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APPROXIMATION THEOREMS IN PROBABILISTIC NORMED SPACES[1]

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Abstract. We study some properties of a general class of probabilistic normed space and prove approximations theorems for functions with values into such a space. By taking into account that time series and random signals are, in fact, such particular functions, these results give some tools for the analysis of time series and random signals

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1. Introduction

In [10], K. Menger replaced the number $d(p, q)$, the distance between two points p, q, by a probabilistic distribution function $F_{p,q}$, and defined probabilistic metric spaces. That idea has led to a remarkable development of probabilistic analysis. Applications to systems possessing hysteresis, mixture processes, the measuring error were also given. In [14], A. N. Šerstnev used the K. Menger's idea and endowed a set having an algebraic structure of linear space with a probabilistic norm. For an extensive view of this subject we refer to [6, 8, 13].

In [1], C. Alsina, B. Schweizer and A. Sklar gave a new definition of probabilistic normed spaces which is based on a characterization of normed spaces by means of a betweenness relation and generalized the definition of A. N. Šerstnev. Important results relating the issues of probabilistic analysis were also obtained by O. Hadžić [6, 7].

In the second section of this paper we give some properties of a new class of probabilistic normed spaces defined in [5], which also generalize the probabilistic normed spaces defined by A. N. Šerstnev which become a particular case. We have generalized the axiom which gives a connection between the distribution functions associated to a vector and the distribution function associated to the product of the vector by a scalar. The third section is devoted to the study of functions with values into such a probabilistic normed space. By using the properties of the probabilistic norm we obtain some approximation theorems.

Throughout this paper we shall use standard symbols and basic notions [6, 13]. Let $\mathbb{R}$ denote the the set of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $I = [0, 1]$ the closed unit interval. A mapping $F : \mathbb{R} \to I$ is called a distribution function if it is non decreasing, left continuous with $\inf F = 0$ and

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$\sup F = 1$. D^+ denotes the set of all distribution functions for that $F(0) = 0$. Let F, G be in D^+, then we write $F \leq G$ if $F(t) \leq G(t)$, for all $t \in \mathbb{R}$. If $a \in \mathbb{R}_+$ then H_a will be the element of D^+ defined by $H_a(t) = 0$ if $t \leq a$ and $H_a(t) = 1$ if $t > a$. It is obvious that $H_0 \geq F$, for all $F \in D^+$. The set D^+ will be endowed with the natural topology defined by the modified Lévy metric d_L [13]. A t-norm T is a two-place function $T : I \times I \to I$ which is associative, commutative, non decreasing in each place and such that $T(a, 1) = a$, for all $a \in [0, 1]$. The most used t-norms in probabilistic metric spaces theory are : $T = Min$, $Min(a, b) = \min\{a, b\}$; $T = Prod$, $Prod(a, b) = a \cdot b$; $T = T_m$, $T_m(a, b) = \max\{a + b - 1, 0\}$. A triangle function τ is a binary operation on D^+ which is commutative, associative, non-decreasing in each place and for which H_0 is the identity, that is, $\tau(F, H_0) = F$, for every $F \in D^+$. T-norms and triangle functions have made possible the writing appropriate probabilistic triangle inequalities.

2. On generalized probabilistic normed spaces

The following functions $\varphi : \mathbb{R} \to \mathbb{R}$ unify normed spaces with p-normed spaces, probabilistic normed spaces with probabilistic p-normed spaces and give uncountable possibilities in defining of probabilistic (or deterministic) normed spaces. The functions φ satisfy the following conditions : (a_1) $\varphi(-t) = \varphi(t)$, for every $t \in \mathbb{R}$; (a_2) φ is strict increasing and continuous on $[0, \infty)$, (a_3) $\varphi(0) = 0$, $\varphi(1) = 1$; and $\lim_{\alpha \to \infty} \varphi(\alpha) = \infty$.

Examples of such functions are: (e_1) $\varphi(\alpha) = |\alpha|$; (e_2) $\varphi(\alpha) = |\alpha|^p$, $p \in (0, \infty)$; (e_3) $\varphi(\alpha) = \frac{2\alpha^{2n}}{|\alpha|+1}$, $n \in \mathbb{N}^+$.

