Novi Sad J. Math.
Vol. 38, No. 3, 2008, 33-38

ON GRAPHS ASSOCIATED TO RINGS[1]

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Abstract. We review some methods of associating graphs to rings. The main emphasis is on zero-divisor graphs and comaximal ones. Some new results in both directions are presented.

AMS Mathematics Subject Classification (2000): 13A99, 05C99, 16S50

Key words and phrases: graphs, commutative rings, zero-divisors, maximal ideals

1. Introduction

It was Beck (see [4]) who first introduced the notion of a *zero-divisor graph* for a commutative ring. This notion was later redefined in [1]. Since then, there has been a lot of interest in this subject and various papers were published establishing different properties of these graphs as well as relations between graphs of various ring extensions (see, e.g. [2], [3]). The notion of a zero-divisor graph was extended to non-commutative rings in [11] and various properties were established in [11] and [12]. We discuss zero-divisor graphs in Section 2.

In [13], the authors have introduced the notion of a *comaximal graph* of a commutative ring with identity (without actually giving it this name). This investigation was later continued in [7] and [10]. We discuss comaximal graphs in Section 3.

In order to make this paper easier to follow, we recall in this section various notions from graph theory which will be used in the sequel.

For a graph Γ by $E(\Gamma)$ and $V(\Gamma)$ we denote the set of all edges and vertices respectively. We recall that a graph is connected if there exists a path connecting any two distinct vertices. The distance between two distinct vertices a and b, denoted by $d(a,b)$, is the length of the shortest path connecting them (if such a path does not exist, then $d(a,b)=\infty$). The diameter of a graph Γ, denoted ny $\mathrm{diam}(\Gamma)$, is equal to

$$\sup\{d(a,b)\,|\,a,b\in V(\Gamma)\}.$$

The girth of a graph Γ, denoted $g(\Gamma)$ is the length of the shortest cycle in Γ. *Eccentricity* of a vertex a is defined as a $\sup\{d(a,x) : x\in V(\Gamma)\}$. If the diameter of a graph is finite, it is interesting to see what is the smallest eccentricity of a vertex in Γ. Vertices of Γ with this smallest eccentricity form the *center* of this graph. Center of the graph is one of the so-called central sets of a graph.

[1]This work was partially supported by the Ministry of Science and Environmental Protection of Republic of Serbia Project #144018 and Project #144020

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Another one is the *median* of a graph. This notion is primarily interesting for finite graphs. Namely, we define the *status* of a vertex x, $s(x)$ as the sum of distances of that vertex to all the other vertices: $s(x) := \sum_{y \in V(G)} d(x, y)$. The elements with the smallest status form the *median* of the graph in question. The last notion related to the centrality of a graph is that of a *dominating set.* A subset $S \subseteq V(\Gamma)$ is called a dominating set if every vertex in $V(\Gamma)$ is either in S or adjacent to a member of S. The *dominating number* is the size of the smallest dominating set. One also speaks of *connected dominating sets* (*connected dominating number*) if the subgraph of Γ generated by a dominating set is connected (see [12]).

Coloring of a graph is an assignment of *colors* to vertices of this graph in such a way that adjacent vertices should be assigned different colors. A graph is n-colorable if it is possible to give such a coloring with n colors. The *chromatic number* of a graph Γ, denoted by $\chi(\Gamma)$, is the smallest n such that the graph Γ is n-colorable. If such an n does not exist (which means that it is not possible to color this graph with only finitely many colors) one puts $\chi(\Gamma) = \infty$.

A *complement* of a vertex x is a vertex y which is adjacent to x and is such that no other vertex is adjacent to both x and y. Graph Γ is *complemented* if every vertex of Γ has a complement. Complemented graph is *uniquely complemented* if any two complements of a vertex are adjacent to the same vertices (see [6]).

A subset $S \subseteq V(\Gamma)$ is a *clique* if the subgraph generated by this set is complete.

Finally, let us define a blow-up of a graph (see [9] — we give a more general definition). Let Γ be a graph and $(\kappa_x)_{x \in V(\Gamma)}$ be a collection of non-zero cardinals. By $\Gamma((\kappa_x)_{x \in V(\Gamma)})$ we denote the graph which we get from Γ by replacing any vertex x of Γ by a set V_x of cardinality κ_x and any edge $\{x, y\} \in E(\Gamma)$ by a complete bipartite graph whose vertex classes are V_x and V_y. Note that two blow-ups $\Gamma((\kappa_x)_{x \in V(\Gamma)})$ and $\Gamma((\kappa'_x)_{x \in V(\Gamma)})$ of the same graph Γ are isomorphic if $\kappa_x = \kappa'_x$ for all $x \in V(\Gamma)$.

2. Zero-divisor graphs

If R is an arbitrary ring, let $Z(R)$ denote the set of zero-divisors of R and let $Z(R)^*$ denote the set of nonzero zero-divisors of R. If the ring R is non-commutative, we let $Z_L(R)$ and $Z_R(R)$ denote the sets of left and right zero-divisors of R, respectively.

