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EXISTENCE OF GLOBAL WEAK SOLUTIONS TO COUPLED NAVIER–STOKES–FOKKER–PLANCK SYSTEMS: A BRIEF SURVEY[1]

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Abstract. We present a brief survey of recent results concerning the existence of global-in-time weak solutions in a bounded Lipschitz domain in $\mathbb{R}^d$, $d \in \{2, 3\}$, to a class of kinetic models for dilute polymeric liquids with noninteracting polymer chains. The mathematical model is a coupled Navier–Stokes–Fokker–Planck system. The velocity and the pressure of the fluid satisfy a Navier–Stokes-like system of partial differential equations, with an elastic extra-stress tensor appearing on the right-hand side of the momentum equation. The elastic extra-stress tensor stems from the random movement and stretching of the polymer chains and is defined through the associated probability density function, which satisfies a Fokker–Planck type parabolic equation, a crucial feature of which is the presence of a centre-of-mass diffusion term, an unbounded drift term, and microscopic cut-off in the drag term. The Fokker–Planck equation admits a general class of unbounded spring-force potentials, including in particular the FENE (Finitely Extensible Nonlinear Elastic) potential.

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1. Introduction

Polymer solutions exhibit a range of non-Newtonian flow properties: in particular, the stress endured by a fluid element depends upon the history of deformations experienced by that element; the rheological properties of non-Newtonian fluids are governed by the flow-induced evolution of their internal microstructure. Following Keunings [9], a relevant feature of the microstructure is the *conformation* of the macromolecules, i.e., their orientation and the degree of stretching they experience. From the macroscopic viewpoint it is only the statistical distribution of conformations that matters: the macroscopic stress carried by each fluid element is governed by the distribution of polymer conformations within that element. Motivated by this observation, kinetic theories

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of polymeric fluids ignore quantum mechanical and atomistic effects and focus on "coarse-grained" models of the polymeric conformations. Depending on the level of coarse graining, one may arrive at a hierarchy of kinetic models. As has been noted in [9], such coarse-grained models are not meant to capture the detailed structure of the polymer; rather, they are intended to describe, in more or less detail, the evolution of polymer conformations in a macroscopic flow.

Many of the interesting properties of dilute polymer solutions can be understood by modelling them as suspensions of simple coarse-grained objects, such as dumbbells, in an incompressible Newtonian fluid. A dumbbell is simply a pair of beads of small mass, connected with an elastic spring. This paper is devoted to the mathematical analysis of dumbbell models which are nonlinearly coupled Navier–Stokes–Fokker–Planck systems of partial differential equations: from the technical viewpoint such relatively simple models already exemplify many of the analytical difficulties encountered in the study of more complex dilute polymer models.

2. Statement of the dumbbell model

Given $T \in \mathbb{R}_{>0}$, the evolution of the macroscopic properties of the fluid over the time interval $[0,T]$ is governed by the following set of partial differential equations:

$$\frac{\partial \underset{\sim}{u}}{\partial t} + (\underset{\sim}{u}\cdot\underset{\sim}{\nabla}_x)\underset{\sim}{u} - \nu\Delta_x\underset{\sim}{u} + \underset{\sim}{\nabla}_x p = \underset{\sim}{f} + \underset{\sim}{\nabla}_x\cdot\underset{\approx}{\tau}(\psi) \qquad \text{in } \Omega\times(0,T], \tag{1}$$

$$\underset{\sim}{\nabla}_x\cdot\underset{\sim}{u} = 0 \qquad \text{in } \Omega\times(0,T], \tag{2}$$

$$\underset{\sim}{u} = \underset{\sim}{0} \qquad \text{on } \partial\Omega\times(0,T], \tag{3}$$

$$\underset{\sim}{u}(\underset{\sim}{x},0) = \underset{\sim}{u}_0(\underset{\sim}{x}) \qquad \text{for all } \underset{\sim}{x}\in\Omega, \tag{4}$$

