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A NOTE ON DISTRIBUTION SPACES ON MANIFOLDS[1]

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Abstract. Twenty-six different concrete representations of the space of vector valued distributions on a smooth manifold of dimension n are presented systematically, most of them new. In the particular case of representations as module homomorphisms acting on sections of the dual bundle resp. on n-forms, the continuity of these homomorphisms is already a consequence of their algebraic properties.

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1. Introduction

For a smooth paracompact orientable Hausdorff manifold M of dimension n, the space $\mathcal{D}'(M)$ of distributions on M is defined as the topological dual of the (LF)-space $\Omega^n_c(M)$ of compactly supported n-forms on M (for detailed definitions of these and the following spaces, see Section 3 of [4]). On many occasions, however, distributions also taking "values" in a vector bundle $E \xrightarrow{\pi} M$ (for short: E) are required—being aware, of course, that the notion of the value attained at a point only makes sense for regular distributions given by, say, continuous functions on M. By [4], 3.1.4 (cf. also [2], Ch. 1, Def. 7.5; [6], Ch. VI §1, p. 303; [7], Section 1; [8], (1.4.7)) the space of E-valued distributions on M is defined as

$$\mathcal{D}'(M,E) := (\Gamma_c(M, E^* \otimes \textstyle\bigwedge^n \mathrm{T}^*M))' \tag{1}$$

where E^* denotes the dual bundle of E and $\Gamma_c(M, E^* \otimes \bigwedge^n \mathrm{T}^*M)$ the space of compactly supported smooth sections of $E^* \otimes \bigwedge^n \mathrm{T}^*M$ equipped with the usual inductive limit topology. Equation (1) then defines $\mathcal{D}'(M,E)$ as the topological dual of this latter space of sections. A brief motivation for this definition will be given after having presented formula (3). Note that we have replaced $\mathrm{Vol}(M)$ occurring in [4] by $\bigwedge^n \mathrm{T}^*M$ since we are assuming M to be orientable.

In order to make working with $\mathcal{D}'(M,E)$ more comfortable various equivalent representations are used in practice. In a first step, one can rewrite

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the section space $\Gamma_c(M, E^* \otimes \bigwedge^n \mathrm{T}^*M)$ as $\Gamma(M, E^*) \otimes_{\mathcal{C}^\infty(M)} \Gamma_c(\bigwedge^n \mathrm{T}^*M) = \Gamma(M, E^*) \otimes_{\mathcal{C}^\infty(M)} \Omega^n_c(M)$, yielding

$$\mathcal{D}'(M, E) \cong (\Gamma(M, E^*) \otimes_{\mathcal{C}^\infty(M)} \Omega^n_c(M))'. \tag{2}$$

By (2), we can confine ourselves to plugging simple tensors $v \otimes \omega$ (with $v \in \Gamma(M, E^*)$ and $\omega \in \Omega^n_c(M)$) as arguments into $u \in \mathcal{D}'(M, E)$. Moreover, one can show (cf. [4], 3.1.12) that $(\Gamma(M, E^*) \otimes_{\mathcal{C}^\infty(M)} \Omega^n_c(M))'$, as a $\mathcal{C}^\infty(M)$-module, is isomorphic to the space $\mathrm{L}_{\mathcal{C}^\infty(M)}(\Gamma(M, E^*), \mathcal{D}'(M))$ of $\mathcal{C}^\infty(M)$-linear maps from $\Gamma(M, E^*)$ into $\mathcal{D}'(M)$, the correspondence between elements $u \in (\Gamma(M, E^*) \otimes_{\mathcal{C}^\infty(M)} \Omega^n_c(M))'$ and $\tilde{u} \in \mathrm{L}_{\mathcal{C}^\infty(M)}(\Gamma(M, E^*), \mathcal{D}'(M))$ given by

$$\langle u, v \otimes \omega \rangle = \langle \tilde{u}(v), \omega \rangle$$

(we always will use angular brackets to denote the action of any kind of distribution on its respective arguments). This results in

$$\mathcal{D}'(M, E) \cong \mathrm{L}_{\mathcal{C}^\infty(M)}(\Gamma(M, E^*), \mathcal{D}'(M)) \tag{3}$$

which is also often used in practice.

In order to briefly motivate (2) (and, thereby, Definition (1)) we start with recalling the definition of (smooth) regular distributions on M. Define $\rho : \mathcal{C}^\infty(M) \to \mathcal{D}'(M)$ by $\langle \rho(f), \omega \rangle := \int_M f\omega \in \mathbb{R}$ (for $f \in \mathcal{C}^\infty(M)$, $\omega \in \Omega^n_c(M)$). Considering, for the sake of simplicity, the vector bundle $E := \mathrm{T}M$ and the space $\mathfrak{X}(M)$ of its smooth sections (i.e., of vector fields on M), we aim at obtaining $\rho : \Gamma(M, E) = \mathfrak{X}(M) \to \mathcal{D}'(M, \mathrm{T}M)$ by something like

$$\langle \rho(X), \boxed{?} \rangle := \int_M X \boxed{?} \quad \in \mathbb{R} \qquad (X \in \mathfrak{X}(M)).$$

