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FUNDAMENTAL THEOREMS OF ANALYSIS IN FORMALLY REAL FIELDS

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Abstract. The theory of real closed fields admits elimination of quantifiers, therefore fundamental theorems of analysis hold for all definable functions in all real closed fields. We consider the opposite problem: If we assume that in a formally real field some fundamental theorem of analysis holds, would the formally real field be actually real closed?

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1. Introduction

It is proved in [1] that the theory of really closed fields is submodel complete, therefore this theory admits elimination of quantifiers (A. Tarski). This implies that all real closed fields are elementarily equivalent. Therefore, every differential property of definable functions that can be expressed in the first-order predicate calculus, which is true in the field of real numbers, is also true in all real closed fields. In this paper we shall consider the opposite problem: Assuming that in a formally real field some fundamental theorem of analysis holds, would the formally real field be actually real closed?

First, we review some notion and notation from model theory taken from [2] and theory of formally real fields. A field F is *formally real* if -1 is not a sum of squares, or equivalently, $\mathbf{F}$ admits an ordering which makes $\mathbf{F}$ an ordered field. The examples include all real number fields: rational numbers $\mathbf{Q}$, real numbers $\mathbf{R}$, real algebraic numbers $\mathbf{A} \cap \mathbf{R}$ ($\mathbf{A}$ is the field of algebraic numbers) and the field of rational expressions $\mathbf{F}(x)$ over a formally real field $\mathbf{F}$. A field F is *real closed* if it is formally real with no proper formally real algebraic extensions. We remind that every formally real field is contained in a real closed field. The examples of real closed fields are $\mathbf{R}$ and $\mathbf{A} \cap \mathbf{R}$

Let $\Delta_{\mathbf{A}}$ be the diagram of a model $\mathbf{A}$. A theory T is *submodel complete* if $T \cup \Delta_{\mathbf{A}}$ is complete for every substructure $\mathbf{A}$ of a model of T. The next theorem characterizes first-order theories which admit elimination of quantifiers.

Theorem 1.1. *(A. Robinson, L. Blum) Let T be a theory in the language L. Then the following conditions are equivalent.*
1. *T is submodel complete, i.e. if* $\mathbf{B}$ *and* $\mathbf{C}$ *are models of* T *and* $\mathbf{A}$ *is a submodel*

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of **B** *and* **C**, *then* **B** *and* **C** *are elementary equivalent over* **A**.
2. T *admits elimination of quantifiers.*

Theorem 1.2. *(A. Tarski) Theory RCF (Real Closed Fields) admits elimination of quantifiers.*

Colorallary 1.1. *Theory RCF is submodel complete.*

2. Fundamental theorems of analysis in formally real fields

In this section we shall consider the status of fundamental theorems of differential calculus in formally real fields. Namely, assuming that such a theorem is true in a formally real field **F**, then what are the properties of **F**? The following theorem, see [3], is a typical statement of this kind.

Theorem 2.1. *(D. Marker) An ordered field* **F** *is real closed iff whenever* $p(x) \in \mathbf{F}[X]$ *change the sign on the interval (a,b),* $a, b \in F$*, then there is* $c \in F$ *such that* $a < c < b$ *and* $p(c) = 0$.

Now we list some fundamental theorems of analysis (or, precisely, fundamental theorems of differential calculus). All of them have natural formulation in the first order predicate calculus if they are restricted to definable functions.

Theorem 2.2. *(Fermat) Let* f *be a function that is continous on* $[a, b]$*, differentiable on* (a, b)*, and suppose that* f *has local extremum at the point* $c \in (a, b)$*. Then* $f'(c) = 0$.

Theorem 2.3. *(Rolle) Let* f *be a function that is continuous on* $[a, b]$*, differentiable on* (a, b) *and suppose that* $f(a) = f(b)$*. Then there exists* $c \in (a, b)$ *such that* $f'(c) = 0$.

Theorem 2.4. *(Lagrange) Let* f *be a function that is continous on na* $[a, b]$ *and differentiable on* (a, b)*. Then there exists* $c \in (a, b)$ *such that* $\frac{f(b)-f(a)}{b-a} = f'(c)$.