where $\underset{\sim}{u} : (\underset{\sim}{x},t) \in \overline{\Omega}\times[0,T] \to \underset{\sim}{u}(\underset{\sim}{x},t) \in \mathbb{R}^d$, $d\in\{2,3\}$, is the velocity of the fluid, $p : (\underset{\sim}{x},t)\in\Omega\times(0,T]\to p(\underset{\sim}{x},t)\in\mathbb{R}$ is its pressure, $\nu\in\mathbb{R}_{>0}$ its kinematic viscosity, $\underset{\sim}{f} : (\underset{\sim}{x},t)\in\Omega\times[0,T]\to\underset{\sim}{f}(\underset{\sim}{x},t)\in\mathbb{R}^d$ the density of body forces, and $\underset{\sim}{u}_0 : \underset{\sim}{x}\in\overline{\Omega}\to\underset{\sim}{u}_0(\underset{\sim}{x})\in\mathbb{R}^d$ is a specified initial velocity field. The flow domain Ω is a bounded open set in $\mathbb{R}^d$ with Lipschitz continuous boundary. Further, $\underset{\approx}{\tau}(\psi) : (\underset{\sim}{x},t)\in\mathbb{R}^{d+1}\mapsto\underset{\approx}{\tau}(\underset{\sim}{x},t)\in\mathbb{R}^{d\times d}$ denotes the symmetric elastic extra stress tensor, defined by $\underset{\approx}{\tau}(\psi) := k\mu\left(\underset{\approx}{C}(\psi)-\rho(\psi)\underset{\approx}{I}\right)$. Here $k,\mu\in\mathbb{R}_{>0}$ are, respectively, the Boltzmann constant and the absolute temperature, $\underset{\approx}{I}$ is the unit $d\times d$ tensor, $\underset{\approx}{C}(\psi)(\underset{\sim}{x},t)$ is defined by the Kramers expression

$$\underset{\approx}{C}(\psi)(\underset{\sim}{x},t) := \int_D \psi(\underset{\sim}{x},\underset{\sim}{q},t)U'(\tfrac{1}{2}|\underset{\sim}{q}|^2)\,\underset{\sim}{q}\underset{\sim}{q}^{\mathrm{T}}\,\mathrm{d}\underset{\sim}{q} \quad\text{and}\quad \rho(\psi)(\underset{\sim}{x},t) := \textstyle\int_D \psi(\underset{\sim}{x},\underset{\sim}{q},t)\,\mathrm{d}\underset{\sim}{q}.$$

The set D is the ball in $\mathbb{R}^d$ centred at the origin and of radius $\sqrt{b}$, where $b>2$; it represents the set of all admissible elongation vectors $\underset{\sim}{q}$ of the polymer molecules immersed in the Newtonian solvent.

Consider the spring potential $U\in C^\infty([0,b/2);\mathbb{R}_{\geq 0})$. We shall suppose that $U(0)=0$ and that U is monotonic increasing and unbounded on $[0,b/2)$. The

elastic spring-force $\underset{\sim}{F} : D \subset \mathbb{R}^d \mapsto \mathbb{R}^d$ is then defined by $\underset{\sim}{F}(\underset{\sim}{q}) := U(\frac{1}{2}|\underset{\sim}{q}|^2)\underset{\sim}{q}$. We shall further suppose that there exist constants $c_i > 0$, $i = 1, 2, 3, 4$, and $\gamma > 1$ such that, for all $\underset{\sim}{q} \in D$, the (normalized) Maxwellian M, defined by

$$M(\underset{\sim}{q}) := Z^{-1}\mathrm{e}^{-U(\frac{1}{2}|\underset{\sim}{q}|^2)}, \quad \text{where} \quad Z := \int_D \mathrm{e}^{-U(\frac{1}{2}|\underset{\sim}{q}|^2)}\,\mathrm{d}\underset{\sim}{q},$$

and the associated potential U satisfy

$$c_1[\mathrm{dist}(\underset{\sim}{q}, \partial D)]^\gamma \le M(\underset{\sim}{q}) \le c_2[\mathrm{dist}(\underset{\sim}{q}, \partial D)]^\gamma, \quad c_3 \le [\mathrm{dist}(\underset{\sim}{q}, \partial D)]U(\tfrac{1}{2}|\underset{\sim}{q}|^2) \le c_4.$$

