Novi Sad J. Math.
Vol. 38, No. 3, 2008, 81-89

AN EXTENSION OF STOLARSKY MEANS

Slavko Simić[1]

Abstract. In this article we give an extension of the well known Stolarsky means to the multi-variable case in a simple and applicable way. Some basic inequalities concerning this matter are also established with applications in Analysis and Probability Theory.

AMS Mathematics Subject Classification (2000): 26A51, 60E15

Key words and phrases: Logarithmic convexity, extended mean values, generalized power means, shifted Stolarsky means

1. Introduction

1.1.

There is a huge number of papers (cf. [2], [3], [6], [7], [8]) investigating properties of the so-called Stolarsky (or extended) two-parametric mean value, defined for positive values of x, y by the following

$$E_{r,s}(x,y) = \begin{cases} \left(\frac{r(x^s-y^s)}{s(x^r-y^r)}\right)^{1/(s-r)}, & rs(r-s) \neq 0 \\ \exp\left(-\frac{1}{s} + \frac{x^s \log x - y^s \log y}{x^s-y^s}\right), & r = s \neq 0 \\ \left(\frac{x^s-y^s}{s(\log x - \log y)}\right)^{1/s}, & s \neq 0, r = 0 \\ \sqrt{xy}, & r = s = 0, \\ x, & x = y > 0. \end{cases}$$

In this form it is introduced by Keneth Stolarsky in [1].

Most of the classical two variable means are special cases of the class E. For example, $E_{1,2} = \frac{x+y}{2}$ is the arithmetic mean, $E_{0,0} = \sqrt{xy}$ is the geometric mean, $E_{0,1} = \frac{x-y}{\log x - \log y}$ is the logarithmic mean, $E_{1,1} = (x^x/y^y)^{\frac{1}{x-y}}/e$ is the identric mean, etc. More generally, the r-th power mean $\left(\frac{x^r+y^r}{2}\right)^{1/r}$ is equal to $E_{r,2r}$.

Recently, several papers have been produced trying to define an extension of the class E to n, $n > 2$ variables. Unfortunately, this is done in a highly artificial mode (cf. [4], [5], [9]), without a practical background. Here is an

[1] Mathematical Institute SANU, Kneza Mihaila 36, 11000 Belgrade, Serbia, e-mail: ssimic@turing.mi.sanu.ac.yu

illustration of this point; recently J. Merikowski ([9]) has proposed the following generalization of the Stolarsky mean $E_{r,s}$ to several variables

$$E_{r,s}(X) := \Big[\frac{L(X^s)}{L(X^r)}\Big]^{\frac{1}{s-r}}, \quad r \neq s,$$

where $X = (x_1, \cdots, x_n)$ is an n-tuple of positive numbers and

$$L(X^s) := (n-1)! \int_{E_{n-1}} \prod_{i=1}^{n} x_i^{su_i} du_1 \cdots du_{n-1}.$$

The symbol E_{n-1} stands for the Euclidean simplex which is defined by

$$E_{n-1} := \{(u_1, \cdots, u_{n-1}) : u_i \geq 0, 1 \leq i \leq n-1;\ u_1 + \cdots + u_{n-1} \leq 1\}.$$

In this paper we give another attempt to generalize Stolarsky means to the multi-variable case in a simple and applicable way. The proposed task can be accomplished by founding a "weighted" variant of the class E, wherefrom the mentioned generalization follows naturally.

In the sequel we shall need notions of the weighted geometric mean $G = G(p, q; x, y)$ and weighted r-th power mean $S_r = S_r(p, q; x, y)$, defined by

$$G := x^p y^q; \quad S_r := (px^r + qy^r)^{1/r},$$

where

$$p, q, x, y \in R^+;\ p + q = 1;\ r \in R/\{0\}.$$

Note that $(S_r)^r > (G)^r$ for $x \neq y,\ r \neq 0$ and $\lim_{r\to 0} S_r = G$.

1.2.