Theorem 2.5. *(Cauchy) Let* f *and* g *be functions continous on* $[a, b]$*, differentiable on* (a, b) *and suppose that* $g'(x) \neq 0$ *on* (a, b)*. Then, there exists* $c \in (a, b)$ *such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Theorem 2.6. *(Darboux) Let* f *be a function differentiable on* $[a, b]$ *and let* $f'(a) < z < f'(b)$*. Then, there exists* $c \in (a, b)$ *such that* $z = f'(c)$.

Theorem 2.7. *(Bolzano-Cauchy) Let* f *be a function continous on* $[a, b]$ *and let* $min\{f(a), f(b)\} < z < max\{f(a), f(b)\}$*. Then there exists* $c \in (a, b)$ *such that* $z = f(c)$.

It is not difficult to show that if f and g are definable functions, then each of these theorems can be represented by a first order formula. For example, we prove that this statement for Rolle's theorem:

Since f is definable, there is a formula $\varphi(x, y, \bar{b})$ that defines "$f(x) = y$". Also, since $|x| < a$ iff $-a < x < a$, "$|f(x) - f(s)| < \varepsilon$" there is a formula $\psi(x, s, \varepsilon, \bar{c})$ that defines it. So, the following sentences can be written as first-order formulas:

Let $N(f, s)$ mean "f is continuous at the point s ". Then, $N(f, s)$ can be written as: $\forall\varepsilon\exists\delta\forall x\ (\varepsilon > 0 \Rightarrow (\delta > 0 \wedge (|x - s| < \delta \Rightarrow |f(x) - f(s)| < \varepsilon)))$.

Let $N(f, a, b)$ mean "f is continuous on $[a, b]$" Then, $N(f, a, b)$ can be written as: $\forall s(a \leq s \leq b \Rightarrow N(f, s))$.

Let $L(f, x, d)$ mean: $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = d$. $L(f, x, d)$ can be written by the formula: $\forall\varepsilon\exists\delta\forall h\ (\varepsilon > 0 \Rightarrow (\delta > 0 \wedge (0 < h < \delta \Rightarrow |\frac{f(x+h)-f(x)}{h} - d| < \varepsilon)))$.

Let $D(f, a, b)$ mean: "f is differentiable on (a, b)". Then, $D(f, a, b)$ can be written: $\forall x(a \leq x \leq b \Rightarrow \exists d L(f, x, d))$.

Now, we can write Rolle's theorem as:

$(N(f, a, b) \wedge D(f, a, b) \wedge f(a) = f(b)) \Rightarrow \exists c(a < c < b \wedge L(f, c, 0))$

As a consequence of Tarski quantifier elimination Theorem, we can define the transfer principle which enables transfer of all first-order properties from $\mathbf{R}$ to any real closed field.

The transfer principle for real closed fields. If $\mathbf{R}$ and $\mathbf{F}$ are real closed fields, $\mathbf{R} \subseteq \mathbf{F}$ then $\mathbf{R} \prec \mathbf{F}$, i.e. if $\mathbf{R} \subseteq \mathbf{F}$, φ is a first-order formula in the language $RCF \cup \Delta_{\mathbf{R}}$ and μ a valuation over $\mathbf{R}$, then $\mathbf{R} \models \varphi[\mu]$ iff $\mathbf{F} \models \varphi[\mu]$

Applying the transfer principle and the fact that any fundamental theorem of analysis can be written as first-order formulas we conclude: *Fundamental theorems of analysis hold for all definable functions in all real closed fields.*

3. Formally real fields with Rolle's theorem

Now we shall consider the opposite problem: Assuming that in a formally real field a fundamental theorem of analysis holds, would the formally real field be actually real closed? It can easily be proved, using Theorem 1 that if $\mathbf{F}$ is formally real field with Bolzano-Cauchy theorem then $\mathbf{F}$ is real closed. We are focusing here on Rolle's theorem.

There are different ways for the formulation of this problem. We can assume that Rolle's theorem holds for:

1. Polynomials,
2. Rational functions.

In this paper we consider the case 1. Let $\mathbf{F}$ be a formally real field where Rolle's theorem holds for polynomials. We shall not prove that $\mathbf{F}$ is real closed, but we prove that certain polynomials have roots in $\mathbf{F}$.

Lemma 3.1. *For every $n \in \mathbf{N}$, $\sqrt[n]{n+1} \in \mathbf{F}$.*

Proof. We can elementary prove Lagrange's theorem using Rolle's theorem (observing the polynomial $q(x) = p(x) - \frac{p(b)-p(a)}{b-a}(x - a)$).