For a commutative ring R, we consider the undirected graph $\Gamma(R)$ with vertices in the set $Z(R)^*$, such that for distinct vertices a and b there is an edge connecting them if and only if $ab = 0$.

If R is a non-commutative ring, we define a directed zero-divisor graph $\Gamma(R)$ in a similar way (this definition was introduced in [11]). A directed graph is connected if there exists a directed path connecting any two distinct vertices. The distance and the diameter are defined in a similar way as well, having in mind that all paths in question are directed.

In [11], the notion of an undirected zero-divisor graph of a non-commutative ring R, denoted by $\bar{\Gamma}(R)$ was also introduced. The set of vertices is the set $Z(R)^*$ and for distinct vertices a and b there is an edge connecting them if and only if $ab = 0$ or $ba = 0$.

The results from the following theorems were proved in [1] and [8] (commutative case) and [12] (non-commutative case).

Theorem 2.1. *Let R be a commutative ring, with $Z(R)^* \neq \emptyset$. Then $\Gamma(R)$ is always connected,* $\mathrm{diam}(\Gamma(R)) \leq 3$ *and* $g(\Gamma(R)) \leq 4$.

Theorem 2.2. *Let R be a non-commutative ring, with $Z(R)^* \neq \emptyset$. Then $\Gamma(R)$ is connected if and only if $Z_L(R) = Z_R(R)$. If $\Gamma(R)$ is connected, then* $\mathrm{diam}(\Gamma(R)) \leq 3$.

The graph $\bar{\Gamma}(R)$ is connected, $\mathrm{diam}(\bar{\Gamma}(R)) \leq 3$ *and* $g(\bar{\Gamma}(R)) \leq 4$.

In the paper [5], the authors have investigated the zero-divisor graph $\Gamma(M_n(R))$, where $M_n(R)$ is the ring of square matrices of order n over a commutative ring with identity R. It has been established in that paper that this graph is always connected and that in the case when $Z(R)^*$ is not empty $\mathrm{diam}(\Gamma(R)) \leq \mathrm{diam}(\Gamma(M_n(R)))$. Since we know that the diameter $\Gamma(R)$ may only be 1, 2 or 3, it is only interesting to look at the case when diameter of $\Gamma(R)$ is 1 or 2. Theorem 4.1 from [5] establish that if the ring R is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathrm{diam}(\Gamma(R)) = 1$, then $\mathrm{diam}(\Gamma(M_n(R))) = 2$.

By Theorem 2.7 of [2], if R is a commutative ring and $\mathrm{diam}(\Gamma(R)) = 2$, then exactly one of the following holds:

(1) $Z(R)$ is a prime ideal in R, or

(2) $T(R) = K_1 \times K_2$, where both K_i are fields

(by $T(R)$ we denote the total quotient ring of R — we localize with respect to all elements which are not zero-divisors).

In [5], the authors have proved that in the second case we always have that $\mathrm{diam}(\Gamma(R)) = 3$. The first case appears to be more difficult. In the same paper it was established that if R is a McCoy ring (we remind the reader that R is a McCoy ring if every finitely generated ideal contained in $Z(R)$ has a nonzero annihilator) such that $\mathrm{diam}(\Gamma(R)) = 2$ with $Z(R)$ a prime ideal, then $\mathrm{diam}(\Gamma(M_n(R))) = 2$.

3. Comaximal graphs

Let R be a commutative ring with identity. The vertices of the comaximal graph $\Gamma(R)$ of this ring are elements of R and two vertices a and b are adjacent if and only if the ideal generated by these two elements is the ring R itself. It is easy to see that the invertible elements in R are adjacent to all elements and that elements from the Jacobson radical $J(R)$ are adjacent only to the invertible ones. Namely, it follows directly from the definition of the comaximal graph that ring elements a and b are adjacent if and only if they are not contained in the same maximal ideal of R. Let us introduce the notation $V(x)$ for a set of all maximal ideals of R containing $x \in R$. Our observation is therefore

$$a \text{ and } b \text{ are adjacent } \textbf{iff } V(a) \cap V(b) = \emptyset \tag{1}$$

Therefore, the most interesting part consists of non-invertible elements which are not contained in the $J(R)$. We use the notation $\Gamma_2'(R)$ to denote the subgraph of $\Gamma(R)$ spanned by these elements.

In [13], the main result is that $\chi(\Gamma(R))$ is finite if and only if the ring R itself is finite. In [7] the authors have shown that the subgraph $\Gamma_2'(R)$ is always connected, with diameter at most 3, that is a complete bipartite graph iff it contains exactly two maximal ideals (if it contains n maximal ideals then it is n-partite), gave the necessary and sufficient conditions for a diameter of this graph to be 2 and have shown that for some types of rings the fact that $\Gamma(R) \cong \Gamma(S)$ implies that $R \cong S$.