Definition 1. Let L be a linear space, τ a triangle function and let $\mathcal{F}$ be a mapping from L into D^+. If the following conditions are satisfied:

(1) $F_x = H_0$, if and only if $x = \theta$;

(2) $F_{\alpha x}(t) = F_x(\frac{t}{\varphi(\alpha)})$, for every $t > 0, \alpha \in \mathbb{R}$ and $x \in L$;

(3) $F_{x+y} \geq \tau(F_x, F_y)$, whenever $x, y \in L$;

then $\mathcal{F}$ is called a probabilistic φ-norm on L and the triple $(L, \mathcal{F}, \tau)$ is called a probabilistic φ-normed space (of Šerstnev type). The pair $(L, \mathcal{F})$ is said to be probabilistic φ-seminormed space if the mapping $\mathcal{F} : L \to D^+$ satisfies the conditions (1) and (2). We have made the conventions : $F_x(\frac{t}{0}) = 1$, for $t > 0$, $F_x(\frac{0}{0}) = 0$. $\mathcal{F}(x)$ is denoted by F_x.

If (1)-(2) are satisfied and the probabilistic triangle inequality (3) is given under a t-norm T :

(4) $F_{x+y}(t_1 + t_2) \geq T(F_x(t_1), F_y(t_2))$, for all $x, y \in L$ and $t_1, t_2 \in \mathbb{R}_+$, then $(L, \mathcal{F}, T)$ is called a Menger φ-normed space.

Proposition 1. *If T is a left continuous t-norm and τ_T is the triangle function defined by $\tau_T(F, G)(t) = \sup_{t_1+t_2<t} T(F(t_1), G(t_2))$, $t > 0$, then $(L, \mathcal{F}, \tau_T)$ is a probabilistic φ-normed space if and only if $(L, \mathcal{F}, T)$ is a Menger φ-normed space.*

If we define $\mathcal{F}^m(x,y) = F_{x-y}$ then a probabilistic φ-normed space $(L, \mathcal{F}^m, \tau)$ becomes a probabilistic metric space under the same triangle function τ. We consider probabilistic φ-normed spaces under continuous triangle function $\tau \geqslant \tau_{T_m}$, where $(T_m(a,b) = Max\,\{a+b-1, 0\})$. This condition insures the existence of a linear topology on L. The functions φ can be used in a similar generalization of the deterministic normed spaces.

Definition 2. By a φ-normed space we mean a pair $(L, ||\cdot||)$, where L is a linear space, $||\cdot||$ is a real-valued mapping defined on L, such that the following conditions are satisfied:

(5) $||x|| \geqslant 0$ for all $x \in L$, $||x|| = 0$ if and only if $x = \theta$;

(6) $||\alpha \cdot x|| = \varphi(\alpha)||x||$, whenever $x \in L$, $\alpha \in \mathbb{R}$ and φ is a function with the above properties;

(7) $||x + y|| \leq ||x|| + ||y||$, for all $x, y \in L$.

Remark 1. For $\varphi(\alpha) = |\alpha|^p$, one obtains a p-normed space [2], [12] and for $\varphi(\alpha) = |\alpha|$ one obtains an ordinary normed space.

The mapping $d_\varphi : L \times L \to \mathbb{R}$, $d_\varphi(x,y) = ||x - y||$ defines a translation invariant metric on L. It is known that the topology of any Hausdorff locally bounded topological vector space is given by such a p-norm.

Remark 2. It is easy to check that every φ-normed space $(L, ||\cdot||)$ can be, in a natural way, made a probabilistic φ-normed space by setting $F_x(t) = H_0(t - ||x||)$, for every $x \in L$, $t \in \mathbb{R}_+$ and $T = Min$

Proposition 2. *Let $G \in D^+$ be different from H_0, let $(L, ||\cdot||)$ be a φ-normed space. If we define $\mathcal{F} : L \to D^+$ by $F_\theta = H_0$ and, if $x \neq \theta$ by*

$$F_x(t) = G(\frac{t}{||x||}) \quad (t \in \mathbb{R}_+)$$

then the triple $(L, \mathcal{F}, T)$ becomes a probabilistic φ-normed space under the t-norm $T = Min$. If $\varphi(\alpha) = |\alpha|^p$, $p \in (0, \infty)$, then $(L, ||\cdot||)$ is a p-normed space and $(L, \mathcal{F}, T)$ becomes a probabilistic p-normed space. These spaces are called simple generated probabilistic φ-normed spaces generated by the distribution function G and the φ-normed space $(L, ||\cdot||)$.