In the FENE (Finitely Extensible Nonlinear Elastic) dumbbell model, for example, the spring force is given by $\underset{\sim}{F}(\underset{\sim}{q}) = (1-|\underset{\sim}{q}|^2/b)^{-1}\underset{\sim}{q}$, $\underset{\sim}{q} \in D := B(\underset{\sim}{0}, \sqrt{b})$, corresponding to $U(s) = -\frac{b}{2}\ln\left(1 - \frac{2s}{b}\right)$, $s \in [0, b/2)$. Direct calculations show that the Maxwellian M and the elastic potential U of the FENE model satisfy the above inequalities with $\gamma = b/2$ provided that $b > 2$. It is interesting to note that in the (equivalent) stochastic version of the FENE model a solution to the system of stochastic differential equations associated with the Fokker–Planck equation exists and has trajectorial uniqueness if, and only if, $b > 2$. Thus, the assumption $\gamma > 1$ can be seen as the weakest reasonable requirement on the decay-rate of the Maxwellian M in the inequalities above as $\mathrm{dist}(\underset{\sim}{q}, \partial D) \to 0$.

The function $\psi : (\underset{\sim}{x}, \underset{\sim}{q}, t) \in \mathbb{R}^{2d+1} \mapsto \psi(\underset{\sim}{x}, \underset{\sim}{q}, t) \in \mathbb{R}$ featuring in the definition of the elastic extra stress tensor $\underset{\approx}{\tau}$ above is a probability density function satisfying the Fokker–Planck equation with microscopic cut-off:

$$\begin{aligned} &\frac{\partial\psi}{\partial t} + (\underset{\sim}{u}\cdot\underset{\sim}{\nabla}_x)\psi + \underset{\sim}{\nabla}_q \cdot \left[\underset{\approx}{\sigma}(\underset{\sim}{u})\underset{\sim}{q}M\beta^L\left(\frac{\psi}{M}\right)\right] \\ &\qquad = \frac{1}{2\lambda}\underset{\sim}{\nabla}_q \cdot \left(M\underset{\sim}{\nabla}_q\left(\frac{\psi}{M}\right)\right) + \varepsilon\Delta_x\psi \qquad \text{in } \Omega\times D\times(0,T], \end{aligned} \tag{5}$$

with $\underset{\approx}{\sigma}(\underset{\sim}{v}) := \underset{\approx}{\nabla}_x \underset{\sim}{v}$, where $\underset{\approx}{\nabla}_x \underset{\sim}{v}(\underset{\sim}{x}, t) \in \mathbb{R}^{d\times d}$ and $\{\underset{\approx}{\nabla}_x \underset{\sim}{v}\}_{ij} := \frac{\partial v_i}{\partial x_j}$. The positive real number $\varepsilon := \ell_0^2/(8\lambda)$ is the centre-of-mass diffusion coefficient of the dumbbells, $\ell_0 \ll \mathrm{diam}(\Omega)$ is the characteristic microscopic length scale (i.e. the characteristic dumbbell size) and $\lambda = \zeta/(4H)$; the parameter $\lambda \in \mathbb{R}_{>0}$ characterizes the elastic relaxation property of the fluid, $\zeta > 0$ is the drag coefficient and $H > 0$ is a spring-constant.

We impose the following boundary and initial conditions on ψ:

$$\frac{M}{2\lambda}\underset{\sim}{\nabla}_q\left(\frac{\psi}{M}\right)\cdot\frac{\underset{\sim}{q}}{|\underset{\sim}{q}|} - \underset{\approx}{\sigma}(\underset{\sim}{u})\underset{\sim}{q}M\beta^L\left(\frac{\psi}{M}\right) = 0 \qquad \text{on } \Omega\times\partial D\times(0,T], \tag{6}$$

$$\varepsilon\underset{\sim}{\nabla}_x\psi\cdot\underset{\sim}{n} = 0 \qquad \text{on } \partial\Omega\times D\times(0,T], \tag{7}$$

$$\psi(\underset{\sim}{x}, \underset{\sim}{q}, 0) = \psi_0(\underset{\sim}{x}, \underset{\sim}{q}) \qquad \text{for all } (\underset{\sim}{x}, \underset{\sim}{q}) \in \Omega\times D, \tag{8}$$

where $\underset{\sim}{q}$ is normal to ∂D, as D is a bounded ball centred at the origin, and $\underset{\sim}{n}$ is the unit outward normal vector to $\partial\Omega$. Here $\int_D \psi_0(\underset{\sim}{x}, \underset{\sim}{q})\,\mathrm{d}\underset{\sim}{q} = 1$ for a.e. $\underset{\sim}{x} \in \Omega$, $\psi \ge 0$ for a.e. $(\underset{\sim}{x}, \underset{\sim}{q}) \in \Omega\times D$ and, for any $L > 0$, we define the cut-off

