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OSCILLATION PROPERTIES OF A CLASS OF SECOND ORDER EQUATIONS WITH VARIABLE COEFFICIENTS[1]

Zornitza Petrova[2]

Abstract. We establish sufficient conditions for oscillation of the following equations:

$$z''(t)+\sum_{i=0}^{n}\theta_i(t)z'(t-\tau_i)+\sum_{k=0}^{\tilde{n}}\beta_k(t)z(t-\sigma_k)+G(z''(t),z'(t),z(t))=F(t).$$

We suppose that $n,\ \tilde{n}\in\mathbf{N}$, $\tau_i\geq 0,\ \forall\ i=\overline{0,n}$ and $\sigma_k\geq 0,\ \forall\ k=\overline{0,\tilde{n}}$ are given constants as well as $T\geq 0$ is a large enough constant such that all the functions $\{\theta_i(t)\}_{i=0}^{n}$, $\{\beta_k(t)\}_{k=0}^{\tilde{n}}$ and $F(t)$ are of the class $C([T,\infty);\mathbf{R})$. Also, $G(z''(t),z'(t),z(t))\in C([T,\infty)^3;\mathbf{R})$. We obtain two types of results: the first is concerned with the monotonicity of the solutions, and the second one is a sufficient condition for the distributions of their zeros.

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1. Introduction

In general, differential equations appear from the proper mathematical models and the most popular of them come from physics. Both ordinary and partial differential equations of second order attract the attention of many specialists of different areas.

The present paper treats the oscillation behavior of the equation:

$$(1)\qquad z''(t)+\sum_{i=0}^{n}\theta_i(t)z'(t-\tau_i)+\sum_{k=0}^{\tilde{n}}\beta_k(t)z(t-\sigma_k)+G(z''(t),z'(t),z(t))=F(t),$$

which is a natural generalization of the following one:

$$(2)\qquad Az''(t)+\sum_{i=1}^{n}\theta_i z'(t-\tau_i)+\sum_{k=1}^{\tilde{n}}\beta_k z(t-\sigma_k)+Bz(t)=F(t)$$

since all the coefficients in (2) are given constants. Also, the delays are non-negaive constants both in (1) and (2).

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[2]Faculty of Applied Mathematics and Informatics, Technical University of Sofia, 8 Kl. Ohridski Blvd., Sofia-1000, Bulgaria e-mail: zap@tu-sofia.bg

2. Preliminary result

Recently, Petrova [4] established two types of oscillational results for (2). The first one is about the absence of both eventually positive and monotonically non-decreasing solutions of (2) as well as of eventually negative and monotonically non-increasing solutions of (2). There we stated the following assumptions:

(3) $$A > 0 \quad \text{and} \quad B > 0;$$

(4) $$\theta_i \geq 0, \quad \tau_i \geq 0, \quad \forall\, i = \overline{1,n},\ n \in \mathbf{N}, \quad \tau = \max_{1\leq i\leq n} \tau_i$$

(5) $$\text{and} \quad \beta_k \geq 0, \quad \sigma_k \geq 0, \quad \forall\, k = \overline{1,\tilde{n}},\ \tilde{n} \in \mathbf{N}, \quad \sigma = \max_{1\leq k\leq \tilde{n}} \sigma_k$$

for the first type of sufficient conditions for oscillation, which are concerned with the monotonicity of the solutions.

The second type of oscillational results in [4] are sufficient conditions for the distributions of the zeros of the following particular case of (2):

(6) $$Az''(t) + \theta z'(t) + \sum_{k=1}^{\tilde{n}} \beta_k z(t - \sigma_k) + Bz(t) = F(t).$$

More exactly, we found a sequence of semiopen intervals, which consist of at least one zero of every solution of (6), where the assumptions (3) and (5) hold again, but (4) is replaced with a new one:

(7) $$\theta \in \mathbf{R} \quad \text{and} \quad 4AB > \theta^2.$$

Moreover, in the paper [4] we explained that the roles of the delays $\{\tau_i\}_{i=1}^n$ and $\{\sigma_k\}_{k=1}^{\tilde{n}}$ are different from oscillational point of view.

Let us begin with two definitions before explaining briefly the situation. In the sequel we suppose that $c \in \mathbf{R}$ is a constant.

Definition 2.1. We say that the function $\varphi(t) \in C([c,\infty);\mathbf{R})$, $\varphi(t) \neq 0$, is **oscillating** when $t \to \infty$, if there exists a sequence $\{t_m\}_{m=1}^{\infty}$ such that

$$\lim_{m\to\infty} t_m = \infty \qquad \text{and} \qquad \varphi(t_m) = 0.$$

Definition 2.2. We say that the function $\varphi(t) \in C([c,\infty);\mathbf{R})$ is **eventually positive (respectively eventually negative)**, if there exists a constant $\tilde{c} \geq c$ such that

$$\varphi(t) > 0 \quad (\text{respectively} \quad \varphi(t) < 0), \quad \forall\, t \in [\tilde{c},\infty).$$

Approximately twenty years ago Yoshida [6] obtained sufficient conditions for oscillation of an initial characteristic value problem and the next lemma was the main tool there.