Theorem 1. *Let $(L, \mathcal{F}, T)$ be a Menger φ-normed space under a continuous t-norm T such that $T \geq T_m$, then : (a) The family of subsets of L*

$$\mathcal{V} = \{V(\varepsilon, \lambda)\} : \varepsilon > 0, \lambda \in (0,1)\},$$

where

$$V(\varepsilon, \lambda) = \{x \in L : F_x(\varepsilon) > 1 - \lambda\}$$

is a complete system of neighborhoods of null vector for a linear topology on L generated by the φ-norm $\mathcal{F}$.

(b) *The family of subsets of $L \times L$:*

$$\mathcal{U} = \{U(\varepsilon, \lambda)\} : \varepsilon > 0, \lambda \in (0,1)\},$$

where

$$U(\varepsilon, \lambda) = \{(x, y) \in L \times L : F_{x-y}(\varepsilon) > 1 - \lambda\}$$

is a complete system of neighborhoods for a uniformity on L.

Theorem 2. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in L and let $(L, \mathcal{F}, T)$ be a Menger φ-normed space under a continuous t-norm T, then:*

(a) $\{x_n\}$ converges to x in the topology generated by the probabilistic φ-norm $\mathcal{F}$ on L if, and only if, $F_{x_n - x}(t)$, converges to $H_0(t)$, for every $t > 0$;

(b) $\{x_n\}$ is a Cauchy sequence in the uniformity generated by the probabilistic φ-norm $\mathcal{F}$ on L if, and only if, $F_{x_n - x_m}(t)$ converges to $H_0(t)$ for all $t > 0$.

3. Approximation theorems in generalized probabilistic normed spaces

In what follows we consider that $(\Omega, \mathcal{K}, P)$ is a complete probability measure space, i.e., the set Ω is a nonempty abstract set, $\mathcal{K}$ is a σ-algebra of subsets of Ω and P is a complete probability measure on $\mathcal{K}$. Let $(X, \mathcal{B})$ be a measurable space, where $(X, ||\cdot||)$ is a separable Banach space and $\mathcal{B}$ is the σ-algebra of the Borel subsets of $(X, ||\cdot||)$.

A mapping $x : \Omega \to X$ is said to be random variable with values in X if, $x^{-1}(B) \in \mathcal{K}$, for all $B \in \mathcal{B}$. Two X-valued random variables $x(\omega)$ and $y(\omega)$ are said to be equivalent if $x(\omega)$ and $y(\omega)$ are equal in the probability P, i.e.,

$$P(\{\omega \in \Omega : x(\omega) = y(\omega)\}) = 1.$$

Let us denote by $\mathcal{X}$ the space of all classes of equivalent random variables with values in a separable Banach space X. The separability of Banach space $(X, ||\cdot||)$ implies that $\mathcal{X}$ together with additive operation and multiplicaton with real or complex scalars becomes a linear space.

Now, we consider the following mapping $\mathcal{F} : \mathcal{X} \to D^+$ given by

$$F_x(t) = P(\{\omega \in \Omega : \quad ||x(\omega)|| < t\})$$

It is known that the triple $(\mathcal{X}, \mathcal{F}, T_m)$, is a complete probabilistic normed space of Šerstnev type.

Remark 3. One can also verify that, if $(X, ||\cdot||)$ is a p-normed space, $\mathcal{X}$ is a linear subspace of random variables with values into p-normed space $(X, ||\cdot||)$ then $(\mathcal{X}, \mathcal{F}, T_m)$, is a probabilistic p-normed space. Furthermore, the (ε, λ)-topology on $\mathcal{X}$ induced by the probabilistic norm $\mathcal{F}$ is equivalent to the topology of the convergence in probability on $\mathcal{X}$.