We introduce here a class W of weighted two parameters means which includes the Stolarsky class E as a particular case. Namely, for $p, q, x, y \in R_+, p + q = 1, rs(r-s)(x-y) \neq 0$, we define

$$W = W_{r,s}(p, q; x, y) := \Big(\frac{r^2}{s^2}\frac{(S_s)^s - (G)^s}{(S_r)^r - (G)^r}\Big)^{\frac{1}{s-r}} = \Big(\frac{r^2}{s^2}\frac{px^s + qy^s - x^{ps}y^{qs}}{px^r + qy^r - x^{pr}y^{qr}}\Big)^{\frac{1}{s-r}}.$$

Various identities concerning the means W can be established; some of them are the following

$$W_{r,s}(p, q; x, y) = W_{s,r}(p, q; x, y)$$

$$W_{r,s}(p, q; x, y) = W_{r,s}(q, p; y, x);\ W_{r,s}(p, q; y, x) = xyW_{r,s}(p, q; x^{-1}, y^{-1});$$

$$W_{ar,as}(p, q; x, y) = (W_{r,s}(p, q; x^a, y^a))^{1/a},\ a \neq 0.$$

Note that

$$W_{2r,2s}(1/2, 1/2; x, y) = \Big(\frac{r^2}{s^2}\frac{x^{2s} + y^{2s} - 2(\sqrt{xy})^{2s}}{x^{2r} + y^{2r} - 2(\sqrt{xy})^{2r}}\Big)^{1/2(s-r)}$$

$$= \Big(\frac{r^2}{s^2}\frac{(x^s - y^s)^2}{(x^r - y^r)^2}\Big)^{1/2(s-r)} = E(r, s; x, y).$$

Hence $E \subset W$.

The weighted means from the class W can be extended continuously to the domain

$$D = \{(r, s, x, y)|r, s \in R,\ x, y \in R_+\}.$$

This extension is given by

$$W_{r,s}(p,q;x,y) = \begin{cases} \Big(\frac{r^2}{s^2}\frac{px^s+qy^s-x^{ps}y^{qs}}{px^r+qy^r-x^{pr}y^{qr}}\Big)^{1/(s-r)}, & rs(r-s)(x-y) \neq 0 \\ \Big(2\frac{px^s+qy^s-x^{ps}y^{qs}}{pqs^2\log^2(x/y)}\Big)^{1/s}, & s(x-y)\neq 0, r=0 \\ \exp\Big(\frac{-2}{s} + \frac{px^s\log x+qy^s\log y}{px^s+qy^s-x^{ps}y^{qs}} \\ \qquad - \frac{(p\log x+q\log y)x^{ps}y^{qs}}{px^s+qy^s-x^{ps}y^{qs}}\Big), & s(x-y)\neq 0, r=s \\ x^{(p+1)/3}y^{(q+1)/3}, & x\neq y,\ r=s=0 \\ x, & x=y. \end{cases}$$

1.3.

A natural generalization to the multi-variable case gives

$$W_{r,s}(\mathbf{p};\mathbf{x}) = \begin{cases} \Big(\frac{r^2(\sum p_i x_i^s - (\prod x_i^{p_i})^s)}{s^2(\sum p_i x_i^r - (\prod x_i^{p_i})^r)}\Big)^{1/(s-r)}, & rs(s-r)\neq 0; \\ \Big(\frac{2}{s^2}\frac{\sum p_i x_i^s - (\prod x_i^{p_i})^s}{\sum p_i \log^2 x_i - (\sum p_i \log x_i)^2}\Big)^{1/s}, & r=0, s\neq 0; \\ \exp\Big(\frac{-2}{s} + \frac{\sum p_i x_i^s \log x_i - (\sum p_i \log x_i)(\prod x_i^{p_i})^s}{\sum p_i x_i^s - (\prod x_i^{p_i})^s}\Big), & r=s\neq 0; \\ \exp\Big(\frac{\sum p_i \log^3 x_i - (\sum p_i \log x_i)^3}{3(\sum p_i \log^2 x_i - (\sum p_i \log x_i)^2)}\Big), & r=s=0. \end{cases}$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in R_+^n$, $n \geq 2$, $\mathbf{p}$ is an arbitrary positive weight sequence associated with $\mathbf{x}$ and $W_{r,s}(\mathbf{p};\mathbf{x}_0) = a$ for $\mathbf{x}_0 = (a, a, \cdots, a)$.

