom{2}{0}{}^{(2)}B^a_{a'} & \binom{2}{1}{}^{(1)}B^a_{a'} & \binom{2}{2}{}^{(0)}B^a_{a'} & \cdots & 0\\ \vdots & \vdots & \vdots & & \\ \binom{k}{0}{}^{(k)}B^a_{a'} & \binom{k}{1}{}^{(k-1)}B^a_{a'} & \cdots & \cdots & \binom{k}{k}{}^{(0)}B^a_{a'}\end{bmatrix}.$$

Here s means small, because $[B^{(a)}_{(a')}]_s$ is the part of $[B^{(a)}_{(a')}]$, which was introduced in (1.8).

We shall obtain $[p_{a'}]$, $[p_{\alpha'}]$, $[p_{\hat\alpha'}]$ if in (2.23) a, α and $\hat\alpha$ substitute by a', α' and $\hat\alpha'$ respectively. In a similar way, $[B^{(\alpha)}_{(\alpha')}]_{(s)}$, $[B^{(\hat\alpha)}_{(\hat\alpha')}]_{(s)}$ can be obtained from (2.24) if everywhere a, a' are substituted by α, α' or $\hat\alpha$, $\hat\alpha'$, respectively.

The matrices $[B^{(a)}_{(\alpha)}]_s$ and $[B^{(a)}_{(\hat\alpha)}]_s$ can be obtained from (2.24) if everywhere a' is substituted by α or $\hat\alpha$, respectively.

Using the above notations the transformation (1.1) in H can be written in the form

$$(2.25)\qquad x^{a'}=x^{a'}(x^a),\quad [p_{a'}]=[B^{(a)}_{(a')}]_{(s)}[p_a].$$

The transformations in H_1 and H_2 prescribed by (2.20) are now given by

$$(2.26)\qquad u^{0\alpha'}=u^{0\alpha'}(u^\alpha),\quad [p_{(\alpha')}]=[B^{(\alpha)}_{(\alpha')}]_{(s)}[p_{(\alpha)}],$$

$$(2.27)\qquad v^{0\hat\alpha'}=v^{0\hat\alpha'}(v^{0\hat\alpha}),\quad [p_{(\hat\alpha')}]=[B^{(\hat\alpha)}_{(\hat\alpha')}]_{(s)}[p_{(\hat\alpha)}].$$

To obtain the connection between the ambient space H and subspaces H_1 and H_2 we need matrices $[B^{(\alpha)}_{(a)}]_{(s)}$ and $[B^{(\hat\alpha)}_{(a)}]_{(s)}$, which can be obtained from (2.24) if a, a' are substituted by α, a or $\hat\alpha$, a, respectively.

The relations between the coordinates of H, H_1 and H_2, which are explicitly written in (2.21), (2.22) now can be expressed as:

$$x^a=x^a(u^\alpha)+x^a(v^{\hat\alpha})\Leftrightarrow u^\alpha=u^\alpha(x^a),\quad v^{\hat\alpha}=v^{\hat\alpha}(x^a)$$

$$[p_{(\alpha)}] = [B^{a}_{(\alpha)}]_{(s)}[p_{(a)}], \quad [p_{(\hat\alpha)}] = [B^{(a)}_{(\hat\alpha)}]_{(s)}[p_{(a)}] \tag{2.28}$$

$$[p_{(a)}] = [B^{(\alpha)}_{(a)}]_{(s)}[p_{(\alpha)}] + [B^{(\hat\alpha)}_{(a)}]_{(s)}[p_{(\hat\alpha)}]. \tag{2.29}$$

3. The natural and adapted bases in $T(H_1)$, $T(H_2)$, $T^*(H_1)$ and $T^*(H_2)$

The natural bases $\bar B_1$ of $T(H_1)$ and $\bar B_2$ of $T(H_2)$ are given by:

$$\bar B_1 = \{\partial_\alpha, \partial^{0\alpha}, \partial^{1\alpha}, \dots, \partial^{k\alpha}\}, \quad \partial_\alpha = \frac{\partial}{\partial u^\alpha}, \partial^{A\alpha} = \frac{\partial}{\partial p_{A\alpha}} \tag{3.1}$$

$$\bar B_2 = \{\partial_{\hat\alpha}, \partial^{0\hat\alpha}, \partial^{1\hat\alpha}, \dots, \partial^{k\hat\alpha}\}, \quad \partial_{\hat\alpha} = \frac{\partial}{\partial v^{\hat\alpha}}, \partial^{A\hat\alpha} = \frac{\partial}{\partial p_{A\hat\alpha}}. \tag{3.2}$$

We shall use the matrices $[B^{(\alpha)}_{(\alpha')}]$, $[B^{(\hat\alpha)}_{(\hat\alpha')}]$ which are obtained from $[B^{(a)}_{(a')}]$ defined by (1.8) if a, a', x are substituted α, α', u and $\hat\alpha$, $\hat\alpha'$, v, respectively.