In the paper [10], we use the observation (1) in order to give the complete structure of the comaximal graph in the case when $Max(R)$, the set of all maximal ideals of R is finite. The reason why we are able to find this structure rests on the well-known fact that if the ideal is contained in the *finite* union of *maximal* ideals it is contained in one of them. This need not be true in general—one may find an example in [10] showing a commutative ring with identity in which every maximal ideal is contained in the union of other maximal ideals. Therefore, the structure of the comaximal graph in this case is much harder to establish in this case (some results for special types of rings may be found in [10]).

In order to show the structure of the comaximal graph in the case of finitely many maximal ideals it is useful to introduce an auxiliary graph $\mathbb{G}_n$ whose vertices are subsets of the set $\{1, \ldots, n\}$ and two subsets are adjacent if and only if their intersection is empty. We denote by $\mathbb{G}_n'$ the subgraph spanned by proper non-empty subsets. In [10] we prove that in the case when $Max(R)$ is finite $\Gamma(R)$ ($\Gamma_2'(R)$) is a blow-up of $\mathbb{G}_n$ ($\mathbb{G}_n'$) and from this we are able to determine the radius and center of $\Gamma_n'(R)$, the median (in the case of finite ring R), show that this graph is uniquely complemented and determine its (connected) dominating number. For details as well as the proof, the reader is referred to [10].

As an illustration of these results, in the following picture one can see the schematic presentation of the comaximal graph $\Gamma_2'(R)$ in the case when $|Max(R)| = 3$ (this is a picture of the graph $\mathbb{G}_3'$ where we put dotted circles instead of vertices to show that our graph is a blow-up of $\mathbb{G}_3'$ and "inside" the dotted circles are elements of the appropriate intersections of maximal ideals).

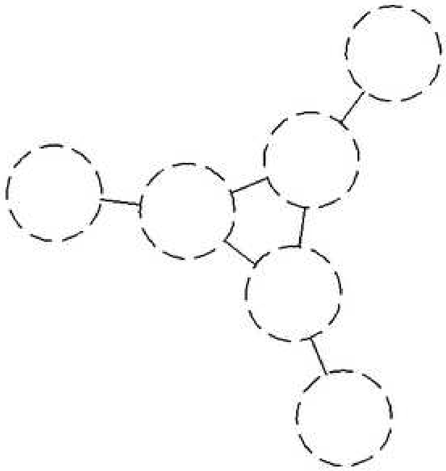

Figure 1: Comaximal graph

In order to illustrate some of the methods from [10] we prove the following proposition.

Proposition 3.1. *Let the commutative ring with identity R has only finitely many maximal ideals. If r and s are adjacent in the comaximal graph $\Gamma(R)$, there exists t such that $r + st$ is invertible.*

Proof. Since r and s are adjacent, $V(r) \cap V(s) = \emptyset$. Since the ring R has only finitely many maximal ideals, there exists t such that $V(t) = \mathrm{Max}(R) \setminus (V(r) \cup V(s))$ (if $V(r) \cup V(s) = \mathrm{Max}(R)$ we may take $t = 1$). It is easy to see that $V(r + st) = V(r) \cap V(st)$. Namely, if $\mathfrak{m} \in V(r) \cap V(st)$ then $r, st \in \mathfrak{m}$, therefore $r + st \in \mathfrak{m}$. On the other hand, if $\mathfrak{m} \in V(r + st)$, since we know that $\mathrm{Max}(R) = V(r) \cup V(s) \cup V(t) = V(r) \cup V(st)$ (every maximal ideal is prime), $\mathfrak{m} \in V(r)$ or $\mathfrak{m} \in V(st)$. In both cases we conclude that $\mathfrak{m} \in V(r) \cap V(st)$. Since $V(r)$ is disjoint with both $V(s)$ and $V(t)$, we conclude that $V(r + st) = \emptyset$ and therefore, that $r + st$ is invertible. □

Remark 3.1. Since r and s are adjacent in $\Gamma(R)$, we know that there exist u and v such that $ru + sv$ is invertible. It is not clear why one of these elements may be taken to be the identity. The previous proposition exactly tells us that it can be in the case of finitely many maximal ideals. In the case of infinitely many ideals this need not be true. For example, one may take the ring $\mathbb{Z}$ and elements 7 and 10.

In the paper [7], the authors also address the question of whether the isomorphism of comaximal graphs implies the isomorphism of the rings in question. They prove that in some cases of finite rings $\Gamma(R) \cong \Gamma(S)$ implies $R \cong S$. They also claim that $\Gamma(R) \cong \Gamma(S)$ implies $R/J(R) \cong S/J(S)$. While this is valid for the case of finite rings, it is not true in general. One can find an example showing this in [10].

Acknowledgement

The authors would like to thank the organizers of the conference for a pleasant experience.

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Received by the editors October 1, 2008