We also write $\sum(\cdot), \prod(\cdot)$ instead of $\sum_1^n(\cdot), \prod_1^n(\cdot)$.

The above formulae are obtained by an appropriate limit process, implying continuity. For example $W_{s,s}(\mathbf{p},\mathbf{x}) = \lim_{r\to s} W_{r,s}(\mathbf{p},\mathbf{x})$ and $W_{0,0}(\mathbf{p},\mathbf{x}) = \lim_{s\to 0} W_{0,s}(\mathbf{p},\mathbf{x})$.

2. Results and applications

Our main result is contained in the following

Proposition 1. *The means $W_{r,s}(\mathbf{p},\mathbf{x})$ are monotone increasing in both variables r and s.*

Passing to the continuous variable case, we get the following definition of the class $\bar{W}_{r,s}(p,x)$.

Assuming that all integrals exist

$$\bar{W}_{r,s}(p,x)=\begin{cases}\left(\frac{r^2(\int p(t)x^s(t)dt-\exp(s\int p(t)\log x(t)dt))}{s^2(\int p(t)x^r(t)dt-\exp(r\int p(t)\log x(t)dt)}\right)^{1/(s-r)}, & rs(s-r)\neq 0;\\ \left(\frac{2}{s^2}\frac{\int p(t)x^s(t)dt-\exp(s\int p(t)\log x(t)dt)}{\int p(t)\log^2 x(t)dt-(\int p(t)\log x(t)dt)^2}\right)^{1/s}, & r=0, s\neq 0;\\ \exp\Bigg(\frac{-2}{s}+\frac{\int p(t)x^s(t)\log x(t)dt}{\int p(t)x^s(t)dt-\exp(s\int p(t)\log x(t)dt)} \\ \qquad -\frac{(\int p(t)\log x(t)dt)\exp(s\int p(t)\log x(t)dt)}{\int p(t)x^s(t)dt-\exp(s\int p(t)\log x(t)dt)}\Bigg), & r=s\neq 0;\\ \exp\left(\frac{\int p(t)\log^3 x(t)dt-(\int p(t)\log x(t)dt)^3}{3(\int p(t)\log^2 x(t)dt-(\int p(t)\log x(t)dt)^2)}\right), & r=s=0\end{cases}$$

where $x(t)$ is a positive integrable function and $p(t)$ is a non-negative function with $\int p(t)dt=1$.

¿From our former considerations a very applicable assertion follows

Proposition 2. $\bar{W}_{r,s}(p,x)$ *is monotone increasing in either* r *or* s.

As an illustration we give the following

Proposition 3. *The function* $w(s)$ *defined by*

$$w(s):=\begin{cases}\left(\frac{12}{(\pi s)^2}(\Gamma(1+s)-e^{-\gamma s})\right)^{1/s}, & s\neq 0;\\ \exp(-\gamma-\frac{4\xi(3)}{\pi^2}), & s=0,\end{cases}$$

is monotone increasing for $s\in(-1,\infty)$.

In particular, for $s\in(-1,1)$ *we have*

$$\Gamma(1-s)e^{-\gamma s}+\Gamma(1+s)e^{\gamma s}-\frac{\pi s}{\sin(\pi s)}\leq 1-\frac{(\pi s)^4}{144},$$

where $\Gamma(\cdot),\xi(\cdot),\gamma$ *stands for the Gamma function, Zeta function and the Euler's constant, respectively.*

Applications in Probability Theory

For a random variable X and an arbitrary distribution with support on $(-\infty,+\infty)$, it is well known that

$$Ee^X\geq e^{EX}.$$

Denoting the central moment of order k by $\mu_k=\mu_k(X):=E(X-EX)^k$, we improve the above inequality to the following

Proposition 4. *For an arbitrary probability law with support on* $\mathbb{R}$, *we have*

$$Ee^X\geq(1+(\mu_2/2)\exp(\mu_3/3\mu_2))e^{EX}.$$

Proposition 5. *We also have that*

$$\Big(\frac{Ee^{sX}-e^{sEX}}{s^2\sigma_X^2/2}\Big)^{1/s}$$

is monotone increasing in s.