function $\beta^L(s) := (s-L)_- + L$. The boundary conditions (6), (7) for ψ on $\Omega \times \partial D \times (0,T]$ and $\partial\Omega \times D \times (0,T]$ have been chosen so as to ensure that $\frac{1}{|\Omega|}\int_{\Omega\times D} \psi(\underset{\sim}{x},\underset{\sim}{q},t)\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} = \frac{1}{|\Omega|}\int_{\Omega\times D}\psi_0(\underset{\sim}{x},\underset{\sim}{q})\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} = 1$ for all $t \geq 0$. We note that if ψ/M is bounded above, then, for $L \in \mathbb{R}_{>0}$ sufficiently large, the drag term in the Fokker–Planck (5) is equal to

$$\underset{\sim}{\nabla}_q \cdot \left[\underset{\approx}{\sigma}(\underset{\sim}{u})\underset{\sim}{q} M \beta^L\left(\frac{\psi}{M}\right)\right] = \underset{\sim}{\nabla}_q \cdot \left[\underset{\approx}{\sigma}(\underset{\sim}{u})\underset{\sim}{q} M\left(\frac{\psi}{M}\right)\right] = \underset{\sim}{\nabla}_q \cdot \left[\underset{\approx}{\sigma}(\underset{\sim}{u})\underset{\sim}{q}\psi\right],$$

the expression on the right being the standard form of the drag term. Hence the terminology *microscopic cut-off*. The cut-off β^L proposed by Barrett and Süli in [3] and recalled here has several attractive properties. We observe that the couple $(\underset{\sim}{u}_\infty, \psi_\infty)$ defined by $\underset{\sim}{u}_\infty(\underset{\sim}{x}) := \underset{\sim}{0}$ and $\psi_\infty(\underset{\sim}{x},\underset{\sim}{q}) := M(\underset{\sim}{q})$, is an equilibrium solution of the Fokker–Planck equation with cut-off (for all $L > 0$) and without cut-off, when $\underset{\sim}{f} = \underset{\sim}{0}$. Thus, unlike the truncation of the (unbounded) potential U proposed in El-Kareh and Leal [7], the introduction of the cut-off function β^L into the Fokker–Planck equation does not alter the equilibrium solution $(\underset{\sim}{u}_\infty, \psi_\infty)$ of the original Navier–Stokes–Fokker–Planck system.

Motivated by the form of the Fokker–Planck equation and the boundary conditions, we introduce the following substitutions: $\widehat{\psi} := \frac{\psi}{M}$, $\widehat{\psi}_0 := \frac{\psi_0}{M}$. Similarly, for any function φ, we shall write $\widehat{\varphi}$ instead of φ/M.

Since $\underset{\sim}{u}$, p and ψ all depend on the cut-off parameter L as well as on the centre-of-mass diffusion coefficient ε, we shall henceforth write $\underset{\sim}{u}_{\varepsilon,L}$, $p_{\varepsilon,L}$ and $\psi_{\varepsilon,L}$ instead of $\underset{\sim}{u}$, p and ψ, respectively, in order to highlight the dependence of these functions on ε and L.

We close this section by noting that in the simple Hookean dumbbell model the spring force is defined by $\underset{\sim}{F}(\underset{\sim}{q}) = \underset{\sim}{q}$, with $\underset{\sim}{q} \in D = \mathbb{R}^d$, corresponding to $U(s) = s$, $s \in [0,\infty)$. Unfortunately, this model is physically unrealistic as it admits arbitrarily large extensions. It is more appropriate to assume therefore that D is a bounded open set in $\mathbb{R}^d$ containing the origin. Since in the case of a general flow velocity all directions of stretching $\underset{\sim}{q}$ are equally likely, we have assumed that D is an open ball in $\mathbb{R}^d$ centred at the origin $\underset{\sim}{0} \in \mathbb{R}^d$.

