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SOME RESULTS FOR UNIVALENT FUNCTIONS DEFINED WITH RESPECT TO N-SYMMETRIC POINTS[1]

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Abstract. The criteria that embed a normalized analytic function in the class of functions that are starlike with respect to N-symmetric points are presented. The criteria are based on the quotient of analytical representations of starlikeness and convexity with respect to N-symmetric points.

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1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of analytic functions in the unit disk $\mathcal{U} = \{z : |z| < 1\}$ normalized so that $f(0) = f'(0) - 1 = 0$, i.e., of type $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Also let $\mathcal{S}$ be the class of functions in $\mathcal{A}$ that are univalent in $\mathcal{U}$.

In [5], K. Sakaguchi introduced the class of functions that are *starlike with respect to N-symmetric points*, $N = 1, 2, 3, \dots$, as follows

$$\mathcal{SSP}_N = \left\{ f \in \mathcal{A} : \mathrm{Re}\frac{zf'(z)}{f_N(z)} > 0, z \in \mathcal{U} \right\},$$

where

$$f_N(z) = z + \sum_{m=2}^{\infty} a_{m\cdot N+1} z^{m\cdot N+1}.$$

In order to give geometric characterization of the class $\mathcal{SSP}_N$ for $N \geq 2$ we define $\varepsilon := \exp(2\pi i/N)$ and consider the weighted mean of $f \in \mathcal{A}$,

$$M_{f,N}(z) = \frac{1}{\sum_{j=1}^{N-1} \varepsilon^{-j}} \cdot \sum_{j=1}^{N-1} \varepsilon^{-j} \cdot f(\varepsilon^j z).$$

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It is easy to verify that

$$\frac{f(z)-M_{f,N}(z)}{N}=\frac{1}{N}\cdot\sum_{j=0}^{N-1}\varepsilon^{-j}\cdot f(\varepsilon^j z)=f_N(z)$$

and further

$$\begin{aligned} f_N(\varepsilon^j z)&=\varepsilon^j f_N(z),\\ f_N'(\varepsilon^j z)&=f_N'(z)=\frac{1}{N}\sum_{j=0}^{N-1}f'(\varepsilon^j z),\\ \varepsilon^j f_N''(\varepsilon^j z)&=f_N''(z)=\frac{1}{N}\sum_{j=0}^{N-1}\varepsilon^j f''(\varepsilon^j z). \end{aligned}$$

Now, the class $\mathcal{SSP}_N$ is collection of functions $f\in\mathcal{A}$ such that for any r close to 1, $r<1$, the angular velocity of $f(z)$ about the point $M_{f,N}(z_0)$ is positive at $z=z_0$ as z traverses the circle $|z|=r$ in the positive direction.

For $N=1$ we obtain the well-known class of *starlike functions*, $S^*\equiv\mathcal{SSP}_1$, such that $f(\mathcal{U})$ is a starlike region with respect to the origin, i.e., $t\omega\in f(\mathcal{U})$ whenever $\omega\in f(\mathcal{U})$ and $t\in[0,1]$ (for more details see [1]). One of its subclasses is the class of *strongly starlike functions of order* α, $0<\alpha\le 1$, defined by

$$\widetilde{S}^*(\alpha)=\left\{f\in\mathcal{A}:\left|\arg\frac{zf'(z)}{f(z)}\right|<\frac{\alpha\pi}{2},z\in\mathcal{U}\right\}.$$

For $N=2$ we obtain $2f_2(z)=f(z)-f(-z)$, $M_{f,2}(z)=f(-z)$ and

$$\mathcal{SSP}\equiv\mathcal{SSP}_2=\left\{f\in\mathcal{A}:\ \mathrm{Re}\,\frac{zf'(z)}{f(z)-f(-z)}>0,z\in\mathcal{U}\right\}$$

is the class of *starlike functions with respect to symmetric points.*

As for the inclusion properties of the class $\mathcal{SSP}_N$, in [3] it is shown that $\mathcal{SSP}_N\not\subseteq S^*$ and $S^*\not\subseteq\mathcal{SSP}_N$ for $N\ge 2$. Coefficient estimates for $f\in\mathcal{SSP}_N$ are obtained in [5] and [7] and two-sided estimates for $|f(z)|$ and $|f'(z)|$ in [3] and [7].

Further, a function $f\in\mathcal{A}$ belongs to the class $\mathcal{K}_N$ of *convex functions with respect to N-symmetric points* if

$$\mathrm{Re}\frac{[zf'(z)]'}{f_N'(z)}>0,\quad z\in\mathcal{U}.$$

For $N=1$ we obtain the usual class of *convex functions*

$$K\equiv\mathcal{K}_1=\left\{f\in\mathcal{A}:\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>0,z\in\mathcal{U}\right\}.$$

In this paper we study the classes defined above and obtain sufficient conditions for starlikeness with respect to N-symmetric points in terms of the operator

$$I(f,N,a,b;z)=\frac{a+bzf''(z)/f'(z)}{zf_N'(z)/f_N(z)}.$$

For $a = b = 1$ we receive

$$I(f, N, 1, 1; z) = \frac{1 + zf''(z)/f'(z)}{zf'_N(z)/f_N(z)} = \frac{[zf'(z)]'/f'_N(z)}{zf'(z)/f_N(z)}$$

which is the quotient of analytical representations of starlikeness and convexity with respect to $N-$symmetric points. The classical case when $N = 1$ is studied in the following papers: $N = a = b = 1$ in [6], [4], [10]; $N = b = 1$ and a real in [8], [9]; and the most general case when $N = 1$ and a, b real in [11].

