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ON THE NON-COMMUTATIVE NEUTRIX PRODUCT INVOLVING SLOWLY VARYING FUNCTIONS[1]

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Abstract. Let $L(x)$ be a slowly varying function at both zero and infinity. The existence of the non-commutative neutrix convolution product of the distributions $x_+^\lambda L(x)$ and x_-^μ is proved, where λ, μ are real numbers such that $\lambda, \mu \notin -\mathbb{N}$ and $\lambda+\mu \notin -\mathbb{Z}$. Some other products of distributions are obtained.

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1. Introduction

In the following we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. The convolution product $f * g$ of two distributions f and g in $\mathcal{D}'$ is then usually defined as follows, see [2].

Definition 1.1. *Let f and g be distributions in $\mathcal{D}'$, satisfying at least one of the following conditions:*
(a) either f or g has bounded support,
(b) the supports of f and g are bounded on the same side.
Then the convolution product $f * g$ is defined by

$$\langle (f * g)(x), \varphi(x)\rangle = \langle f(y), \langle g(x), \varphi(x + y)\rangle\rangle$$

for an arbitrary φ in $\mathcal{D}$.

It follows that if the convolution product $f * g$ exists by this definition then

$$f * g = g * f, \tag{1}$$

$$(f * g)' = f * g' = f' * g. \tag{2}$$

The convolution product of distributions may be defined in a more general way yet, without the restrictions on the supports given above in a) or b). However, the convolution product in the sense of any of these definitions does not

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exist for many pairs of distributions. In [3] (and other papers) the *neutrix convolution product* was defined so it exists for a considerably larger class of pairs of distributions. In that definition unit sequences of functions in $\mathcal{D}$ are used, which allows one to approximate a given distribution by a sequence of distributions of bounded support.

To recall the definition of the neutrix convolution product we, first of all, let τ be a fixed function in $\mathcal{D}$ with the following properties:

(i) $\tau(x) = \tau(-x)$,
(ii) $0 \leq \tau(x) \leq 1$,
(iii) $\tau(x) = 1$, for $|x| \leq \frac{1}{2}$,
(iv) $\tau(x) = 0$, for $|x| \geq 1$.

Next we define the unit sequence $\{\tau_n\}_{n\in\mathbb{N}}$ of functions, setting

$$\tau_n(x) = \begin{cases} 1, & |x| \leq n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n. \end{cases}$$

In order to define the neutrix convolution product, we need the following definition given by Van der Corput (see [1]):

Definition 1.2. *A neutrix N is a commutative additive group of functions $\nu : N^{'} \to N^{''}$ (where the domain $N^{'}$ is a set and the range $N^{''}$ is a commutative additive group) with the property that if ν is in N and $\nu(\xi) = \gamma$ for all ξ in $N^{'}$ then $\gamma = 0$. The functions in N are said to be negligible.*

Now suppose that $N^{'}$ is contained in a topological space with a limit point b which is not in $N^{'}$, and let N be a commutative additive group of functions $\nu : N^{'} \to N^{''}$ with the property that if N contains a function of ξ which tends to a finite limit γ as ξ tends to b, then $\gamma = 0$. It follows that N is a neutrix. If now $f : N^{'} \to N^{''}$ and there exists a constant β such that $f(\xi) - \beta$ is negligible in N, then β is called the neutrix limit of $f(\xi)$ as ξ tends to b and we write $\mathrm{N}{-}\lim_{\xi\to b} f(\xi) = \beta$. Note that if a neutrix limit exists, then it is unique, since if $f(\xi) = \beta$ and $f(\xi) = \beta^{'}$ are in N, then the constant function $\beta - \beta^{'}$ is also in N and so $\beta = \beta^{'}$.

In the following we let N be the neutrix having domain $N^{'} = \mathbb{N} = \{1, 2, \ldots, n, \ldots\}$, range the real numbers, with negligible functions finite linear sums of the functions

$$n^{\lambda} \ln^{r-1} n,\ \ln^{r} n, \quad (\lambda \neq 0,\ r = 1, 2, \ldots)$$

and all functions which converge to zero in the usual sense as n tends to infinity.