Let $(u^{0\alpha'}, p_{0\alpha'}, \dots, p_{k\alpha'})$ be transformed coordinates of the same point $(u^{0\alpha}, p_{0\alpha}, \dots, p_{k\alpha})$ from H_1 and

$$[\partial^{(\alpha)}] = [\partial_\alpha \partial^{0\alpha}\partial^{1\alpha}\dots\partial^{k\alpha}].$$

Theorem 3.1. *The connection between two natural bases $\bar B_1$ and $\bar B_1'$ of $T(H_1)$ are given by*

$$[\partial^{(\alpha)}] = [\partial^{(\alpha')}][B^{(\alpha)}_{(\alpha')}]. \tag{3.3}$$

The elements of the natural bases $\bar B_2$ and $\bar B_2'$ of $T(H_2)$ are transforming as follows:

$$[\partial^{(\hat\alpha)}] = [\partial^{(\hat\alpha')}][B^{(\hat\alpha)}_{(\hat\alpha')}]. \tag{3.4}$$

The natural bases $\bar B_1^*$ of $T^*(H_1)$ and $\bar B_2^*$ of $T(H_2)$ are given by

$$\bar B_1^* = \{du^{0\alpha}, dp_{0\alpha}, dp_{1\alpha}, \dots, dp_{k\alpha}\}, \tag{3.5}$$

$$\bar B_2^* = \{dv^{0\hat\alpha}, dp_{0\hat\alpha}, dp_{1\hat\alpha}, \dots, dp_{k\hat\alpha}\}. \tag{3.6}$$

The bases $\bar B_1^{*\prime}$ and $\bar B_2^{*\prime}$ are obtained from $\bar B_1^*$ and $\bar B_2^*$ if in (3.5) α is substituted by α' and in (3.6) $\hat\alpha$ is substituted by $\hat\alpha'$.

We shall use the notations

$$[d_{(\alpha)}] = \begin{bmatrix} du^{0\alpha} \\ dp_{0\alpha} \\ dp_{1\alpha} \\ \vdots \\ dp_{k\alpha} \end{bmatrix}, \quad [d_{(\hat{\alpha})}] = \begin{bmatrix} dv^{0\hat{\alpha}} \\ dp_{0\hat{\alpha}} \\ dp_{1\hat{\alpha}} \\ \vdots \\ dp_{k\hat{\alpha}} \end{bmatrix}$$

(similar for $[d_{(\alpha')}]$ and $[d_{(\hat{\alpha}')}]$).

Theorem 3.2. *The connection between two natural bases $\bar{B}_1^*$ and $\bar{B}_1^{*\prime}$ of $T^*(H_1)$ are given by*

$$[d_{(\alpha')}] = [B^{(\alpha)}_{(\alpha')}][d_{(\alpha)}]. \tag{3.7}$$

The elements of the natural bases $\bar{B}_2^*$ and $\bar{B}_2^{*\prime}$ of $T(H_2)$ are transforming as follows:

$$[d_{(\hat{\alpha}')}] = [B^{(\hat{\alpha})}_{(\hat{\alpha}')}][d_{(\hat{\alpha})}]. \tag{3.8}$$

It is obvious that the elements of the natural bases $\bar{B}_1$, $\bar{B}_1^*$, $\bar{B}_2$, $\bar{B}_2^*$ are not transforming as tensors, so it is necessary to construct the adapted bases B_1, B_2 of $T(H_1)$, $T(H_2)$ and B_1^*, B_2^* of $T^*(H_1)$, $T^*(H_2)$. The explicit forms of these bases are:

$$B_1 = \{\delta_\alpha, \delta^{0\alpha}, \delta^{1\alpha}, \ldots, \delta^{k\alpha}\}$$

$$B_2 = \{\delta_{\hat{\alpha}}, \delta^{0\hat{\alpha}}, \delta^{1\hat{\alpha}}, \ldots, \delta^{k\hat{\alpha}}\}$$

$$B_1^* = \{\delta u^{0\alpha}, \delta p_{0\alpha}, \delta p_{1\alpha}, \ldots, \delta p_{k\alpha}\}$$

$$B_2^* = \{\delta v^{0\hat{\alpha}}, \delta p_{0\hat{\alpha}}, \delta p_{2\hat{\alpha}}, \ldots, \delta p_{k\hat{\alpha}}\}.$$

We introduce the notations

$$[\delta^{(\alpha)}] = [\delta_\alpha \delta^{0\alpha} \delta^{1\alpha} \ldots \delta^{k\alpha}],$$

$$[\delta^{(\hat{\alpha})}] = [\delta_{\hat{\alpha}} \delta^{0\hat{\alpha}} \delta^{1\hat{\alpha}} \ldots \delta^{k\hat{\alpha}}].$$