Remark 4. Let $L_p(\Omega, \mathcal{K}, P)$ be the space of all X-valued random variables for which

$$\int_\Omega ||x(\omega)||^p dP < \infty.$$

Then, the mapping $||\cdot|| : L_p(\Omega, \mathcal{K}, P) \to \mathbb{R}_+$ by :

$$||x(\omega)|| = \int_\Omega ||x(\omega)||^p dP,$$

where $p \in (0, 1]$ is a p-norm. By Proposition 2 $L_p(\Omega, \mathcal{K}, P)$ becomes a simple generated probabilistic φ-normed space under different distribution functions G.

By a random function defined on a subset A with values in a separable Banach space X we mean a function $f : \Omega \times A \to X$ such that, for every $t \in A$ the mapping $f(t, \cdot) : \Omega \to X$ is X-valued random variable. Two X-valued random functions f and g are said to be equivalent if $f(t, \omega)$ and $g(t, \omega)$ are equal almost surely for every $t \in A$.

Random functions have a special importance in the probability theory, as well as in its applications. If the set A of real line is a succession of moments, then a random function becomes a random time series or a random signal. In what follows we associate to a random function a function with values into a probabilistic φ-normed space.

Let f be an X-valued random function defined on $A \subset \mathbb{R}$, then one can define the mapping $\widetilde{f}$ on A with values in the random normed space $(\mathcal{X}, \mathcal{F}, T_m)$ by $A \ni t \to \widetilde{f}(t)$, where $[\widetilde{f}(t)](\omega) = f(t, \omega)$. Conversely, for every function $\widetilde{f} : A \to (\mathcal{X}, \mathcal{F}, T_m)$ one can define the X-valued random function on A by $f(t, \omega) = [\widetilde{f}(t)](\omega)$, for every $t \in A$ and $\omega \in \Omega$. Furthermore, the correspondence

$$(f : A \times \Omega \to X) \leftrightarrows (\widetilde{f} : A \to (\mathcal{X}, \mathcal{F}, T_m)$$

is one to one and onto. By this correspondence some results obtained for functions with values in a probabilistic φ-normed space can be applied to study random functions with values in a separable Banach space. So, the study of functions with values into a probabilistic φ-normed space gives useful results applicable to random time series and random signals.

In the sequel we approach the approximation issues for such functions.

For each pair of integers $m, n, 0 \le m \le n, n \ne 0$, let $g_{m,n}$ be the Borel functions defined on the unit interval $[0, 1]$ by:

$$g_{m,n} = \begin{cases} 0 & if \quad 0 \le t \le \frac{m-1}{n} \quad \text{or} \quad t \ge \frac{m+1}{n} \\ (nt - m + 1)f(\frac{m}{n}) & if \quad \frac{m-1}{n} \le t \le \frac{m}{n} \\ (-nt + m + 1)f(\frac{m}{n}) & if \quad \frac{m}{n} \le t \le \frac{m+1}{n} \end{cases} \tag{8}$$

Let f be a function defined on $[0, 1]$ with values in a Menger φ-normed space $(L, \mathcal{F}, T)$. We consider the sequence of functions $\{f_n\}_{n \in N}$ defined by

$$f_n(t) = \sum_{m=0}^{n} g_{mn}(t). \tag{9}$$

Theorem 3. *If the function $f : [0, 1] \longrightarrow (L, \mathcal{F}, T)$ is continuous on $[0, 1]$ then, the sequence of functions $\{f_n\}_{n \in N}$ defined by (9) is uniformly convergent on $[0, 1]$ to the function f.*

Proof. Let $\varepsilon > 0$ and $\lambda \in (0,1)$. Then by continuity of the function f it follows that there exist $N(\varepsilon,\lambda) \in N$ and $\eta > 0$ such that $F_{f(t')-f(t'')}(\frac{\varepsilon}{2} \geq 1-\frac{\eta}{2} > 1-\frac{\lambda}{2}$ for all $t', t'' \in [0,1]$ with the property that $|t'-t''| < \frac{1}{N(\varepsilon,\lambda)}$.