Definition 1.3. *Let f and g be distributions in $\mathcal{D}'$ and let $f_n = f\tau_n$ for $n = 1, 2, \ldots$. Then the non-commutative neutrix convolution product $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_n * g\}_{n\in\mathbb{N}}$, provided that the limit h exists in the sense that*

$$\mathop{\mathrm{N}{-}\lim}_{n\to\infty} \langle f_n * g, \varphi \rangle = \langle h, \varphi \rangle,$$

for all φ in $\mathcal{D}$, where N is the neutrix described in Definition 1.2

Note that in this definition the convolution product $f_n * g$ is in the sense of Definition 1.1. Namely, the distribution f_n has bounded support since the support of τ_n is contained in the interval $[-n-n^{-n}, n+n^{-n}]$. From the following theorem, proved in [3], (see also [4]), it follows that the neutrix convolution product from Definition 1.3 is a proper generalization of the "classical" convolution product of distributions from Definition 1.1.

Theorem 1.1. *Let f and g be distributions in $\mathcal{D}'$ satisfying either condition (a) or condition (b) of Definition 1.1. Then the neutrix convolution product $f \circledast g$ exists and*

$$f \circledast g = f * g.$$

On using Definition 1.3, one can find several important neutrix products of distributions, see [4].

Theorem 1.2. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the neutrix convolution $f \circledast g$ exists. Then the neutrix convolution $f \circledast g'$ exists and*

$$(f \circledast g)' = f \circledast g'.$$

If $\mathrm{N}-\lim\limits_{n\to\infty}\langle (f\tau_n') * g, \varphi\rangle$ *exists and equals $\langle h, \varphi\rangle$ for all φ in $\mathcal{D}$, then $f' \circledast g$ exists and*

$$(f \circledast g)' = f' \circledast g + h.$$

Here and also throughout this paper $L : (0,\infty) \to (0,\infty)$ is a given locally integrable function which satisfies the following condition:

$$\lim_{x\to 0^+} \frac{L(kx)}{L(x)} = 1, \qquad \text{for any} \quad k > 0, \tag{3}$$

$$\lim_{x\to \infty} \frac{L(kx)}{L(x)} = 1, \qquad \text{for any} \quad k > 0. \tag{4}$$

A positive locally integrable function satisfying (3), (resp.(4)) is called *resp. slowly varying at zero (slowly varying at infinity).* The first example of a function satisfying the relations (3) and (4) is the logarithm; other examples are the positive powers and the iterations of the logarithm, e.g., $\ln^3$ and $\ln\ln$. An exposition of the theory of slowly varying functions can be found in [6].

The distribution $x_+^\lambda L(x)$ is defined for different values of the real parameter λ by:

$$\begin{aligned} \langle x_+^\lambda L(x), \varphi(x)\rangle &= \int_0^\infty x^\lambda L(x)\varphi(x)dx, \quad \text{if} \quad \lambda > -1, \\ \langle x_+^\lambda L(x), \varphi(x)\rangle &= \int_0^\infty x^\lambda L(x)\left[\varphi(x) - \sum_{i=0}^{k-1}\frac{x^i}{i!}\varphi^i(0)\right]dx, \end{aligned} \tag{5}$$

if $-k-1<\lambda<-k$ and $k\in\mathbb{N}$.

The aim of this paper is to analyze several neutrix product involving slowly varying functions. For that reason we now take the set of negligible functions obtained by replacing the logarithmic function ln with the slowly varying function L. More precisely, our new neutrix, again denoted by N, will have the domain $N'=\{1,2,\dots,n,\dots\}$, the range of the real numbers $\mathbb{R}$, with negligible functions finite linear sums of the functions

$$n^{\lambda},\ n^{\lambda}L(n),\ L^{r}(n)$$

for all real $\lambda\neq 0$ and $r\in\mathbb{N}$ and all functions which converge to zero in the usual sense, as n tends to infinity. In this way, we obtain a wide range of neutrix products, involving the corresponding slowly varying functions.

2. Main Results

Before we turn to the announced neutrix products, we cite three statements that we need later on.

Lemma 2.1. *Let $L(x)$ be a slowly varying function at infinity. Then $K(x)=L(\frac{1}{x})$ is a slowly varying function at zero.*

Theorem 2.1. *Let L be a slowly varying function at infinity and let f be a locally integrable function on the interval $[a,b]$ with the property that*

$$\int_a^b x^{\delta}|f(x)|\,dx<\infty \qquad \text{for some} \quad \delta>0.$$

Then the integral

$$\Phi(t)=\int_a^b f(x)L(tx)\,dx$$

exists and

$$\Phi(t)\sim L(t)\int_a^b f(x)\,dx \qquad \text{as} \quad t\to+\infty.$$

Theorem 2.2. *Let $x_+^{\lambda}L(x)$ be given by (5) for $-k-1<\lambda<-k$, $k\in\mathbb{N}$ and L a slowly varying function at zero and at infinity. Then there exists a locally integrable function $K:(0,\infty)\to\mathbb{R}$ which is both slowly varying at zero and at infinity, and satisfies the following conditions:*

$$\left(x_+^{\lambda+k}K(x)\right)^{(k)}=x_+^{\lambda}L(x), \qquad K(x)\sim((\lambda+1)\cdots(\lambda+k))^{-1}L(x),$$

as $x\to 0^+$ and as $x\to+\infty$.