$[N^{(\alpha)}_{(\beta)}]$ is obtained from $[N^{(a)}_{(b)}]$, given by (1.5), if a is substituted by α and b by β (similar for $[N^{(\hat{\alpha})}_{(\hat{\beta})}]$). $[{}^{(0)}B^{(\alpha)}_{(\alpha')}]([{}^{0}B^{(\hat{\alpha})}_{(\hat{\alpha}')}])$ is obtained from $[{}^{(0)}B^{a}_{(a')}]$ (see (1.22)) if a is substituted by $\alpha(\hat{\alpha})$ and a' by $\alpha'(\hat{\alpha}')$.

The elements of B_1 are transforming as d-tensors if they satisfy the relations

$$\delta_\alpha = {}^{(0)}B^{\alpha'}_{\alpha}\delta_{\alpha'} \quad \delta^{A\alpha} = {}^{(0)}B^{\alpha}_{\alpha'}\delta^{A\alpha'}, \quad A = \overline{0,k}. \tag{3.9}$$

The above equations can be written in the matrix form:

$$[\delta^{(\alpha)}] = [{}^{(0)}B^{(\alpha)}_{(\alpha')}][\delta^{(\alpha')}]. \tag{3.10}$$

The elements of B_2 are transforming as d-tensors if the equations of the type (3.9) and (3.10) are valid when α, α' are substituted by $\hat{\alpha}$, $\hat{\alpha}'$.

Definition 3.1. . *The adapted basis B_1 of $T(H_1)$ (B_2 of $T(H_2)$) is defined by*

$$[\delta^{(\alpha)}] = [\partial^{(\beta)}][N^{(\alpha)}_{(\beta)}] \tag{3.11}$$

$$([\delta^{(\hat{\alpha})}] = [\partial^{(\hat{\beta})}][N^{(\hat{\alpha})}_{(\hat{\beta})}]). \tag{3.12}$$

Theorem 3.3. *The necessary and sufficient conditions for the elements of the adapted basis $B_1(B_2)$ of $T(H_1)(T(H_2)$ to transform as d-tensors are the following matrix equations*

$$[N^{(\alpha')}_{(\beta')}][{}^{(0)}B^{(\alpha)}_{(\alpha')}] = [B^{(\beta)}_{(\beta')}[N^{(\alpha)}_{(\beta)}] \tag{3.13}$$

$$([N^{(\hat{\alpha}')}_{(\hat{\beta}')}][{}^{(0)}B^{(\hat{\alpha})}_{(\hat{\alpha}')}] = [B^{(\hat{\beta})}_{(\hat{\beta}')}][N^{(\hat{\alpha})}_{(\hat{\beta})}]. \tag{3.14}$$

The proof is similar to the proof of Theorem 1.6.

Let us denote by $[M^{(\beta)}_{(\alpha)}]([M^{(\hat{\beta})}_{(\hat{\alpha})}])$ the matrix obtained from $[M^{(b)}_{(a)}]$ defined in (1.25) if a is substituted by $\alpha(\hat{\alpha})$ and b by $\beta(\hat{\beta})$.

Definition 3.2. . *The adapted basis $B_1^*(B_2^*)$ of $T^*(H_1)(T^*(H_2))$ is defined by*

$$[\delta_{(\alpha)}] = [M^{(\beta)}_{(\alpha)}][d_{(\beta)}] \tag{3.15}$$

$$[\delta_{(\hat{\alpha})}] = [M^{(\hat{\beta})}_{(\hat{\alpha})}][d_{(\hat{\beta})}], \tag{3.16}$$

where

$$[\delta_{(\alpha)}] = \begin{bmatrix} \delta u^{0\alpha} \\ \delta p_{0\alpha} \\ \delta p_{1\alpha} \\ \vdots \\ \delta p_{k\alpha} \end{bmatrix}, \quad [\delta_{(\hat{\alpha})}] = \begin{bmatrix} \delta v^{0\hat{\alpha}} \\ \delta p_{0\hat{\alpha}} \\ \delta p_{1\hat{\alpha}} \\ \vdots \\ \delta p_{k\hat{\alpha}} \end{bmatrix}.$$