Let $n \geq N(\varepsilon,\lambda)$ and $s \in [0,1]$, and let us choose the integer k such that $0 \leq k \leq n$ and $\frac{k-1}{n} \leq s \leq \frac{k}{n^n}$. Then we have $s = \frac{k}{n} - \frac{u}{n}$, where $0 \leq u \leq 1$ and $f_n(s) = \sum\limits_{m=0}^{n} g_{m,n}(s) f(\frac{m}{n}) = uf(\frac{k-1}{n}) + (1-u)f(\frac{k}{n})$.

Thus we have:

$$F_{f(s)-f_n(s)}(\varepsilon) = F_{(1-u)f(s)+uf(s)-f_n(s)}(\varepsilon) \geq T(F_{(1-u)[f(s)-f(\frac{k}{n}]}(\frac{\varepsilon}{2})),$$

$$F_{u[f(s)-f(\frac{k-1}{n}]}(\frac{\varepsilon}{2}) = T(F_{f(s)-f(\frac{k}{n})}(\frac{\varepsilon}{2\varphi(1-u)})),$$

$$F_{f(s)-f(\frac{k-1}{n})}(\frac{\varepsilon}{2\varphi(u)})) \geq T(F_{f(s)-f(\frac{k}{n})}(\frac{\varepsilon}{2}), F_{f(s)-f(\frac{k-1}{n})}(\frac{\varepsilon}{2})) \geq$$

$$\geq T_m(1-\frac{\varepsilon}{2}, 1-\frac{\varepsilon}{2}) > 1-\eta > 1-\lambda.$$

By these inequalities it follows the conclusion of the theorem.

Remark 5. Notice that if $f \in C_{[a,b]}$, we can take $u = \frac{t-a}{b-a}$ to translate the approximation problem to the interval $[0,1]$ and use the above results.

Because every linear metric space [9] is a probabilistic metric space under the t-norm T_m [11] we can give the following consequence of Theorem 3.

Corollary 1. *If f is a continuous function defined on unit interval $[0,1]$ with values in a linear metric space (L,d) then, the sequence of functions $\{f_n\}_{n\in N}$ defined by (9), is uniformly convergent on $[0,1]$, to the function f.*

Let $(L,\mathcal{F},T)$ be a probabilistic normed space. For each $p \geq 1$ we consider the set $L^p(L,\mathcal{F},T) = \{x \in L : \int_0^\infty t^p dF_x(t) < \infty\}$, the L_p-space induced by the probabilistic normed space $(L,\mathcal{F},T)$. The set $L^p(L,\mathcal{F},T)$ is a linear space and the mapping

$$||\cdot|| : L^p(L,\mathcal{F},T) \to \mathbb{R}, \quad ||x|| = (\int_0^\infty t^p dF_x(t))^{\frac{1}{p}}$$

is a quasi-norm [3], [4]. At the same time, $L^p(L,\mathcal{F},T)$, endowed with the above quasi-norm, is a metric linear space and, by the Corollary 1, it results in the following statement.

Corollary 2. *If f is a continuous function defined on the unit interval $[0,1]$ with values in the quasi-normed space $L^p(L,\mathcal{F},T)$ then, the sequence of functions $\{f_n\}_{n\in N}$ defined by (9) is, uniformly convergent on $[0,1]$, to the function f.*

In the same time for the probabilistic normed space $(\mathcal{X},\mathcal{F},T_m)$ of the random variable defined on the probability measure space $(\Omega,\mathcal{K},P)$ with values in the separable Banach space $(X,||\cdot||)$ the quasi-normed space $L^p(\mathcal{X},\mathcal{F},T_m)$ is equivalent with the space $L_p(\Omega,\mathcal{K},P)$ with the usual norm. By the Theorem 3 the following statement follows.

Corollary 3. *If $f : [0,1] \to L_p(\Omega, \mathcal{K}, P)$ is a continuous, in the sense of the order p, functions on the unit interval $[0,1]$ then, the sequence of functions $\{f_n\}_{n \in N}$ defined by (9), is uniformly convergent in the sense of the order p on $[0,1]$, to the function f.*

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