Especially interesting is studying of the *shifted Stolarsky means* E^*, defined by

$$E^*_{r,s}(x,y) := \lim_{p\to 0^+} W_{r,s}(p,q;x,y).$$

Their analytic continuation to the whole (r,s) plane is given by

$$E^*_{r,s}(x,y)=\begin{cases}\Big(\frac{r^2(x^s-y^s(1+s\log(x/y)))}{s^2(x^r-y^r(1+r\log(x/y)))}\Big)^{1/(s-r)}, & rs(r-s)(x-y)\neq 0;\\ \Big(\frac{2}{s^2}\frac{x^s-y^s(1+s\log(x/y))}{\log^2(x/y)}\Big)^{1/s}, & s(x-y)\neq 0,\ r=0;\\ \exp\Big(\frac{-2}{s}+\frac{(x^s-y^s)\log x-sy^s\log y\log(x/y)}{x^s-y^s(1+s\log(x/y))}\Big), & s(x-y)\neq 0, r=s;\\ x^{1/3}y^{2/3}, \quad r=s=0;\\ x, \quad x=y.\end{cases}$$

Main results concerning the means E^* are the following

Proposition 6. *Means $E^*_{r,s}(x,y)$ are monotone increasing in either r or s for each fixed $x,y\in\mathbb{R}^+$.*

Proposition 7. *Means $E^*_{r,s}(x,y)$ are monotone increasing in either x or y for each $r,s\in\mathbb{R}$.*

The well known result of Feng Qi ([11]) states that the means $E_{r,s}(x,y)$ are logarithmically concave for each fixed $x,y>0$ and $r,s\in[0,+\infty)$; also, they are logarithmically convex for $r,s\in(-\infty,0]$.

According to this, we propose the following

3. Open question

Is there any compact interval I, $I\subset\mathbb{R}$ such that the means $E^*_{r,s}(x,y)$ are logarithmically convex (concave) for $r,s\in I$ and each $x,y\in\mathbb{R}^+$?

A partial answer to this problem is given in the next

Proposition 8. *On any interval I which includes zero and $r,s\in I$,*

(i) $E^*_{r,s}(x,y)$ are not logarithmically convex (concave);

(ii) $W_{r,s}(p,q;x,y)$ are logarithmically convex (concave) if and only if $p=q=1/2$.

4. Proofs

We prove first a global theorem concerning log-convexity of the Jensen's functional with a parameter, which can be very usable (cf [10]).

Theorem 1. *Let $f_s(x)$ be a twice continuously differentiable function in x with a parameter s. If $f''_s(x)$ is log-convex in s for $s \in I := (a, b)$; $x \in J := (c, d)$, then the form*

$$\Phi_f(w, x; s) = \Phi(s) := \sum w_i f_s(x_i) - f_s(\sum w_i x_i),$$

is log-convex in s for $s \in I$, $x_i \in J, i = 1, 2, \cdots$, where $w = \{w_i\}$ is any positive weight sequence.

At the beginning we need some preliminary lemmas.

Lemma 1. *A positive function f is log-convex on I if and only if the relation*

$$f(s)u^2 + 2f(\frac{s+t}{2})uw + f(t)w^2 \geq 0,$$

holds for each real u, w and $s, t \in I$.

This assertion is nothing more than the discriminant test for the nonnegativity of second-order polynomials.

Another well known assertions are the following (cf [12], p. 74, 97-98),

Lemma 2 (Jensen's inequality)**.** *If $g(x)$ is twice continuously differentiable and $g''(x) \geq 0$ on J, then $g(x)$ is convex on J and the inequality*

$$\sum w_i g(x_i) - g(\sum w_i x_i) \geq 0$$

holds for each $x_i \in J$, $i = 1, 2, \cdots$ and any positive weight sequence $\{w_i\}$, $\sum w_i = 1$.