3. Existence of global weak solutions

An early contribution to the existence and uniqueness of local-in-time solutions to a family of bead-spring type polymeric flow models is due to Renardy [15]. While the class of potentials $\underset{\sim}{F}(\underset{\sim}{q})$ considered by Renardy does include the case of Hookean dumbbells, it excludes the practically relevant case of the FENE model. We refer to E, Li, and Zhang [6], Li, Zhang, and Zhang [10] and [17] concerning local existence of regular solutions to FENE-type models. All of these papers require high regularity of the initial data. Constantin [4] considered the Navier–Stokes equations coupled to nonlinear Fokker–Planck equations describing the evolution of the probability distribution of the particles interacting with the fluid. The necessary relationship (eq. (2.14) in [4]) for the existence of a Lyapunov function in the sense of Theorem 2.2 of [4] does not hold for the

polymer models considered herein. Otto and Tzavaras [14] have investigated the Doi model (which is similar to a Hookean model (except that $D = S^2$) for suspensions of rod-like molecules in the dilute regime. They considered the evolution of possibly large perturbations of stationary flows and proved that, even in the absence of a microscopic cut-off, discontinuities in the velocity gradient cannot occur in finite time. Jourdain, Lelièvre, and Le Bris [8] studied the existence of solutions to the FENE model in the case of a simple Couette flow. By using tools from the theory of stochastic differential equations, they established the existence of a unique local-in-time solution to the FENE model in two space dimensions ($d = 2$) when the velocity field $\underset{\sim}{u}$ is unidirectional and of the particular form $\underset{\sim}{u}(x_1, x_2) = (u_1(x_2), 0)^{\mathrm{T}}$.

In the case of Hookean dumbbells, and with $\varepsilon = 0$, the coupled microscopic-macroscopic model described above yields, formally, taking the second moment of $\underset{\sim}{q} \mapsto \psi(\underset{\sim}{q}, \underset{\sim}{x}, t)$, the fully macroscopic, Oldroyd-B model of viscoelastic flow. Lions and Masmoudi [11] showed the existence of global-in-time weak solutions to the Oldroyd-B model in a simplified corotational setting (i.e. with $\sigma(\underset{\sim}{u}) = \underset{\approx}{\nabla}_x \underset{\sim}{u}$ replaced by $\frac{1}{2}(\underset{\approx}{\nabla}_x \underset{\sim}{u} - (\underset{\approx}{\nabla}_x u)^{\mathrm{T}})$). It is not known if an identical global existence result for the Oldroyd-B model also holds in the absence of the crucial assumption that the drag term is corotational. We note in passing that, assuming $\varepsilon > 0$, the coupled microscopic-macroscopic model above yields, taking the appropriate moments in the case of Hookean dumbbells, a dissipative version of the Oldroyd-B model. In this sense, the Hookean dumbbell model has a macroscopic closure: it is the Oldroyd-B model when $\varepsilon = 0$, and a dissipative version of Oldroyd-B when $\varepsilon > 0$ (cf. Barrett and Süli [2]). In contrast, the FENE model is not known to have an exact closure at the macroscopic level, though Du, Yu, and Liu [5] and Yu, Du, and Liu [16] have recently considered the analysis of approximate closures of the FENE model. Lions and Masmoudi [12] proved the global existence of weak solutions for the corotational FENE model, once again corresponding to the case of $\varepsilon = 0$, and the Doi model, also called the rod model. As in Lions and Masmoudi [11], the proof is based on propagation of compactness; see also the related paper of Masmoudi [13].

Previously, El-Kareh and Leal [7] had proposed a macroscopic model, with added dissipation in the equation that governs the evolution of the conformation tensor $\underset{\approx}{A}(\underset{\sim}{x}, t) := \int_D \underset{\sim}{q}\,\underset{\sim}{q}^{\mathrm{T}} U'(\frac{1}{2}|\underset{\sim}{q}|^2)\,\psi(\underset{\sim}{x}, \underset{\sim}{q}, t)\,\mathrm{d}\underset{\sim}{q}$, in order to account for Brownian motion across streamlines; the model can be thought of as an approximate macroscopic closure of a FENE-type micro-macro model with centre-of-mass diffusion.