Theorem 3.4. *The elements of the adapted basis B_1^* of $T^*(H_1)$ (B_2^* of $T^*(H_2)$) are transforming as d-tensors, i.e.*

$$du^{0\alpha'} = {}^{(0)}B^{\alpha'}_{\alpha} du^{0\alpha}, \quad \delta p_{A\alpha'} = {}^{(0)}B^{\alpha}_{\alpha'} \delta p_{A\alpha} \quad A = \overline{0,k}$$

$$dv^{0\hat{\alpha}'} = {}^{(0)}B^{\hat{\alpha}}_{\hat{\alpha}'} dv^{0\hat{\alpha}}, \delta p_{A\hat{\alpha}'} = {}^{(0)}B^{\hat{\alpha}}_{\hat{\alpha}'} \delta p_{A\hat{\alpha}} \quad A = \overline{0,k}$$

if and only if the matrices M are transforming in the following way:

$$[{}^{(0)}B^{(\alpha)}_{(\alpha')}][M^{(\beta)}_{(\alpha)}] = [M^{(\beta')}_{(\alpha')}][B^{(\beta)}_{(\beta')}] \tag{3.17}$$

$$([{}^{(0)}B^{(\hat{\alpha})}_{(\hat{\alpha}')}][M^{(\hat{\beta})}_{(\hat{\alpha})}] = [M^{(\hat{\beta}')}_{(\hat{\alpha}')}][B^{(\hat{\beta})}_{(\hat{\beta}')}]. \tag{3.18}$$

The proof is similar to the proof of Theorem 1.7.

Theorem 3.5. *The adapted bases B_1^* and B_1 are dual to each other when $\bar{B}_1^*$ and $\bar{B}_1$ are dual to each other and*

$$[M^{(\gamma)}_{(\alpha)}][N^{(\beta)}_{(\gamma)}] = \delta^{\beta}_{\alpha} I \tag{3.19}$$

i.e. $[M^{(\alpha)}_{(\beta)}]$ is the inverse matrix of $[N^{(\alpha)}_{(\beta)}]$.

Theorem 3.6. *The adapted bases B_2^* and B_2 are dual to each other when $\bar{B}_2^*$, and $\bar{B}_2$ are dual to each other and*

$$[M^{(\hat{\gamma})}_{(\hat{\alpha})}][N^{(\hat{\beta})}_{(\hat{\gamma})}] = \delta^{\hat{\beta}}_{\hat{\alpha}} I, \tag{3.20}$$

i.e. $[M^{(\hat{\alpha})}_{(\hat{\beta})}]$ is the inverse matrix of $[N^{(\hat{\alpha})}_{(\hat{\beta})}]$.

The proof of Theorems 3.5 and 3.6 is similar to the proof of Theorem 1.8.

Theorem 3.7. *The elements of the natural bases $\bar{B}_1(\bar{B}_2)$ of $T(H_1)(T(H_2))$ can be expressed as functions of the adapted bases $B_1(B_2)$ of $T(H_1)(T(H_2))$ in the following way*

$$[\partial^{(\alpha)}] = [\delta^{(\beta)}][M^{(\alpha)}_{(\beta)}] \tag{3.21}$$

$$[\partial^{(\hat{\alpha})}] = [\delta^{(\hat{\beta})}][M^{(\hat{\alpha})}_{(\hat{\beta})}]. \tag{3.22}$$

Proof. From (3.11) and (3.19) it follows

$$[\delta^{(\alpha)}][M^{(\gamma)}_{(\alpha)}] = [\partial^{(\beta)}][N^{(\alpha)}_{(\beta)}][M^{(\gamma)}_{(\alpha)}] = [\partial^{(\beta)}]\delta^{\gamma}_{\beta} I = [\partial^{(\gamma)}]$$

From (3.12) and (3.20) we get

$$[\delta^{(\hat{\alpha})}][M^{(\hat{\gamma})}_{(\hat{\alpha})}] = [\partial^{(\hat{\beta})}][N^{(\hat{\alpha})}_{(\hat{\beta})}][M^{(\hat{\gamma})}_{(\hat{\alpha})}] = [\partial^{\hat{\beta}}]\delta^{\hat{\gamma}}_{\hat{\beta}} I = [\partial^{\hat{\gamma}}].$$

□

Theorem 3.8. *The elements of the natural bases $\bar{B}_1^*(\bar{B}_2^*)$ of $T^*(H_1)(T^*(H_2))$ can be expressed as functions of the adapted bases $B_1^*(B_2^*)$ of $T^*(H_1)(T^*(H_2))$ in the following way*

$$[d_{(\alpha)}] = [N_{(\alpha)}^{(\beta)}][\delta_{(\beta)}] \tag{3.23}$$

$$[d_{(\hat{\alpha})}] = [N_{(\hat{\alpha})}^{(\hat{\beta})}][\delta_{(\hat{\beta})}]. \tag{3.24}$$

Proof. The proof follows from (3.15), (3.16), (3.19) and (3.20). □

As $\dim H = (k+2)n$, $\dim H_1 = (k+2)m$, $\dim H_2 = (k+2)(n-m)$ and H_1, H_2 are the subspaces of H, we can construct the adapted bases

$$B' = B_1 \cup B_2 \text{ of } T(H) \text{ and } B^{*\prime} = B_1^* \cup B_2^* \text{ of } T^*(H).$$

