Novi Sad J. Math.
Vol. 38, No. 3, 2008, 1-6

ANTILINEAR HILBERT-SCHMIDT OPERATORS THAT MAP ONE HILBERT SPACE INTO ANOTHER ISOMORPHIC TO TWO-PARTICLE VECTORS

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Abstract. In this article the antilinear-operator representation of two-particle state vectors (wave functions) in quantum mechanics and its application in distant correlations is reviewed. All proofs are omitted. They are contained in the references.

2003 PACS classification scheme: 03.65.Ca (Formalism); 03.65.Db (Functional analytical methods; 03.65Fd (Algebraic methods):

Key words and phrases: Bipartite entanglement; distant correlations

1. Introduction

Two detailed articles [2] and [3] on the topic from the title are shortly reviewed.

2. The isomorphism

Theorem 2.1. *The partial scalar product*

$$\forall\phi_1 : \quad \Big(\hat{O}_a\phi_1\Big)_2 \equiv \Big(\phi_1, \Psi_{12}\Big)_1 \tag{1}$$

associates with each vector Ψ_{12} from the tensor product $\Big(\mathcal{H}_1 \otimes \mathcal{H}_2\Big)$ an antilinear Hilbert-Schmidt operator (AHSO) $\hat{O}_a$ that maps $\mathcal{H}_1$ into $\mathcal{H}_2$, and the map $\Psi_{12} \rightarrow \hat{O}_a$ is an isomorphism of the state space (the Hilbert space) of the two-particle system $\Big(\mathcal{H}_1 \otimes \mathcal{H}_2\Big)$ onto the Hilbert space of all AHSO's $\hat{O}_a$ that map $\mathcal{H}_1$ into $\mathcal{H}_2$.

To understand the theorem, one should note the following remarks. By $\Big(\dots,\dots\Big)$ is meant the scalar product; $\Big(\dots,\dots\Big)_1$ denotes the partial scalar product, and $\Big(\dots\Big)_i$, $i = 1, 2, 12$ stands for a vector in the space $\mathcal{H}_i$, $i = 1, 2, 12$. The index shows in which space the entity is (except in case of the partial scalar product). Finally, the operators are denoted by a hat to distinguish them from the vectors, etc.

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Each AHSO $\hat{O}_a$ determines its *adjoint* AHSO $\hat{O}_a^\dagger$ by the scalar products in the two spaces:

$$\forall \psi_1, \phi_2 : \quad \Big(\hat{O}_a \psi_1, \phi_2\Big)_2 = \Big(\psi_1, \hat{O}_a^\dagger \phi_2\Big)_1^*,$$

where the asterisk denotes complex conjugation. One should note that $\hat{O}_a^\dagger$ maps $\mathcal{H}_2$ into $\mathcal{H}_1$.

An antilinear operator $\hat{O}_a$ is said to be a Hilbert-Schmidt one if $\mathrm{tr}(\hat{O}_a^\dagger \hat{O}_a) < \infty$. The scalar product in the Hilbert space of all AHSO's $\hat{O}_a$ is defined by

$$\Big(\hat{O}_a, \hat{O}'_a\Big) \equiv \mathrm{tr}[(\hat{O}'_a)^\dagger \hat{O}_a)].$$

In the rest of this review we confine ourselves to infinite-dimensional Hilbert spaces because they are the mathematically interesting case.

3. Role of AHSO's in quantum-mechanical distant correlations

Let $\{\phi_1^p : p = 1, 2, \ldots, \infty\}$ be an arbitrary complete orthonormal basis in $\mathcal{H}_1$. One can expand an arbitrary two-particle state vector $\Psi_{12} \in \mathcal{H}_1 \otimes \mathcal{H}_2$ in it, and the (generalized) expansion coefficients, which are vectors in $\mathcal{H}_2$, turn out to be the $\hat{O}_a$ images of the corresponding basis vectors. Naturally, $\hat{O}_a$ is the AHSO representative of Ψ_{12}. One has

$$\Psi_{12} = \sum_p \phi_1^p \otimes \Big(\hat{O}_a \phi_1^p\Big)_2. \tag{2}$$

Distant correlations are concerned with the second particle. Its quantum-mechanical state (reduced density operator) is $\hat{\rho}_2 \equiv \mathrm{tr}_1 \hat{P}_{\Psi_{12}}$, where "$\mathrm{tr}_1$" denotes the partial trace over $\mathcal{H}_1$, and "$\hat{P}_{\ldots}$" stands for the projector onto the (one-dimensional) space spanned by the vector in the index.