Lemma 2.1. *([6]) If there is a number $s \geq \rho$ such that*

$$\int_s^{s+\pi/L} F(t) \sin L(t-s) dt \leq 0, \tag{8}$$

then the ordinary differential inequality

$$z''(t) + L^2 z(t) \leq F(t) \tag{9}$$

has no positive solution in $(s, s + \pi/L]$.

Later Yoshida [7] dealt with another problem. There he just mentioned that he applies Lemma 2.1 immediately to the inequality

$$z''(t) + L^2 z(t) + \beta_1 z(t-\sigma) + \beta_2 z(t-\tau) \leq F(t), \tag{10}$$

where $\beta_1 \geq 0$ and $\beta_2 \geq 0$ are proper constants. He did not write anything for the distribution of the zeros of both equations

$$z''(t) + L^2 z(t) = F(t) \quad \text{and}$$
$$z''(t) + L^2 z(t) + \beta_1 z(t-\sigma) + \beta_2 z(t-\tau) = F(t)$$

although he could do it in [6] and [7] respectively.

The excellent publications of Yoshida are devoted absolutely to concrete partial differential equations and the application of ordinary differential inequalities to them is not connected with the exact formulation of the conclusions for the respective ordinary differential equations.

We point out that it is very difficult to find a proper generalization of Lemma 2.1.

Vel'misov, Gârnefska and Milusheva [5] considered an equation of a pipeline under four boundary conditions and the application of Galerkin's method to these boundary problems led to the equation

$$A_k z''(t) + B_k z(t) + E_k z'(t-\tau_1) + F_k z'(t-\tau_2) + G_k z(t-\tau_3) = 0, \quad \text{where} \tag{11}$$

$$A_k > 0,\ B_k \in \mathbf{R},\ E_k \geq 0,\ F_k \geq 0,\ G_k \geq 0,\ \forall\ k \in \mathbf{N}. \tag{12}$$

In [1] we realized for the first time a combination between Galerkin's method and oscillation theory. There we considered the equation (11) under the particular case of (12), where the assumption $B_k \in \mathbf{R}$ was replaced by $B_k > 0,\ \forall\ k \in \mathbf{N}$. In this way, we succeeded to find sufficient conditions for the absence of both eventually positive and monotonically non-decreasing solutions of (11) and eventually negative and monotonically non-increasing solutions of (11).

Further, we formulated the following generalization of Lemma 2.1, which allows us to have a first derivative of the unknown function, as well as to omit the property monotonicity of the solution. There $\rho \geq 0$ is a constant.

Theorem 2.1. *([3]) Let p_* and q_* be constants such that*

$$p_*,\ q_* \in \mathbf{R}, \quad 4q_* > p_*^2 \quad \textit{and let} \quad L_* = \frac{\sqrt{4q_* - p_*^2}}{2}. \tag{13}$$

If there is a number $s \geq \rho$ such that

$$\int_{s}^{s+\pi/L_*} F(t)e^{\frac{p_*}{2}t} \sin L_*(t-s)dt \leq 0, \tag{14}$$

then the inequality

$$z''(t) + p_* z'(t) + q_* z(t) \leq F(t) \tag{15}$$

has no positive solution in $(s, s+\pi/L_*]$.

Remark 2.1. The proof of Theorem 2.1 is based on Lemma 2.1 since the function $z_1(t)$ is a solution of (14) if and only if the function $z_2(t) = z_1(t)e^{-\frac{p_*}{2}t}$ is a solution of

$$z''(t) + L_*^2 z(t) \leq F(t)e^{\frac{p_*}{2}t},$$

which is a particular case of (9).

We mention that in [2] we investigated a non-homogeneous generalization of the equation of a pipiline and there we applied the inequality

$$Az''(t) + Bz(t) + kz'(t-\tau_1) + gz'(t-\tau_2) + \beta z(t-\tau_3) \leq F(t),$$
$$\text{where} \quad A > 0, \quad B > 0, \quad k \geq 0, \quad g \geq 0 \quad \text{and} \quad \beta > 0$$

are concrete constants in [2].

Now we are ready to come back to the present equation (1) with variable coefficients. As it will become clear bellow, all the conclusions for (1) are based on this one for the respective inequality:

$$z''(t) + \sum_{i=0}^{n} \theta_i(t) z'(t-\tau_i) + \sum_{k=0}^{\tilde{n}} \beta_k(t) z(t-\sigma_k) + G(z''(t), z'(t), z(t)) \leq F(t). \tag{16}$$