Now we have two adapted bases of $T(H)$:

$$B = \{\delta_a, \delta^{0a}, \delta^{01}, \dots, \delta^{ka}\} = [\delta^{(a)}]$$

$$B' = \{\delta_\alpha, \delta_{\hat{\alpha}}, \delta^{0\alpha}, \delta^{0\hat{\alpha}}, \delta^{1\alpha}, \delta^{1\hat{\alpha}}, \dots, \delta^{k\alpha}, \delta^{k\hat{\alpha}}\} = \{\delta^\alpha, \delta^{\hat{\alpha}}\} \tag{3.25}$$

and two adapted bases of $T^*(H)$:

$$B^* = \{\delta x^a, \delta p_{0a}, \delta p_{1a}, \dots, \delta p_{ka}\}$$

$$B^{*\prime} = \{\delta u^\alpha, \delta v^{\hat{\alpha}}, \delta p_{0\alpha}, \delta p_{0\hat{\alpha}}, \delta p_{1\alpha}, \delta p_{1\hat{\alpha}}, \dots, \delta p_{k\alpha}, \delta p_{k\hat{\alpha}}\} = \{\delta_\alpha, \delta_{\hat{\alpha}}\}. \tag{3.26}$$

We want such adapted basis B' of $T(H)$ and $B^{*\prime}$ of $T^*(H)$ which is connected with B and B^* in the following way:

$$\delta_a = B_a^\alpha \delta_\alpha + B_a^{\hat{\alpha}} \delta_{\hat{\alpha}} \quad \delta^{Aa} = B_\alpha^a \delta^{A\alpha} + B_{\hat{\alpha}}^a \delta^{A\hat{\alpha}} \tag{3.27}$$

$$\delta x^a = B_\alpha^a \delta u^\alpha + B_{\hat{\alpha}}^a \delta v^{\hat{\alpha}} \quad \delta p_{Aa} = B_a^\alpha \delta p_{A\alpha} + B_a^{\hat{\alpha}} \delta p_{A\hat{\alpha}}. \tag{3.28}$$

The matrix equation of (3.27) and (3.28) is given by

$$[\delta^{(a)}] = [\delta^{(\alpha)}][{}^{(0)}B_{(\alpha)}^{(a)}] + [\delta^{\hat{\alpha}}][{}^{(0)}B_{(\hat{\alpha})}^{(a)}] \tag{3.29}$$

$$[\delta_{(a)}] = [{}^{(0)}B_{(a)}^{(\alpha)}][\delta_{(\alpha)}] + [{}^{(0)}B_{(a)}^{(\hat{\alpha})}][\delta_{(\hat{\alpha})}]. \tag{3.30}$$

In the former theorems we gave the conditions for $[M]$ and $[N]$, such that the elements of B, B^*, B_1, B_1^*, B_2, B_2^* transform as tensors and B^* be dual to B, B_1^* to B_1 and B_2^* to B_2. The equations (3.27) and (3.28) are new restriction for the adapted bases. It is easy to prove

Theorem 3.9. *If B_1^* is dual to B_1, B_2^* dual to B_2, then B^* is dual to B if (3.27) and (3.28) are satisfied.*

Theorem 3.10. *The elements of the adapted bases B, B^*, B' and $B^{*\prime}$ satisfy (3.29) and (3.30) if different $[M]$ and $[N]$, which appear in their construction are connected by*

(3.31)
$$[^{(0)}B^{(b)}_{(\beta)}][N^{(a)}_{(b)}] = [N^{(\alpha)}_{(\beta)}][^{(0)}B^{(a)}_{(\alpha)}]$$
$$[^{(0)}B^{(b)}_{(\hat{\beta})}][N^{(a)}_{(b)}] = [N^{(\hat{\alpha})}_{(\hat{\beta})}][^{(0)}B^{(a)}_{(\hat{\alpha})}]$$

(3.32)
$$[M^{(b)}_{(a)}][^{(0)}B^{(\beta)}_{(b)}] = [^{(0)}B^{(\alpha)}_{(a)}][M^{(\beta)}_{(\alpha)}]$$
$$[M^{(b)}_{(a)}][^{(0)}B^{(\hat{\beta})}_{(b)}] = [^{(0)}B^{(\hat{\alpha})}_{(a)}][M^{(\hat{\beta})}_{(\hat{\alpha})}].$$