2. Main result and consequences

To prove the main result of this section we need the following lemmas.

Lemma 2.1. *[2] Let Ω be a subset of the complex plane $\mathbb{C}$ and let function $\psi : \mathbb{C}^2 \times \mathcal{U} \to \mathbb{C}$ satisfies $\psi(ix, y; z) \notin \Omega$ for all real x, $y \leq -\frac{1+x^2}{2}$ and for all $z \in \mathcal{U}$. If the function $p(z)$ is analytic in $\mathcal{U}$, $p(0) = 1$ and $\psi(p(z), zp'(z); z) \in \Omega$ for all $z \in \mathcal{U}$ then* $\mathrm{Re} p(z) > 0$, $z \in \mathcal{U}$.

Lemma 2.2. *Let $f \in \mathcal{A}$ and let a and b be real numbers. Also, let $\Omega = \mathbb{C} \setminus \Omega_1$, where*

$$\Omega_1 = \left\{ b + \frac{f_N(z)}{zf'_N(z)} (a - b + bui) : z \in \mathcal{U}, u \in \mathbb{R}, |u| \geq 1 \right\}.$$

If $I(f, N, a, b; z) \in \Omega$, $z \in \mathcal{U}$, then $f \in \mathcal{SSP}_N$.

Proof. If we let $p(z) = \frac{zf'(z)}{f_N(z)}$ then $p(z)$ is analytic in $\mathcal{U}$ and $p(0) = 1$. Further, for the function $\psi(r, s; z) = b + \frac{f_N(z)}{zf'_N(z)} \left(a - b + b\frac{s}{r}\right)$ we have

$$\psi(p(z), zp'(z); z) = b + \frac{f_N(z)}{zf'_N(z)} \left(a - b + b\frac{zp'(z)}{p(z)} \right) = I(f, N, a, b; z).$$

So, by Lemma 2.1, for proving $f \in \mathcal{SSP}_N$, or equivalently $\mathrm{Re}\, p(z) > 0$, $z \in \mathcal{U}$, it is enough to show that $\psi(ix, y; z) \in \Omega_1$ for all real x, $y \leq -\frac{1+x^2}{2}$ and for all $z \in \mathcal{U}$. Indeed

$$\psi(ix, y; z) = b + \frac{f_N(z)}{zf'_N(z)} \left(a - b - b\frac{y}{x} i \right) \in \Omega_1$$

because y/x attains all real numbers with absolute value greater than or equal to 1. □

This lemma leads to the following criteria for starlikeness with respect to N-symmetric points.

Theorem 2.1. *Let $f \in \mathcal{A}$, $N \in \mathbb{N} \setminus \{1\}$ and let a and b be real numbers.*

(i) If $f_N \in \widetilde{S}^*(\alpha)$ $(0 < \alpha < 1)$, $\frac{|b|}{a-b} > \tan\frac{\alpha\pi}{2}$ *when* $a - b > 0$ *and if*

$$|\arg[I(f,N,a,b;z) - b]| < \lambda_1 \equiv \begin{cases} \arctan\frac{|b|}{a-b} - \alpha\frac{\pi}{2}, & a-b>0 \\ (1-\alpha)\frac{\pi}{2}, & a-b \le 0 \end{cases},$$

for all $z \in \mathcal{U}$ *then* $f \in \mathcal{SSP}_N$.

(ii) If $\left|\frac{zf_N'(z)}{f_N(z)}\right| > \frac{1}{\mu}$ $(\mu > 1)$ *and*

$$|(I(f,N,a,b;z) - b| < \lambda_2 \equiv \mu\sqrt{(a-b)^2 + b^2}$$

for all $z \in \mathcal{U}$ *then* $f \in \mathcal{SSP}_N$.

Proof. Let define two sets of complex numbers $\Sigma_1 = \{w : |\arg(w-b)| < \lambda_1\}$ and $\Sigma_2 = \{w : |w-b| < \lambda_2\}$. In view of Lemma 2.2, for proving (i) and (ii) it is enough to show that $\Sigma_1 \subseteq \Omega$ and $\Sigma_2 \subseteq \Omega$, respectively.