Theorem 2.2 was proved in [7].

We now give our main theorem, which proves the existence of a neutrix convolution product involving a slowly varying function. Similar results involving commutative neutrix convolution product were proved in [5].

Theorem 2.3. *Let L be a slowly varying function both at zero and at infinity, with the parameters λ and μ satisfying the conditions*

$$\lambda, \mu \neq -1, -2, \dots \quad \textit{and} \quad \lambda + \mu \neq 0, \pm 1, \pm 2, \dots . \tag{6}$$

Then the non-commutative neutrix convolution product

$$x_+^\lambda L(x) \circledast x_-^\mu \tag{7}$$

exists, and

$$\left[x_+^\lambda L(x) \circledast x_-^\mu\right]' = \left[x_+^\lambda L(x)\right]' \circledast x_-^\mu . \tag{8}$$

Proof. We will first suppose that $\lambda, \mu > -1$ and $\lambda + \mu \neq -1, 0, 1, 2, \dots$ so that $x_+^\lambda L(x)$ and x_-^μ are locally summable functions. Put

$$\left[x_+^\lambda L(x)\right]_n = x_+^\lambda L(x) \tau_n(x) \ .$$

Then the convolution product $\left[x_+^\lambda L(x)\right]_n * x_-^\mu$ exists by Definition 1.1 and so:

$$\begin{aligned} \left[x_+^\lambda L(x)\right]_n * x_-^\mu &= \int_{-\infty}^{\infty} \left[t_+^\lambda L(t)\right]_n (x-t)_-^\mu \, dt \\ &= \int_0^\infty t^\lambda L(t)(x-t)_-^\mu \tau_n(t) \, dt. \end{aligned}$$

For $0 \leq x \leq n$ we have

$$\begin{aligned} \int_0^\infty t^\lambda L(t)(x-t)_-^\mu \tau_n(t) \, dt &= \int_x^n t^\lambda L(t)(t-x)^\mu \, dt \\ &+ \int_n^{n+n^{-n}} t^\lambda L(t)(t-x)^\mu \tau_n(t) \, dt. \end{aligned} \tag{9}$$

Making the substitution $t = xnu$, we have:

$$\int_x^n t^\lambda L(t)(t-x)^\mu \, dt = x^{\lambda+\mu+1} n^{\lambda+1} \int_{\frac{1}{n}}^{\frac{1}{x}} u^\lambda (nu-1)^\mu L(xnu) \, du.$$

On using Theorem 2 we can see that the right-hand side behaves as:

$$\begin{aligned} &L(xn) n^{1+\lambda} \Big[(1+\lambda+\mu) n^{-\lambda-\mu} x^{(1+\lambda+\mu)} \Gamma(-1-\lambda-\mu)\Gamma(1+\mu) + \\ &+ n\Gamma(-\lambda) {}_2F_1(-1-\lambda-\mu, -\mu; -\lambda-\mu; x/n)\Big] / (1+\lambda+\mu)\Gamma(-\lambda) \end{aligned}$$

where ${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{a_k b_k}{c_k} \frac{z^k}{k!}$ is the hypergeometric function. Now we have:

$$\operatorname*{N-lim}_{n\to\infty} \int_x^n t^\lambda L(t)(t-x)^\mu \, dt = 0. \tag{10}$$

When $-n \leq x \leq 0$, we have

$$\begin{aligned}\int_0^\infty t^\lambda L(t)(x-t)_-^\mu \tau_n(t)\,dt &= \int_0^{x+n} t^\lambda L(t)(t-x)^\mu\,dt \\ &+ \int_n^{n+n^{-n}} t^\lambda L(t)(t-x)^\mu \tau_n(t)\,dt.\end{aligned} \tag{11}$$

Making the substitution $t = xn(1-u)$, we have:

$$\int_0^{x+n} t^\lambda L(t)(t-x)^\mu\,dt = \\ = n^{\lambda+1}x^{\lambda+\mu+1}\int_{1-\frac{1}{n}-\frac{1}{x}}^1 (1-u)^\lambda(n-nu-1)^\mu L\left(xn(1-u)\right)\,du.$$