Proof. Substituting the equations

$$[\delta^{(a)}] = [\partial^{(b)}][N^{(a)}_{(b)}] \quad [\delta_{(a)}] = [M^{(b)}_{(a)}][d_{(b)}]$$
$$[\delta^{(\alpha)}] = [\partial^{(\beta)}][N^{(\alpha)}_{(\beta)}] \quad [\delta_{(\alpha)}] = [M^{(\beta)}_{(\alpha)}][d_{(\beta)}]$$
$$[\delta^{(\hat{\alpha})}] = [\partial^{(\hat{\beta})}][N^{(\hat{\alpha})}_{(\hat{\beta})}] \quad [\delta_{(\hat{\alpha})}] = [M^{(\hat{\beta})}_{(\hat{\alpha})}][d_{(\hat{\beta})}]$$

into (3.29) and (3.30) we get

(3.33)
$$[\partial^{(b)}][N^{(a)}_{(b)}] = [\partial^{(\beta)}][N^{(\alpha)}_{(\beta)}][^{(0)}B^{(a)}_{(\alpha)}] + [\partial^{\hat{\beta}}][N^{\hat{\alpha}}_{\hat{\beta}}][B^{(a)}_{\hat{\alpha}}]$$

(3.34)
$$[M^{(b)}_{(a)}][d_{(b)}] = [^{(0)}B^{(\alpha)}_{(a)}][M^{(\beta)}_{(\alpha)}][d_{(\beta)}] + [^{(0)}B^{(\hat{\alpha})}_{(a)}][M^{(\hat{\beta})}_{(\hat{\alpha})}][d_{\hat{\beta}}].$$

Equations (3.33) and (3.34) are valid for an arbitrary $[\partial^{(b)}]$ and arbitrary $[d_{(b)}]$. If we take:

$$[\partial^{(b)}] = [\partial^{(\beta)}][^{(0)}B^{(b)}_{(\beta)}], \quad [\partial^{(\hat{\beta})}] = 0$$

we obtain the first equation in (3.31) and if we take

$$[\partial^{(b)}] = [\partial^{(\hat{\beta})}][^{(0)}B^{(b)}_{(\hat{\beta})}], \quad [\partial^{(\beta)}] = 0$$

we obtain the second equation in (3.31).

On the other hand, if we take

$$[d_{(b)}] = [^{(0)}B^{(\beta)}_{(b)}][d_{(\beta)}], \quad [d_{(\hat{\beta})}] = 0$$

we obtain the first equation in (3.32), and if we take

$$[d_{(b)}] = [{}^{(0)}B^{(\hat\beta)}_{(b)}][d_{(\hat\beta)}], \quad [d_{(\beta)}] = 0$$

we obtain the second equation of (3.32). □

From (3.31), (3.32) and (2.7) we obtain

Theorem 3.11. *If the elements of B, B' further B^* and $B^{*\prime}$ satisfy (3.29) and (3.30), then*

$$[N^{(a)}_{(c)}] = [{}^{(0)}B^{(\beta)}_{(c)}][N^{(\alpha)}_{(\beta)}][{}^{(0)}B^{(a)}_{(\alpha)}] + [B^{(\hat\beta)}_{(c)}][N^{(\hat\alpha)}_{(\hat\beta)}][{}^{(0)}B^{(a)}_{(\hat\alpha)}] \tag{3.35}$$

$$[M^{(d)}_{(a)}] = [{}^{(0)}B^{(\alpha)}_{(a)}][M^{(\beta)}_{(\alpha)}][{}^{(0)}B^{(d)}_{(\beta)}] + [B^{(\hat\alpha)}_{(a)}][M^{(\hat\beta)}_{(\hat\alpha)}][{}^{(0)}B^{(d)}_{(\hat\beta)}] \tag{3.36}$$

Theorem 3.12. *If (3.35) and (3.36) are valid and B^*_1, is dual to B_1, B^*_2 is dual to B_2, then $[N^{(a)}_{(c)}][M^{(d)}_{(a)}] = \delta^d_c I$, i.e. $B^{*\prime}$ is dual to B'.*

Proof.

$$\begin{aligned}[N^{(a)}_{(c)}][M^{(d)}_{(a)}] &= ([{}^{(0)}B^{(\beta)}_{(c)}][N^{(\alpha)}_{(\beta)}][{}^{(0)}B^{(a)}_{(\alpha)}] + [{}^{(0)}B^{(\hat\beta)}_{(c)}][N^{(\hat\alpha)}_{(\hat\beta)}][{}^{(0)}B^{(a)}_{(\hat\alpha)}])\\ &\quad ([{}^{(0)}B^{(\gamma)}_{(a)}][M^{(\delta)}_{(\gamma)}][{}^{(0)}B^{(d)}_{(\delta)}] + [{}^{(0)}B^{(\hat\gamma)}_{(a)}][M^{(\hat\delta)}_{(\hat\gamma)}][{}^{(0)}B^{(d)}_{(\hat\delta)}]\end{aligned}$$