Lemma 3. *For a convex f, the expression*

$$\frac{f(s) - f(r)}{s - r}$$

is increasing in both variables.

Proof of Theorem 1.

Consider the function $F(x)$ defined as

$$F(x) = F(u, v, s, t; x) := u^2 f_s(x) + 2uv f_{\frac{s+t}{2}}(x) + v^2 f_t(x),$$

where $u, v \in \mathbb{R}$; $s, t \in I$ are real parameters independent of the variable $x \in J$.

Since

$$F''(x) = u^2 f''_s(x) + 2uv f''_{\frac{s+t}{2}}(x) + v^2 f''_t(x),$$

and by the assumption $f_s''(x)$ is log-convex in s, it follows from Lemma 1 that $F''(x) \geq 0,\ x \in J$.

Therefore, by Lemma 2 we get

$$\sum w_i F(x_i) - F(\sum w_i x_i) \geq 0,\ x_i \in J,$$

which is equivalent to

$$u^2\Phi(s) + 2uv\Phi(\frac{s+t}{2}) + v^2\Phi(t) \geq 0.$$

According to Lemma 1 again, this is possible only if $\Phi(s)$ is log-convex and proof is done. □

Proof of Proposition 1.

Define the auxiliary function $g_s(x)$ by

$$g_s(x) := \begin{cases} (e^{sx} - sx - 1)/s^2, & s \neq 0; \\ x^2/2, & s = 0. \end{cases}$$

Since

$$g_s'(x) = \begin{cases} (e^{sx} - 1)/s, & s \neq 0; \\ x, & s = 0, \end{cases}$$

and

$$g_s''(x) = e^{sx}, s \in \mathbb{R},$$

we see that $g_s(x)$ is twice continuously differentiable and that $g_s''(x)$ is a log-convex function for each real s, x.

Applying Theorem 1, we conclude that the form

$$\Phi_g(w, x; s) = \Phi(s) := \begin{cases} (\sum w_i e^{sx_i} - e^{s\sum w_i x_i})/s^2, & s \neq 0; \\ (\sum w_i x_i^2 - (\sum w_i x_i)^2)/2, & s = 0, \end{cases}$$

is log-convex in s.

By Lemma 3, with $f(s) = \log\Phi(s)$, we find out that

$$\frac{\log\Phi(s) - \log\Phi(r)}{s - r} = \log\Big(\frac{\Phi(s)}{\Phi(r)}\Big)^{\frac{1}{s-r}},$$

is monotone increasing either in s or r. Therefore, by changing variable $x_i \to \log x_i$, we finally obtain the proof of Proposition 1. □

Proof of Proposition 2. The assertion of Proposition 2 follows from Proposition 1 by the standard argument (cf [12], pp. 131-134). Details are left to the reader. □

Proof of Proposition 3. The proof follows putting $f(t) = t, p(t) = e^{-t}, t \in (0, +\infty)$ and applying Proposition 2. □

Proof of Proposition 4. By Proposition 2, we get

$$W_{0,1}(\mathbf{p}, e^{\mathbf{x}}) \geq W_{0,0}(\mathbf{p}, e^{\mathbf{x}}),$$

i. e.,

$$\frac{Ee^X - e^{EX}}{\mu_2/2} \geq \exp(\frac{EX^3 - (EX)^3}{3\mu_2}).$$

Using the identity $EX^3 - (EX)^3 = \mu_3 + 3\mu_2 EX$, we obtain the proof of Proposition 4. □

Proof of Proposition 5. This assertion is a straightforward consequence of the fact that $W_{0,s}(\mathbf{p}, e^{\mathbf{x}})$ is monotone increasing in s. □

Proof of Proposition 6 Direct consequence of Proposition 1. □

Proof of Proposition 7 This is left as an easy exercise to the readers. □

Proof of Proposition 8 We prove only the part (ii). The proof of (i) goes along the same lines.