Barrett, Schwab, and Süli [1] established the existence of, global in time, weak solutions to the coupled Navier–Stokes–Fokker–Planck model with $\varepsilon = 0$, an $\underset{\sim}{x}$-mollified velocity gradient in the Fokker–Planck equation and an $\underset{\sim}{x}$-mollified probability density function ψ in the Kramers expression—admitting a large class of potentials U (including the Hookean dumbbell model as well as general FENE-type models); in addition to these mollifications, $\underset{\sim}{u}$ in the $\underset{\sim}{x}$-convective term $(\underset{\sim}{u} \cdot \underset{\sim}{\nabla}_x)\psi$ in the Fokker–Planck equation was also mollified. Unlike Lions and Masmoudi [11], the arguments in Barrett, Schwab, and

Süli [1] did not require the assumption that the drag term was corotational in the FENE case. In Barrett and Süli [2], we derived the coupled Navier–Stokes–Fokker–Planck model with centre-of-mass diffusion stated above. The anisotropic Friedrichs mollifiers, which naturally arise in the derivation of the model in the Kramers expression for the extra stress tensor and in the drag term in the Fokker–Planck equation, were replaced by isotropic Friedrichs mollifiers. We established the existence of global-in-time weak solutions to the model for a general class of spring-force-potentials including in particular the FENE potential. We justified also, through a rigorous limiting process, certain classical reductions of this model appearing in the literature that exclude the centre-of-mass diffusion term from the Fokker–Planck equation on the grounds that the diffusion coefficient is small relative to other coefficients featuring in the equation. In the case of a corotational drag term we performed a rigorous passage to the limit as the mollifiers in the Kramers expression and the drag term converge to identity operators.

In this paper neither the probability density function ψ in the Kramers expression nor the velocity field $\underset{\sim}{u}$ in the drag term appearing in (5) will be mollified. Let $\underset{\sim}{H} := \{\underset{\sim}{w} \in \underset{\sim}{L}^2(\Omega) : \underset{\sim}{\nabla}_x \cdot \underset{\sim}{w} = 0\}$ and $\underset{\sim}{V} := \{\underset{\sim}{w} \in \underset{\sim}{H}^1_0(\Omega) : \underset{\sim}{\nabla}_x \cdot \underset{\sim}{w} = 0\}$, where the divergence operator $\underset{\sim}{\nabla}_x \cdot$ is to be understood in the sense of vector-valued distributions on Ω. Let $\underset{\sim}{V}'$ be the dual of $\underset{\sim}{V}$, with duality pairing $\langle \cdot, \cdot \rangle_V$. Let $L^2_M(\Omega \times D)$ denote the Maxwellian-weighted L^2 space over $\Omega \times D$ with norm

$$\|\widehat{\varphi}\|_{L^2_M(\Omega\times D)} := \left\{ \int_{\Omega\times D} M\, |\widehat{\varphi}|^2 \, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} \right\}^{\frac{1}{2}}.$$

On introducing

$$\|\widehat{\varphi}\|_{H^1_M(\Omega\times D)} := \left\{ \int_{\Omega\times D} M \left[|\widehat{\varphi}|^2 + |\underset{\sim}{\nabla}_x \widehat{\varphi}|^2 + |\underset{\sim}{\nabla}_q \widehat{\varphi}|^2 \right] \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} \right\}^{\frac{1}{2}},$$

we then set

$$\widehat{X} \equiv H^1_M(\Omega \times D) := \left\{ \widehat{\varphi} \in L^1_{\mathrm{loc}}(\Omega \times D) : \|\widehat{\varphi}\|_{H^1_M(\Omega\times D)} < \infty \right\}.$$

We denote by $\widehat{X}'$ the dual space of $\widehat{X}$, with duality pairing $\langle \cdot, \cdot \rangle_{\widehat{X}}$.

In [3], we established the following result concerning the global existence of weak solutions to the coupled Fokker–Planck–Navier–Stokes system (1)–(5).

Theorem 3.1. *Suppose that* $\partial\Omega \in C^{0,1}$, $\underset{\sim}{u}_0 \in \underset{\sim}{H}$, $\widehat{\psi}_0 := M^{-1}\psi_0 \in L^2_M(\Omega \times D)$, $\widehat{\psi}_0 \geq 0$ *a.e. on* $\Omega \times D$, *and* $\underset{\sim}{f} \in L^{\frac{4}{d}}(0,T;V')$. *Suppose further that* $\varepsilon \in (0,1]$ *and* $L > 1$. *Then, there exists* $\underset{\sim}{u}_{\varepsilon,L} \in L^\infty(0,T;\underset{\sim}{L}^2(\Omega)) \cap L^2(0,T;\underset{\sim}{V}) \cap W^{1,\frac{4}{d}}(0,T;\underset{\sim}{V}')$ *and* $\widehat{\psi}_{\varepsilon,L} \in L^\infty(0,T;L^2_M(\Omega\times D)) \cap L^2(0,T;\widehat{X}) \cap W^{1,\frac{4}{d}}(0,T;\widehat{X}')$, *with* $\underset{\approx}{C}(M\,\widehat{\psi}_{\varepsilon,L}) \in L^\infty(0,T;\underset{\approx}{L}^2(\Omega))$, *such that* $\underset{\sim}{u}_{\varepsilon,L}(\cdot,0) = \underset{\sim}{u}_0(\cdot)$, $\widehat{\psi}_{\varepsilon,L}(\cdot,0) =$