Let $\alpha \in \mathbf{F}$, $\alpha > 0$. Let us observe the polynomial $p(x) = x^{n+1}$.
Then, for some $\varepsilon \in (0, \alpha)$:
$\frac{\alpha^{n+1}-0^{n+1}}{\alpha-0} = (n+1)\varepsilon^n$, $\alpha^n = (n+1)\varepsilon^n$, $(\frac{\alpha}{\varepsilon})^n = n+1$.
Hence, $\frac{\alpha}{\varepsilon} = \sqrt[n]{n+1}$. □

Lemma 3.2. *For every $n, k \in \mathbf{N}$, $\sqrt[n]{nk+1} \in \mathbf{F}$.*

Proof. According to previous lemma, for any $n \in \mathbf{N}$ $\sqrt[n]{n+1} \in \mathbf{F}$. Hence, for $n,k \in \mathbf{N}$, $\sqrt[nk]{nk+1} \in \mathbf{F}$. Let c be a solution of the equation $x^{nk} = nk+1$. Then $(c^k)^n = nk+1$, so c^k is a solution of the equation $x^n = nk+1$, ie. $c^k = \sqrt[n]{nk+1}$

Lemma 3.3. *Let $\alpha, n \in \mathbf{N}$. Then there exist a and b such that*

$$a^n + a^{n-1}b + a^{n-2}b^2 + \ldots + a^2b^{n-2} + ab^{n-1} + b^n = \alpha$$

iff the equation $x^n = \alpha$ has a solution.

Proof. Let us consider the polynomial $p(x) = x^{n+1} - (n+1)\alpha x$. Applying Rolle's theorem we have: if there are a and b such that $p(a) = p(b)$, then, there exists $c \in (a, b)$ such that $p'(c) = 0$, i.e. $(n+1)(c^n - \alpha) = 0$.
$p(a) = p(b)$ iff
$a^{n+1} - (n+1)\alpha a = b^{n+1} - (n+1)\alpha b$ iff
$a^{n+1} - b^{n+1} = (n+1)\alpha(a-b)$ iff
$(a-b)(a^n + a^{n-1}b + a^{n-2}b^2 + \ldots + a^2b^{n-2} + ab^{n-1} + b^n - (n+1)\alpha) = 0$ iff
$a^n + a^{n-1}b + a^{n-2}b^2 + \ldots + a^2b^{n-2} + ab^{n-1} + b^n - (n+1)\alpha = 0$
Putting: $a_1 = \frac{a}{\sqrt[n]{n+1}}$, $b_1 = \frac{b}{\sqrt[n]{n+1}}$ last equation becomes:
$a_1^n + a_1^{n-1}b_1 + a_1^{n-2}b_1^2 + \ldots + a_1^2b_1^{n-2} + a_1b_1^{n-1} + b_1^n = \alpha$ □

Lemma 3.4. *If $\alpha \in \mathbf{N}$, $2 \le n \le 6$ then $\sqrt[n]{\alpha} \in \mathbf{F}$.*

Proof. For example, we will prove this theorem for $n = 5$. There are following cases:

1. $\alpha = 5k+1$. Then by Lemma 3.2, $\sqrt[5]{\alpha} \in \mathbf{F}$
2. $\alpha = 5k+2$. Then $3\alpha = 15k+6 = 5(3k+1)+1$, so $\sqrt[5]{3\alpha} \in \mathbf{F}$. Therefore, if $\sqrt[5]{3} \in \mathbf{F}$ then, for every k, $\sqrt[5]{5k+2} \in \mathbf{F}$
3. $\alpha = 5k+3$. Then $2\alpha = 10k+6 = 5(2k+1)+1$, so $\sqrt[5]{2\alpha} \in \mathbf{F}$. Therefore, if $\sqrt[5]{2} \in \mathbf{F}$ then, for every k $\sqrt[5]{5k+3} \in \mathbf{F}$.
4. $\alpha = 5k+4$. Then $4\alpha = 20k+16 = 5(4k+3)+1$, so $\sqrt[5]{4\alpha} \in \mathbf{F}$. Therefore, if $\sqrt[5]{4} \in \mathbf{F}$ then, for every k $\sqrt[5]{5k+4} \in \mathbf{F}$. $\sqrt[5]{4} = (\sqrt[5]{2})^2$, so it suffices to prove that $\sqrt[5]{2} \in \mathbf{F}$.
5. $\alpha = 5k$. Let $k = 5^m k_1$, where 5 is not a divisor of k_1. Then $\alpha = \alpha^{m+1}k_1$ so $\sqrt[5]{\alpha} = (\sqrt[5]{5})^{m_1}\sqrt[5]{k_1}$. $\sqrt[5]{k_1} \in \mathbf{F}$ if any of cases $1-4$ holds, so it suffices to prove that $\sqrt[5]{5} \in \mathbf{F}$.