It follows as above that

$$\operatorname{N-lim}_{n\to\infty}\int_0^{x+n} t^\lambda L(t)(t-x)^\mu\,dt = 0. \tag{12}$$

Further, it is easily seen that
$\int_n^{n+n^{-n}} t^\lambda L(t)(t-x)^\mu \tau_n(t)\tau_n(x-t)dt = O(n^{-n+\lambda+\mu}L(n+n^{-n}))$ and so

$$\lim_{n\to\infty}\int_n^{n+n^{-n}} t^\lambda L(t)(t-x)^\mu \tau_n(t)dt = 0. \tag{13}$$

Now it follows from equations (9), (10), (11), (12) and (13) that the neutrix convolution product $x_+^\lambda L(x) \circledast x_-^\mu$ exists and it is equal to zero, proving the theorem for the case $\lambda, \mu > -1$ and $\lambda + \mu \neq -1, 0, 1, 2, \ldots$.

In order to finish the proof of Theorem 2.3 we still have to consider the case of arbitrary $\lambda < -1$ and $\mu < -1$, satisfying (6).

To that end, let us assume that equation (8) holds and that the product $x_+^\lambda L(x) \circledast x_-^\mu$ exists when $-k < \lambda < -k+1$ and any μ such that $\mu > -1$, $\lambda+\mu \neq -1, 0, 1, \ldots$. This is certainly true from what we have for $k = 1$. If $-k-1 < \lambda < -k$ then the product

$$x_+^\lambda K(x) \circledast x_-^\mu = \left[x_+^{\lambda+1}L(x) \circledast x_-^\mu\right]'$$

exists and it follows from induction that $x_+^\lambda L(x) \circledast x_-^\mu$ exists for $\lambda \neq -1, -2, \ldots$, $\mu > -1$ and $\lambda+\mu \neq -1, 0, 1, \ldots$. Similar induction arguments on μ finally prove the existence of the neutrix convolution product (7).

We still have to prove equation (8). First, integrating by parts we have

$$\begin{aligned}\left[x_+^\lambda L(x)\tau_n^{'}(x)\right]_n * \left[x_-^\mu\right] &= \int_n^{n+n^{-n}} t^\lambda L(t)(t-x)^\mu d\tau_n(t)\\ &= -n^\lambda L(n)(n-x)^\mu +\\ &\quad -\lambda\int_n^{n+n^{-n}} t^{\lambda-1}L(t)(t-x)^\mu\tau_n(t)dt\\ &\quad -\int_n^{n+n^{-n}} t^\lambda L'(t)(t-x)^\mu\tau_n(t)dt\\ &\quad -\mu\int_n^{n+n^{-n}} t^\lambda L(t)(t-x)^{\mu-1}\tau_n(t)dt.\end{aligned}$$

First we have $\operatorname{N-}\lim\limits_{n\to\infty} n^\lambda L(n)(n-x)^\mu = 0\,.$

Next $\left|\int_n^{n+n^{-n}} t^{\lambda-1}L(t)(t-x)^\mu\tau_n(t)dt\right| \le Cn^{-n+\lambda}(|x|+2n)^\mu$ and so

$$\lim_{n\to\infty}\int_n^{n+n^{-n}} t^{\lambda-1}L(t)(t-x)^\mu\tau_n(t)dt = 0.$$

Here C is a constant that we get estimating $L(x)$ and using the property that $L(x)$ is a slowly varying function.

Similarly we have that

$$\left|\int_n^{n+n^{-n}} t^\lambda L'(t)(t-x)^\mu\tau_n(t)dt\right| \le C_1 n^{-n+\lambda}(|x|+2n)^\mu, \quad \text{and}$$
$$\left|\int_n^{n+n^{-n}} t^\lambda L(t)(t-x)^{\mu-1}\tau_n(t)dt\right| \le C_2 n^{-n+\lambda}(|x|+2n)^{\mu-1}\,.$$

So

$$\lim_{n\to\infty}\int_n^{n+n^{-n}} t^\lambda L'(t)(t-x)^\mu\tau_n(t)dt = 0,$$
$$\lim_{n\to\infty}\int_n^{n+n^{-n}} t^\lambda L(t)(t-x)^{\mu-1}\tau_n(t)dt = 0.$$

Using Theorem 1.2 and Theorem 2.2 it follows that

$$\left[x_+^\lambda L(x)\circledast x_-^\mu\right]' = \left[x_+^\lambda L(x)\right]'\circledast x_-^\mu = x_+^{\lambda-1}K(x)\circledast x_-^\mu$$

proving equation (8).

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