One obtains

$$\hat{\rho}_2 = \sum_p \Big|\Big|\hat{O}_a \phi_1^p\Big|\Big|^2 \hat{P}_{(\hat{O}_a \phi_1^p)_2}. \tag{3}$$

If Ψ_{12} is taken to describe an ensemble of two-particle systems (it describes also the individual ones), then $\hat{\rho}_{12}$ describes the, so-called improper, ensemble of second particles. Relation (3) is an ensemble decomposition with $\Big|\Big|\hat{O}_a \phi_1^p\Big|\Big|^2$ as the statistical weights. (They are also the probabilities that in a measurement of the above first-particle basis vectors one obtains ϕ_1^p.)

If one has in mind the individual two-particle systems, then one is dealing with the terms in decomposition (3), i. e., one has the change of single-particle states

$$\hat{\rho}_1 \;\rightarrow\; \phi_1^p, \quad \hat{\rho}_2 \;\rightarrow\; \hat{P}_{(\hat{O}_a \phi_1^p)_2}. \tag{4}$$

Since the mentioned measurement affects dynamically only the first particles, the change in state of the second particle is due exclusively to the quantum correlations in Ψ_{12} called entanglement. There are many physical examples in which the changed state $\hat{P}_{(\hat{O}_a\phi_1^p)}$ is very surprising. Einstein called this "spooky distant action" in 1935. But all interesting cases have been experimentally confirmed.

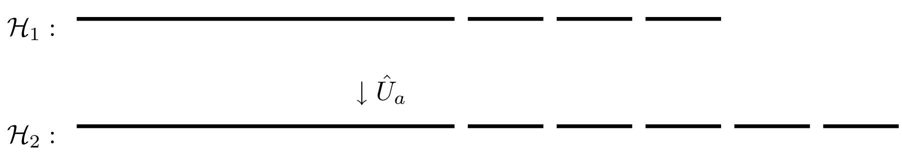
4. Polar factorization of the AHSO

Herbut and Vujčić have shown [2] that *polar factorization* of the AHSO representatives of two-particle state vectors is relevant in quantum mechanics, particularly in distant correlations. The polar factorizations of a given AHSO read:

$$\hat{O}_a = \hat{\rho}_2^{1/2} \circ \hat{U}_a \circ \hat{Q}_1 = \hat{U}_a \circ \hat{\rho}_1^{1/2}. \tag{5}$$

Here " $\circ$ " denotes the application "after" (reading from right to left), $\hat{\rho}_i \equiv \mathrm{tr}_j\Big(\mid \Psi\rangle_{12}\langle\Psi \mid_{12} \Big)$, $i,j = 1,2$, $i \neq j$ are the respective states (reduced density operators) of the single particles, and the trace with an index denotes the corresponding partial trace. $\hat{Q}_1$ is the projector onto the range of ρ_1, and, finally, the antiunitary correlation operator $\hat{U}_a$, which is, as seen, uniquely implied by the bipartite state vector, maps the (topologically) closed range $\bar{\mathcal{R}}(\rho_1)$ of ρ_1 onto that of ρ_2.

The action of the correlation operator is presented on the following diagram.



The lines are meant to represent the orthogonal decompositions

$$\mathcal{H}_i = \bar{\mathcal{R}}(\rho_i) \ \oplus \ \mathcal{N}(\rho_i), \qquad i = 1,2,$$

where $\mathcal{N}(\rho_i)$ stands for the null space of ρ_i.

One should note that the ranges are always equally dimensional (this is represented by the equal length of the lines), but the null spaces need not be.

The correlation operator is the carrier of the correlations in the composite-system state. As far as we know, this is the only case when one can find an entity that is responsible for the correlations.

Let $\{\phi_1^k : \forall k\}$ be a complete orthonormal *eigen-basis* of ρ_1. in $\mathcal{H}_1$. (In quantum mechanics every basis corresponds to a complete first-particle observable.) This means the following in the formalism:

$$\forall k: \quad \rho_1\phi_1^k = r_k\phi_1^k. \tag{6}$$

If one takes the general expansion (2) and one substitutes in it the first-particle basis with the eigen-basis (6) and the AHSO representative with its second polar form (5), one obtains:

$$\Psi_{12} = \sum_k r_k^{1/2}\phi_1^k \otimes \Big(\hat{U}_a\phi_1^k\Big)_2. \tag{7}$$

Decomposition (7) is the so-called canonical Schmidt decomposition. It is bi-orthonormal (disregarding the $r_k^{1/2}$ coefficients). Hence, if one measures on the first particle an observable (Hermitian operator) whose eigen-basis is (6), then *ipso facto*, due to the entanglement in Ψ_{12}, one measures also on particle 2 the observable whose eigen-basis $\{\left(\hat{U}_a\phi_1^k\right)_2 : \forall k\}$ is. This is called *distant measurement*. It plays an important role in distant correlations.

The present author et al. presented the results reviewed so far at the International Mathematical Congress in Helsinki in 1978.