By Lemma 3.1, $\sqrt[5]{6} \in \mathbf{F}$, so it is sufficient to prove that $\sqrt[5]{5} \in \mathbf{F}$ and ($\sqrt[5]{2} \in \mathbf{F}$ or $\sqrt[5]{3}) \in \mathbf{F}$. Hence, we are solving the next equation:

$a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5 = \alpha$.

Using elementary algebraic transformations we have:

$a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5 = \alpha$

$a^4(a+b) + b^4(a+b) + a^2b^2(a+b) = \alpha$

$(a+b)(a^4 + a^2b^2 + b^4) = \alpha$

Let $a + b = x$, $a^4 + a^2b^2 + b^4 = \frac{\alpha}{x}$

Then:

$a^4 + a^2(x-a)^2 + (x-a)^4 = \frac{\alpha}{x}$

$a^4 + a^2(x^2 - 2ax + a^2) + x^4 - 4ax^3 + 6a^2x^2 - 4a^3x + a^4 = \frac{\alpha}{x}$

$a^4 + a^2x^2 - 2a^3x + a^4 + x^4 - 4ax^3 + 6a^2x^2 - 4a^3x + a^4 = \frac{\alpha}{x}$

$3a^4 - 6a^3x + 7a^2x^2 - 4ax^3 + x^4 = \frac{\alpha}{x}$

$a^2(3a^2 - 6ax + 7x^2 - \frac{4x^3}{a} + \frac{x^4}{a^2}) = \frac{\alpha}{x}$

$a^2(3a^2 + \frac{x^4}{a^2} - (6ax + \frac{4x^3}{a}) + 7x^2) = \frac{\alpha}{x}$

$a^2(3a^2 + \frac{x^4}{a^2} - 2x\sqrt{3}(\sqrt{3}a + \frac{2x^2}{\sqrt{3}a}) + 7x^2) = \frac{\alpha}{x}$

$a^2(t^2 - \frac{x^4}{3a^2} - 4x^2 - 2x\sqrt{3}t + 7x^2) = \frac{\alpha}{x}$, where $t = \sqrt{3}a + \frac{2x^2}{\sqrt{3}a}$

$a^2(t^2 - 2x\sqrt{3}t + 3x^2 - \frac{x^4}{3a^2}) = \frac{\alpha}{x}$

$a^2((t - x\sqrt{3})^2 - \frac{x^4}{3a^2}) = \frac{\alpha}{x}$

$(a(t - x\sqrt{3}))^2 - \frac{x^4}{3} = \frac{\alpha}{x}$

$(a(t - x\sqrt{3}))^2 = \frac{\alpha}{x} + \frac{x^4}{3}$

$a(t - x\sqrt{3}) = \pm\sqrt{\frac{\alpha}{x} + \frac{x^4}{3}}$

Let $a(t - x\sqrt{3}) = \sqrt{\frac{\alpha}{x} + \frac{x^4}{3}}$

$a(\sqrt{3}a + \frac{2x^2}{\sqrt{3}a} - x\sqrt{3}) = \sqrt{\frac{\alpha}{x} + \frac{x^4}{3}}$

$\sqrt{3}a^2 - a\sqrt{3}x + \frac{2x^2}{\sqrt{3}} = \sqrt{\frac{\alpha}{x} + \frac{x^4}{3}}$

$\sqrt{3}a^2 - a\sqrt{3}x + \frac{2x^2}{\sqrt{3}} = \sqrt{\frac{3\alpha + x^5}{3x}}$

Multiplying both sides of the equation by $\sqrt{3}$ we have:

$3a^2 - 3ax + 2x^2 = \sqrt{\frac{3\alpha + x^5}{x}}$

So, solving the quadratic equation

$3a^2 - 3ax + 2x^2 - \sqrt{\frac{3\alpha + x^5}{x}} = 0$

we get : $a_{1/2} = \frac{3x \pm \sqrt{12\sqrt{\frac{3\alpha + x^2}{x}} - 15x^2}}{6}$ $\qquad (*)$

1. Let $x = 1$, $\alpha = 5$. Putting these values in the equation $(*)$ we get $a = \frac{3 \pm \sqrt{33}}{6}$. Let $a = \frac{3+\sqrt{33}}{6}$. Then $b = \frac{3-\sqrt{33}}{6}$. So, $\sqrt[5]{5} \in F$.

2. Let $x = 2$, $\alpha = 22$. Putting these values in the equation $(*)$ we get $a = \frac{6 \pm 2\sqrt{6}}{6}$. Let $a = \frac{6+2\sqrt{6}}{6}$. Then $b = \frac{6-2\sqrt{6}}{6}$. so, $\sqrt[5]{22} \in F$. Also, $\sqrt[5]{11} \in F$ since $11 = 5 \cdot 2 + 1$ therefore $\sqrt[5]{2} \in F$. $\square$

Using similar methods we can prove that the field **F** contains the quadratic field over the positive rational numbers, i.e. if $\mathbf{Q}_2 = \mathbf{Q}[\sqrt{2}, \sqrt{3}\ldots]$ then:

Theorem 3.1. $\mathbf{Q}_2 \subseteq S$.

The last result is about existence of root of the polynomial degree 3.

Theorem 3.2. *Let* $f(x) = x^3 + f_2x^2 + f_1x + f_0 \in \mathbf{Q}[x]$. *Then there exists* $a \in \mathbf{F}$ *such that* $f(a) = 0$ *whenever any of these conditions holds:*

1. $f_0 \cdot f_2 < 0$
2. $f_1 < 0$

Proof. Let us observe the polynomial $q(x) = \frac{x^4}{4} + \frac{f_2x^3}{3} + \frac{f_1x^2}{2} + f_0x$. We apply Lagrange's theorem on this polynomial, ie., for any $a \in \mathbf{F}$ there exists $-a < \varepsilon < a$ such that $\frac{q(a)-q(-a)}{2a} = q'(\varepsilon)$. Since $q' = f$, counting the left side of the equation we have:

$\frac{f_2a^2}{3} + f_0 = \varepsilon^3 + f_2\varepsilon^2 + f_1\varepsilon + f_0$ ie.

$\varepsilon^3 + f_2\varepsilon^2 + f_1\varepsilon - \frac{f_2a^2}{3} = 0$

Putting $a^2 = -\frac{3f_0}{f_2}$ we have that ε is the solution of the equation $x^3 + f_2x^2 + f_1x + f_0 = 0$. Since $f_0, f_2 \in \mathbf{Q}$, such a exists whenever $f_0 \cdot f_2 < 0$.

Putting $a^2 = -\frac{3f_0}{f_2}$ we have that ε is the solution of the equation $x^3 + f_2x^2 + f_1x + f_0 = 0$. Since $f_0, f_2 \in \mathbf{Q}$, such a exists whenever $f_0 \cdot f_2 < 0$.

Putting $x = -\frac{1}{y}$ in equation $x^3 + f_2x^2 + f_1x + f_0 = 0$ we get equation $f_0y^3 - f_1y^2 + f_2y - 1 = 0$. Let $q(y) = \frac{f_0y^4}{4} - \frac{f_1y^3}{3} + \frac{f_2y^2}{2} - y$. Then

$\frac{q(a)-q(-a)}{2a} = f_0\varepsilon^3 - f_1\varepsilon^2 + f_2\varepsilon - 1$ i.e. $-\frac{f_1a^2}{3} - 1 = f_0\varepsilon^3 - f_1\varepsilon^2 + f_2\varepsilon - 1$

$f_0\varepsilon^3 - f_1\varepsilon^2 + f_2\varepsilon + \frac{f_1a^2}{3} = 0$.

So, for $a^2 = -\frac{3}{f_1}$, ε is the solution. Such a exists when $f_1 < 0$. □

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