$$[{}^{(0)}B^{(\beta)}_{(c)}][N^{(\alpha)}_{(\beta)}\delta^\gamma_\alpha I[M^{(\delta)}_{(\gamma)}][{}^{(0)}B^{(d)}_{(\delta)}] + [{}^{(0)}B^{(\hat\beta)}_{(c)}][N^{(\hat\alpha)}_{(\hat\beta)}]\delta^{\hat\gamma}_{\hat\alpha} I[M^{(\hat\delta)}_{(\hat\gamma)}][{}^{(0)}B^{(d)}_{(\hat\delta)}] =$$

$$= [{}^{(0)}B^{(\beta)}_{(c)}\delta^\delta_\beta I[{}^{(0)}B^{(d)}_{(\delta)}] + [{}^{(0)}B^{(\hat\beta)}_{(c)}]\delta^{\hat\delta}_{\hat\beta} I[{}^{(0)}B^{(d)}_{(\hat\delta)}] = \delta^d_c I.$$

In the calculation of (3.19), (3.20) and (2.7) was used. □

Conclusion:

If we construct two supplementary family of subspaces H_1 and H_2 of H and construct the adapted bases B_1, B_2 of $T(H_1)$, $T(H_2)$, further B^*_1, B^*_2 of $T^*(H_1)$, $T^*(H_2)$, in such a way, that the duality is valid, then in $T(H)$ there exist one and only one adapted basis B constructed by $[N^a_b]$ given by (3.35) and in $T^*(H)$ the basis $B*$ constructed by $[M^a_b]$ given by (3.36) in such a way that the elements of different adapted bases are connected by (3.29) and (3.30). These equations are tensor equations and are very important for further investigation.

4. Special adapted bases

For the further investigations, especially in the theory of sprays and Jacobi fields the introduced adapted bases are not convenient. We need less variables in the matrices $[N^{(a)}_{(b)}]$ and $[M^{(b)}_{(a)}]$, in such a way that the previous conditions (1.19), (1.21), (1.26), (1.28) and (1.29) are satisfied.

The explicit form of (1.21) and (1.28) have equations of the form:

$$N^{0a'}_{Bb'}{}^{(0)}B^a_{a'} = \binom{B}{B} N^{0a}_{Bb}{}^{(0)}B^b_{b'} + \binom{B}{B-1} N^{0a}_{(B-1)b}{}^{(1)}B^b_{b'} + \cdots + \binom{B}{1} N^{0a}_{1b}{}^{(B-1)}B^b_{b'} - \binom{B}{0}{}^{(B)}B^a_{b'} \tag{4.1}$$

$$\begin{aligned} N^{Aa'}_{(A+B)b'}{}^{(0)}B^a_{a'} &= \binom{A+B}{A+B} N^{Aa}_{(A+B)b}{}^{(0)}B^b_{b'} + \\ &\binom{A+B}{A+B-1} N^{Aa}_{(A+B-1)b}{}^{(1)}B^b_{b'} + \cdots + \\ &\binom{A+B}{A+1} N^{Aa}_{(B-1)b}{}^{(B-1)}B^b_{b'} - \binom{A+B}{A}{}^{(B)}B^a_{b'}, \end{aligned} \tag{4.2}$$

$$\begin{aligned} M^{0b}_{Ba}{}^{(0)}B^a_{a'} &= \binom{0}{0} M^{0b'}_{Ba'}{}^{(0)}B^b_{b'} + \binom{1}{0} M^{1b'}_{Ba'}{}^{(1)}B^b_{b'} + \cdots \\ &\binom{B-1}{0} M^{(B-1)b'}_{Ba'}{}^{(B-1)}B^b_{b'} + \binom{B}{0}{}^{(B)}B^b_{a'}, \ldots \end{aligned} \tag{4.3}$$

$$\begin{aligned} M^{Ab}_{(A+B)a}{}^{(0)}B^a_{a'} &= \binom{A}{A} M^{Ab'}_{(A+B)a'}{}^{(0)}B^b_{b'} + \\ &\binom{A+1}{A} M^{(A+1)b'}_{(A+B)a'}{}^{(1)}B^b_{b'} + \cdots + \\ &\binom{A+B-1}{A} M^{(A+B-1)b'}_{(A+B)a'}{}^{(A+B-1)}B^b_{b'} + \\ &\binom{A+B}{A}{}^{(A+B)}B^b_{a'}. \end{aligned} \tag{4.4}$$