5. Linearly independent density operator decomposition in mathematics

Cassinelli, De Vito, and Levrero published an article in 1997 [1] in which they gave a complete solution of the problem in the title of the section. We outline it.

Now we are in one countably-infinite dimensional Hilbert space, and we are interested in density operators with an infinite-dimensional range, particularly of their decomposition into linearly independent ray projectors. We assume that such a density operator $\hat{\rho}$ is given.

An infinite sequence of vectors $\{\psi^p : p = 1, 2, \ldots, \infty\}$ is *linearly independent* if $\forall p : \quad \psi^p \notin \overline{span}\{\psi^{p'} : p' \neq p\}$.

Let $\{e^p : p = 1, 2, \ldots, \infty\}$ be a complete orthonormal basis in $\bar{\mathcal{R}}(\hat{\rho})$ but such that $\forall p : \quad e^p \in \mathcal{R}(\hat{\rho}^{1/2})$. One should keep in mind that

$$\mathcal{R}(\hat{\rho}) \subset \mathcal{R}(\hat{\rho}^{1/2}) \subset \bar{\mathcal{R}}(\hat{\rho}). \tag{8}$$

If one defines

$$\forall p : \quad \psi^p \equiv \hat{\rho}^{1/2}e^p \Big/ ||\hat{\rho}^{1/2}e^p||, \quad w_p \equiv ||\hat{\rho}^{1/2}e^p||^2, \tag{9}$$

then the infinite sequence $\{\psi^p : p = 1, 2, \ldots, \infty\}$ is linearly independent, one has

$$\hat{\rho} = \sum_{p=1}^{\infty} w_p \hat{P}_{\psi^p}, \tag{10}$$

a decomposition of the density operator into linearly independent ray projectors with the statistical weights w_p, and, finally, every decomposition into linearly independent ray projectors can be obtained via (9) from an orthonormal basis $\{e^p : p = 1, 2, \ldots, \infty\}$. Besides, for each ψ^p that appears in a term of decomposition (10), one has $\psi^p \in \mathcal{R}(\hat{\rho})$.

6. Back to distant correlations

Let it now be divulged that the correlation operator $\hat{U}_a$ introduced in the polar factorizations (5) of the AHSO representative $\hat{O}_a$ of a given two-particle state vector Ψ_{12}, takes, by similarity transformation, the first-particle reduced density operator of Ψ_{12} into that of the second particle:

$$\hat{\rho}_2 = \hat{U}_a \circ \hat{\rho}_1 \circ \hat{U}_a^{-1} \circ \hat{Q}_2, \tag{11}$$

where $\hat{Q}_2$ is the range projector of $\hat{\rho}_2$.

This has the consequence that

$$\mathcal{R}(\hat{\rho}_2^{1/2}) = \hat{U}_a \mathcal{R}(\hat{\rho}_1^{1/2}). \tag{12}$$

The result of the three mathematicians from the preceding section can now be used in distant-correlations theory after modifying it as follows.

We take a complete orthonormal basis $\{e_1^p : p = 1, 2, \ldots, \infty\}$ in $\bar{\mathcal{R}}(\hat{\rho}_1)$ such that $\forall p: \; e_1^p \in \mathcal{R}(\rho_1^{1/2})$. Then we generate $\forall p: \; \psi_2^p \equiv \hat{O}_a e_1^p \Big/ ||\hat{O}_a e_1^p||$.

Utilizing the first polar factorization in (5), we have actually $\forall p: \; \psi_2^p = \hat{\rho}_2^{1/2} \circ \hat{U}_a e_1^p \Big/ ||\hat{O}_a e_1^p||$, and, according to [2], we have a decomposition of of $\hat{\rho}_2$ into linearly independent states:

$$\hat{\rho}_2 = \sum_p ||\hat{O}_a e_1^p||^2 \hat{P}_{(\hat{O}_a e_1^p)}. \tag{13}$$

Thus, performing a direct measurement on the first particle that ascertains in which of the states $\{e_1^p : p = 1, 2, \ldots, \infty\}$ the particle is, though neither the measuring apparatus nor the first particle interact with the second particle, it is decomposed according to (13).

In the selective-measurement version, if in the described measurement the first particle is found in the state e_1^p, then the second particle is *ipso facto* in the state $\hat{P}_{(\hat{O}_a e_1^p)}$ as follows from the general expansion (2), which is now $\Psi_{12} = \sum_p e_1^p \otimes \left(\hat{O}_a e_1^p\right)_2$.

It can also be shown that if one want to steer the second particle into the state $\hat{P}_{(\hat{O}_a e_1^p)}$, though this can be done in different ways, one has the largest probability of success if one does it in the described way that leads, in the non-selective version, to the decomposition (13) into linearly independent states.

Acknowledgement

The author would like to thank the organizers for having invited him to give a talk and for their warm hospitality.

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Received by the editors October 1, 2008