(i) We will show that $\Sigma_1 \subseteq \Omega$ by verifying $\Sigma_1 \cap \Omega_1 = \emptyset$. If $w \in \Omega_1$ then for some $z \in \mathcal{U}$, $u \in \mathbb{R}$ and $|u| \ge 1$, we have

$$\begin{aligned} |\arg(w-b)| &= \left|\arg\frac{f_N(z)}{zf_N'(z)} + \arg(a-b+bui)\right| \\ &\ge \left|\left|\arg\frac{f_N(z)}{zf_N'(z)}\right| - |\arg(a-b+bui)|\right|. \end{aligned}$$

We continue with the proof having in mind that $f_N \in \widetilde{S}^*(\alpha)$ implies $\left|\arg\frac{f_N(z)}{zf_N'(z)}\right| < \alpha\frac{\pi}{2}$, $z \in \mathcal{U}$. In the case when $a - b > 0$ we obtain

$$\arctan\frac{|b|}{a-b} < |\arg(a-b+bui)| < \frac{\pi}{2}$$

and further

$$|\arg(w-b)| \ge \arctan\frac{|b|}{a-b} - \alpha\frac{\pi}{2} = \lambda_1,$$

i.e., $w \notin \Sigma_1$. In the case $a - b \le 0$ we have $w \notin \Sigma_2$ because

$$|\arg(w-b)| \ge \left|\alpha\frac{\pi}{2} - \left(\frac{\pi}{2} + \arctan\frac{|a-b|}{|b_n|}\right)\right| \ge (1-\alpha)\frac{\pi}{2} = \lambda_1.$$

(ii) The proof of $\Sigma_2 \subseteq \Omega$, i.e., $\Sigma_2 \cap \Omega_1 = \emptyset$ goes in a similar manner as in (i). If $w \in \Omega_1$ then $w \notin \Sigma_2$ because of

$$\begin{aligned} |w-b| &= \left|\frac{f_N(z)}{zf_N'(z)} \cdot (a-b+bui)\right| = \left|\frac{f_N(z)}{zf_N'(z)}\right| \cdot \sqrt{(a-b)^2 + b^2u^2} \\ &\ge \mu\sqrt{(a-b)^2 + b^2} = \lambda_2. \end{aligned}$$

□

Remark 2.1. *In the statement of the theorem we impose $N \in \mathbb{N} \setminus \{1\}$ since for $N = 1$ the statement has no sense.*

For $a = b = 1$ we obtain the following

Colorallary 2.1. *Let $f \in \mathcal{A}$ and $N \in \mathbb{N} \setminus \{1\}$.*

(i) If $f_N \in \widetilde{S}^*(\alpha)$ for some $0 < \alpha < 1$ and

$$\left| \arg \left[\frac{[zf'(z)]'/f_N'(z)}{zf'(z)/f_N(z)} - 1 \right] \right| < (1-\alpha)\frac{\pi}{2}$$

for all $z \in \mathcal{U}$ then $f \in \mathcal{SSP}_N$.

(ii) If $\left| \frac{zf_N'(z)}{f_N(z)} \right| > \frac{1}{\mu}$ for some $\mu > 1$ and

$$\left| \frac{[zf'(z)]'/f_N'(z)}{zf'(z)/f_N(z)} - 1 \right| < \mu$$

for all $z \in \mathcal{U}$ then $f \in \mathcal{SSP}_N$.

For $a = 0$ and $b = 1$ we obtain

Colorallary 2.2. *Let $f \in \mathcal{A}$ and $N \in \mathbb{N} \setminus \{1\}$.*

(i) If $f_N \in \widetilde{S}^*(\alpha)$ for some $0 < \alpha < 1$ and

$$\left| \arg \left[\frac{f_N(z)f''(z)}{f_N'(z)f'(z)} - 1 \right] \right| < (1-\alpha)\frac{\pi}{2}$$

for all $z \in \mathcal{U}$ then $f \in \mathcal{SSP}_N$.

(ii) If $\left| \frac{zf_N'(z)}{f_N(z)} \right| > \frac{1}{\mu}$ for some $\mu > 1$ and

$$\left| \frac{f_N(z)f''(z)}{f_N'(z)f'(z)} - 1 \right| < \mu\sqrt{2}$$

for all $z \in \mathcal{U}$ then $f \in \mathcal{SSP}_N$.

For $a = 1$ and $b = -1$ we receive

Colorallary 2.3. *Let $f \in \mathcal{A}$ and $N \in \mathbb{N} \setminus \{1\}$.*

(i) If $f_N \in \widetilde{S}^*(\alpha)$ for some $0 < \alpha < 1$ such that $\tan \frac{\alpha\pi}{2} < \frac{1}{2}$ and if

$$\left| \arg \left[\frac{1 - zf''(z)/f'(z)}{zf_N'(z)/f_N(z)} + 1 \right] \right| < \arctan \frac{1}{2} - \frac{\alpha\pi}{2}$$

for all $z \in \mathcal{U}$ then $f \in \mathcal{SSP}_N$.

(ii) If $\left| \frac{zf_N'(z)}{f_N(z)} \right| > \frac{1}{\mu}$ for some $\mu > 1$ and

$$\left| \frac{1 - zf''(z)/f'(z)}{zf_N'(z)/f_N(z)} + 1 \right| < \mu\sqrt{5}$$

for all $z \in \mathcal{U}$ then $f \in \mathcal{SSP}_N$.

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