If we put in (4.2)

$$N^{Aa}_{(A+B)b} = \binom{A+B}{A} N^{0a}_{Bb},$$

for every $0 \le A \le A+B \le k$, use the properties of binomial coefficients, and compare (4.1) and (4.2) we get

$$N^{Aa'}_{(A+B)b'} = \binom{A+B}{A} N^{0a'}_{1b'}. \tag{4.5}$$

In a similar way if we substitute in (4.4) and (4.3)

$$M^{Ab'}_{(A+B)a'} = \binom{A+B}{A} M^{0b'}_{Ba'}, \tag{4.6}$$

for every $0 \leq A \leq A+B \leq k$ and compare the obtained equations, we get

$$M^{Ab}_{(A+B)a} = \binom{A+B}{A} M^{0b}_{Ba}.$$

Definition 4.1. . *The special adapted basis* $\tilde{B} = \{\delta_a, \delta^{0a}, \delta^{1a}, \ldots, \delta^{ka}\}$ *of* $T(H)$ *is given by*

$$[\delta^{(a)}] = [\partial^b][\tilde{N}^{(a)}_{(b)}], \tag{4.7}$$

where

(4.8)

$$[\tilde{N}^{(a)}_{(b)}] = \begin{bmatrix} \delta^b_a & 0 & 0 & 0 & \cdots & 0 \\ -N_{a0b} & \binom{0}{0}\delta^a_b & 0 & 0 & \cdots & 0 \\ -N_{a1b} & -N^{0a}_{1b} & \binom{1}{1}\delta^a_b & 0 & \cdots & 0 \\ -N_{a2b} & -N^{0a}_{2b} & -\binom{2}{1}N^{0a}_{1b} & \binom{2}{2}\delta^a_b & \cdots & 0 \\ -N_{a3b} & -N^{0a}_{3b} & -\binom{3}{1}N^{0a}_{2b} & -\binom{3}{2}N^{0a}_{1b} & \cdots & 0 \\ \vdots & & & & & \\ -N_{akb} & -N^{0a}_{kb} & -\binom{k}{1}N^{0a}_{(k-1)b} & -\binom{k}{2}N^{0a}_{(k-2)b} & \cdots & \binom{k}{k}\delta^a_b \end{bmatrix}.$$

Definition 4.2. . *The special adapted basis* $\tilde{B}^* = \{dx^a, \delta_{p_{0a}}, \ldots, \delta_{p_{ka}}\}$ *of* $T^*(H)$ *is given by*

$$[\delta_{(a)}] = [\tilde{M}^{(b)}_{(a)}][d_{(b)}], \tag{4.9}$$

where

$$[\tilde{M}^{(b)}_{(a)}] = \begin{bmatrix} \delta^a_b & 0 & 0 & 0 & \cdots & 0 \\ M_{a0b} & \delta^b_a & 0 & 0 & \cdots & 0 \\ M_{a1b} & M^{0b}_{1a} & \binom{1}{1}\delta^b_a & 0 & \cdots & 0 \\ M_{a2b} & M^{0b}_{2a} & \binom{2}{1}M^{0b}_{1a} & \binom{2}{2}\delta^b_a & \cdots & 0 \\ M_{a3b} & M^{0b}_{3a} & \binom{3}{1}M^{0b}_{2a} & \binom{3}{2}M^{0b}_{1a} & \cdots & 0 \\ \vdots & & & & & \\ M_{akb} & M^{0b}_{ka} & \binom{k}{1}M^{0b}_{(k-1)a} & \binom{k}{k-1}M^{0b}_{(k-2)a} & \cdots & \binom{k}{k}\delta^b_a \end{bmatrix}. \tag{4.10}$$

Remark 4.1. If in Definition 4.1 we substitute (a, b) by (α, β) or $(\hat{\alpha}, \hat{\beta})$ we obtain the special adapted basis

$$\tilde{B}_1 = \{\delta_\alpha, \delta^{0\alpha}, \delta^{1\alpha}, \ldots, \delta^{k\alpha}\} \text{ of } T(H_1)$$

or

$$\tilde{B}_2 = \{\delta_{\hat{\alpha}}, \delta^{0\hat{\alpha}}, \delta^{1\hat{\alpha}}, \ldots, \delta^{k\hat{\alpha}}\} \text{ of } T(H_2).$$

Remark 4.2. If we in Definition 4.2 (a, b) substitute by (α, β) or $(\hat{\alpha}, \hat{\beta})$ we obtain the special adapted basis

$$\tilde{B}_1^* = \{du^\alpha, \delta_{p_{0\alpha}}, \delta_{p_{1\alpha}}, \dots, \delta_{p_{k\alpha}}\} \quad \text{of} \quad T^*(H_1)$$

or

$$\tilde{B}_2^* = \{\delta v^{\hat{\alpha}}, \delta_{p_{0\hat{\alpha}}}, \delta_{p_{1\hat{\alpha}}}, \dots, \delta_{p_{k\hat{\alpha}}}\} \quad \text{of} \quad T^*(H_2).$$

Remark 4.3. As the special adapted bases are special cases of adapted bases, so all Theorems 3.3-3.12 are valid for them.

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