Let us write the simultaneous assumptions for (1) and (16). We repeat that $\tau_i \geq 0$, $\forall\ i = \overline{0,n}$ and $\sigma_k \geq 0$, $\forall\ k = \overline{0,\tilde{n}}$ are constants again, but here we need the following new non-negative constants T, T_1, T_2 and $\tilde{T}$, which are large enough. More exactly, we suppose that $\{\theta_i(t)_{i=0}^{n}\}$, $\{\beta_k(t)_{k=0}^{\tilde{n}}\}$ and $F(t)$ are of class $C([T,\infty);\mathbf{R})$ as well as that $G(z''(t), z'(t), z(t)) \in C([T,\infty)^3;\mathbf{R})$. Further, taking into account (3), (4) and (5), this time we assume that

$$\theta_i(t) \geq 0, \quad \forall\ t \geq T_1 \geq T \geq 0, \quad \forall\ i = \overline{0,n}, \tag{17}$$

$$\beta_k(t) \geq 0, \quad \forall\ t \geq T_2 \geq T \geq 0, \quad \forall\ k = \overline{0,\tilde{n}}, \tag{18}$$

$$\tau = \max_{0 \leq i \leq n} \tau_i, \quad \sigma = \max_{0 \leq k \leq \tilde{n}} \sigma_k \quad \text{and} \quad \tilde{\tau} = \max\{\tau, \sigma\}. \tag{19}$$

Further, let $\tilde{T} = \max\{T, T_1, T_2\}$

and let $g > 0$ be a positive constant such that

$$G(x,y,z) \geq g^2 z, \quad (x,y,z) \in \mathbf{R} \times \mathbf{R} \times [0,\infty), \qquad (20)$$
$$G(x,y,z) \leq g^2 z, \quad (x,y,z) \in \mathbf{R} \times \mathbf{R} \times (-\infty,0].$$

In fact, the notations (17) — (20) are concerned with the present continuation of the first type of oscillation results of [4]. The following lemma plays an essential role in this direction.

Lemma 2.2. *Let the assupmtions (17) — (20) be fulfilled. If there is a number $s \geq \tilde{T}$ such that*

$$\int_s^{s+\pi/g} F(t) \sin g(t-s) dt \leq 0, \qquad (21)$$

then the inequality (16) has no positive and monotonically non-decreasing solution in $(s-\tilde{\tau}, s+\pi/g]$.

Theorem 2.2. *Let (17) — (20) be fulfilled and let the function*

$$\Phi_g(s) = \int_s^{s+\pi/g} F(t) \sin g(t-s) dt \qquad (22)$$

be oscillating. Then the equation (1) has neither eventually positive and monotonically non-decreasing solution nor eventually negative and monotonically non-increasing solution.

Proof. Since $\Phi_g(s)$ is oscillating, then

$$\exists \ \{s_m\}_{m=1}^{\infty} : \quad \lim_{m\to\infty} s_m = \infty \quad \text{and} \quad \Phi_g(s_m) = 0,$$

which guarantees that

$$\Phi_g(s_m) \leq 0 \quad \text{and} \quad \Phi_g(s_m) \geq 0.$$

Hence, we apply Lemma 2.2 to establish that the equation (1) has no eventually positive and monotonically non-decreasing solution. Similarly, we could obtain that the equation has no eventually negative and monotonically non-increasing solution.

Finally, we concentrate our attention on the equation:

$$z''(t) + \theta z'(t) + \sum_{k=0}^{\tilde{n}} \beta_k(t) z(t-\sigma_k) + G(z''(t), z'(t), z(t)) = F(t), \qquad (23)$$

which is the particular case of (1), where the constant $\theta \in \mathbf{R}$ is such that (13) is satisfied in the situation:

$$p_* = \theta \quad \text{and} \quad q_* = g^2. \qquad (24)$$

In other words, we suppose that

$$4g^2 > \theta^2 \quad \text{and then} \quad L_* = \tilde{L}_* = \frac{\sqrt{4g^2 - \theta^2}}{2}. \tag{25}$$

Now we replace the pair of assumptions (19) and (20) with the following one:

$$\sigma = \max_{0 \le k \le \tilde{n}} \sigma_k \quad \text{and} \quad \tilde{T} = T_2. \tag{26}$$

Theorem 2.3. *Let (5), (24), (25) and (26) be satisfied and let the function*

$$\tilde{\Phi}_*(s) = \int_s^{s+\pi/\tilde{L}_*} F(t) e^{\frac{\theta t}{2}} \sin \tilde{L}_*(t-s) dt$$

be oscillating. Then every solution of the equation (23) oscillates.

Proof. This time we have that

$$\exists \; \{\tilde{s}_m\}_{m=1}^{\infty} : \quad \lim_{m\to\infty} \tilde{s_m} = \infty \quad \text{and} \quad \tilde{\Phi}_*(\tilde{s}_m) = 0.$$

So that, here we apply a particular case of Theorem 2.1. More exactly, here we obtain that the equation (23) has at least one zero in $(\tilde{s}_m - \sigma, \tilde{s}_m + \pi/\tilde{L}_*]$ for all $m \in \mathbf{N}$.

We mention that it is essential that $\theta \in \mathbf{R}$ is a constant in (23). In fact, if $\theta = \theta(t)$ in (23), then the respective distributions of zeros is still an open problem. The arguments are similar to the ones in Remark 2.1.

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