Suppose that $0 \in (a, b) := I$ and that $E_{r,s}(p, q; x, y)$ are log-convex (concave) for $r, s \in I$ and any fixed $x, y \in \mathbb{R}^+$. Then there should be an $s,\ s > 0$ such that

$$F_s(p, q; x, y) := W_{0,s}(p, q; x, y)W_{0,-s}(p, q; x, y) - (W_{0,0}(p, q; x, y))^2$$

is of constant sign for each $x, y > 0$.

Substituting $(x/y)^s := e^w,\ w \in \mathbb{R}$, after some calculations we get that the above is equivalent to the assertion that $F(p, q; w)$ is of constant sign, where

$$F(p, q; w) := pe^w + q - e^{pw} - e^{\frac{2}{3}(1+p)w}(pe^{-w} + q - e^{-pw}).$$

Developing in power series in w, we get

$$F(p, q; w) = \frac{1}{1620}pq(1 + p)(2 - p)(1 - 2p)w^5 + O(w^6).$$

Therefore, $F(p, q; w)$ can be of constant sign for each $w \in \mathbb{R}$ only if $p = 1/2(= q)$.

Suppose now that I is of the form $I := [0, a)$ or $I := (-a, 0]$. Then there should be an $s, s \neq 0, s \in I$ such that

$$W_{0,0}(p, q; x, y)W_{0,2s}(p, q; x, y) - (W_{0,s}(p, q; x, y))^2$$

is of constant sign for each $x, y \in \mathbb{R}^+$.

Proceeding as above, this is equivalent to the assertion that $G(p, q; w)$ is of constant sign with

$$G(p, q; w) := p^3q^3w^6e^{\frac{2}{3}(p+1)w}(pe^{2w} + q - e^{2pw}) - (pe^w + q - e^{pw})^4.$$

But,

$$G(p, q; w) = \frac{2}{405}p^4q^4(1 + p)(1 + q)(q - p)w^{11} + O(w^{12}).$$

Hence we conclude that $G(p, q; w)$ can be of constant sign for a sufficiently small $w,\ w \in \mathbb{R}$ only if $p = q = 1/2$. Combining this with the Feng Qi theorem, the assertion from Proposition 8 follows. □

References

[1] Stolarsky, K.B., Generalizations of the logarithmic mean. Math. Mag. 48(2) (1975), 87-92.

[2] Stolarsky, K.B., The power and generalized logarithmic means. Amer. Math. Monthly, 87(7) (1980), 545-548.

[3] Leach, E.B., Sholander, M.C., Extended mean values. Amer. Math. Monthly 85 (2) (1978), 84-90.

[4] Leach, E.B., Sholander, M.C., Multi-variable extended mean values. J. Math. Anal. Appl. 104 (1984), 390-407.

[5] Pales, Z. Inequalities for differences of powers. J. Math. Anal. Appl. 131 (1988), 271-281.

[6] Neuman, E., Pales, Z., On comparison of Stolarsky and Gini means. J. Math. Anal. Appl. 278 (2003), 274-284.

[7] Neuman, E., Sandor, J. Inequalities involving Stolarsky and Gini means. Math. Pannonica 14(1) (2003), 29-44.

[8] Czinder, P., Pales, Z., An extension of the Hermite-Hadamard inequality and an application for Gini and Stolarsky means. J. Ineq. Pure Appl. Math. 5(2) (2004), Art. 42.

[9] Merikowski, J.K., Extending means of two variables to several variables. J. Ineq. Pure Appl. Math. 5(3) (2004), Art. 65.

[10] Simić, S. On logarithmic convexity for differences of power meansJ̇. Ineq. Appl. Article ID 37359, 2007.

[11] Qi, F., Logarithmic convexity of extended mean values. Proc. Amer. Math. Soc. 130 (6) (2001), 1787-1796.

[12] Hardy, G.H., Littlewood, J.E., Pólya, G., Inequalities. Cambridge: Cambridge University Press, 1978.

Received by the editors September 16, 2008