$\widehat{\psi}_0(\cdot)$, *and*

$$\int_0^T \left\langle \frac{\partial \underset{\sim}{u}_{\varepsilon,L}}{\partial t}, \underset{\sim}{w} \right\rangle_V \mathrm{d}t + \int_0^T \int_\Omega \left[[(\underset{\sim}{u}_{\varepsilon,L} \cdot \underset{\sim}{\nabla}_x)\underset{\sim}{u}_{\varepsilon,L}] \cdot \underset{\sim}{w} + \nu \underset{\approx}{\nabla}_x \underset{\sim}{u}_{\varepsilon,L} : \underset{\approx}{\nabla}_x \underset{\sim}{w} \right] \mathrm{d}\underset{\sim}{x}\,\mathrm{d}t$$
$$= \int_0^T \langle \underset{\sim}{f}, \underset{\sim}{w} \rangle_V \,\mathrm{d}t - k\,\mu \int_0^T \int_\Omega \underset{\approx}{C}(M\,\widehat{\psi}_{\varepsilon,L}) : \underset{\approx}{\nabla}_x \underset{\sim}{w}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t \qquad \forall \underset{\sim}{w} \in L^{\frac{4}{4-d}}(0,T;\underset{\sim}{V}),$$

$$\int_0^T \left\langle M\,\frac{\partial \widehat{\psi}_{\varepsilon,L}}{\partial t}, \widehat{\varphi} \right\rangle_{\widehat{X}} \mathrm{d}t$$
$$+ \int_0^T \int_{\Omega\times D} M \left[\frac{1}{2\,\lambda} \underset{\sim}{\nabla}_q \widehat{\psi}_{\varepsilon,L} - [\,\underset{\approx}{\sigma}(\underset{\sim}{u}_{\varepsilon,L})\,\underset{\sim}{q}\,]\,\beta^L(\widehat{\psi}_{\varepsilon,L}) \right] \cdot \underset{\sim}{\nabla}_q \widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t$$
$$+ \int_0^T \int_{\Omega\times D} M \left[\varepsilon \underset{\sim}{\nabla}_x \widehat{\psi}_{\varepsilon,L} - \underset{\sim}{u}_{\varepsilon,L}\,\widehat{\psi}_{\varepsilon,L} \right] \cdot \underset{\sim}{\nabla}_x \widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t = 0 \quad \forall \widehat{\varphi} \in L^{\frac{4}{4-d}}(0,T;\widehat{X}).$$

Although we have introduced x-diffusion and a cut-off above to $\widehat{\psi} = \psi/M$ in the drag term in the Fokker–Planck equation through the parameters ε and $L > 1$ in the model compared to the standard polymer model, we stress that the energy estimates (3.78a) in [3] on $\underset{\sim}{u}_{\varepsilon,L}$, the variable of real physical interest, that feature in the proof of Theorem 3.1 above, are independent of these parameters ε and L. We also note that $\int_{\Omega\times D} M\,\widehat{\psi}_{\varepsilon,L}(\underset{\sim}{x},\underset{\sim}{q},s)\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} = \int_{\Omega\times D} M\,\widehat{\psi}_0(\underset{\sim}{x},\underset{\sim}{q})\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} = 1$, for all $s \in (0,T)$. In the case of a corotational model the energy estimates in [3] become independent of L, as one can exploit cancelations due to the skew-symmetry of $\frac{1}{2}(\underset{\approx}{\nabla}_x \underset{\sim}{u} - (\underset{\approx}{\nabla}_x \underset{\sim}{u})^{\mathrm{T}})$. One can then pass to the limit $L \to \infty$ to recover the Fokker–Planck equation, *without* cut-off.

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