Now, for $p \in M$ given, $X(p)$ is a tangent vector at p. In order to render a scalar value for $\int_M X \boxed{?}$, the sought-after object $\boxed{?}$ must be capable of assigning a scalar (depending on p, of course, to each $X(p)$ and then allow integration over M. Thus the most natural choice for $\boxed{?}$ consists in picking a pair (η, ω) consisting of a 1-form $\eta \in \Omega^1(M) = \Gamma(M, E^*)$ and some $\omega \in \Omega^n_c(M)$ and setting $\langle \rho(X), (\eta, \omega) \rangle := \int_M (X \cdot \eta)\, \omega$ $(X \in \mathfrak{X}(M))$. Noting that $(X \cdot \eta)\, \omega$ is $\mathcal{C}^\infty(M)$-bilinear in η and ω, it is obvious that $\Omega^1(M) \otimes_{\mathcal{C}^\infty(M)} \Omega^n_c(M)$ (rather than $\Omega^1(M) \times \Omega^n_c(M)$) is the appropriate choice for the predual of $\mathcal{D}'(M, \mathrm{T}M)$. Similarly, for a general vector bundle E over M, we arrive at $\Gamma(M, E^*) \otimes_{\mathcal{C}^\infty(M)} \Omega^n_c(M)$, as specified at the right-hand side of (2). In the trivial case $E = M \times \mathbb{R}$, this choice also reproduces $\mathcal{D}'(M)$ as its dual.

The purpose of the present paper is to trace back the algebraic roots of the relation between (1), (2) and (3) and to fully exploit these structures in order to obtain as many similar equivalent representations of $\mathcal{D}'(M, E)$ as possible. In fact, one also meets serious topological questions in the course of these investigations: Observe that $u \in (\Gamma(M, E^*) \otimes_{\mathcal{C}^\infty(M)} \Omega^n_c(M))'$ is continuous in the whole space of sections generated by those of the form $v \otimes \omega$, whereas for the corresponding $\tilde{u} \in \mathrm{L}_{\mathcal{C}^\infty(M)}(\Gamma(M,E^*),\mathcal{D}'(M))$, there is no continuity requirement with respect to $v \in \Gamma(M,E^*)$ whatsoever. Finally, let us mention that the main impetus towards the present studies was the necessity of deciding whether ${\mathcal{D}'}^r_s(M) := \mathcal{D}'(M,\mathrm{T}^r_s M)$, the space of distributions on M taking values in the bundle of (r,s)-tensors, could also be represented as $\mathrm{L}_{\mathcal{C}^\infty(M)}(\Omega^n_c(M),\mathcal{T}^s_r(M)')$, for $\mathcal{T}^s_r(M)$ denoting the space of smooth sections of the bundle of (s,r)-tensors over M. The answer to the latter question served to solve a problem arising in the intrinsic construction of generalized tensor fields on smooth manifolds (containing the distributions and allowing for tensor multiplication and Lie derivatives), see the forthcoming paper [5].

Section 2 will lay the algebraic foundation for the 14 equivalent representations of the distributions space $\mathcal{D}'(M,E)$ to be presented in Section 3. In the latter, the question of linear topologies on the respective section spaces will be dealt with, allowing to pass from spaces of linear maps resp. linear functionals to spaces of *continous* linear maps resp. *continuous* linear functionals, as required by the very definition of distribution spaces. Section 4, finally, presents 12 more representations which arise from automatic continuity. This paper being mainly a report on the results obtained, we refer to the forthcoming article [3] for details and complete proofs.

Since throughout the paper, the base manifold of bundles will always be denoted by M, we will, from now on, write section spaces as $\Gamma(E)$, etc. rather than $\Gamma(M,E)$, etc.

2. Module theoretical methods

The algebraic key to obtaining equivalent representations of $\mathcal{D}'(M,E)$ is the relation

$$(V \otimes_A W)^* \cong \mathrm{B}_A(V,W) \cong \mathrm{L}^A(V,W^*) \cong \mathrm{L}_A(W,V^*). \tag{4}$$

Here, A is an algebra over some field $\mathbb{K}$ and V, W are linear spaces over $\mathbb{K}$ such that V is a right and W a left A-module. The module tensor product $V \otimes_A W$ is then defined as the linear space $V \otimes W/K$ with $K = \operatorname{span}\{(va)\otimes w - v\otimes(aw) \mid a\in A,\ v\in V,\ w\in W\}$. Upper stars next to linear spaces, always denoting vector space duals, V^* and W^* become left resp. right [sic!] A-modules by the adjoint action of elements $a \in A$. L^A stands for the respective space of (linear) homorphisms of right A-modules, similarly L_A for "left". $\mathrm{B}_A(V,W)$, finally, denotes the space of A-balanced (i.e., satisfying $S(va,w) = S(v,aw)$ for all $a\in A$, $v\in V$, $w\in W$) bilinear maps from $V\times W$ into $\mathbb{K}$.

The proof of (4), based on standard techniques, is straightforward; compare [1], Chapter II, §4.1 for a similar result (based on abelian groups rather than on linear spaces). In what follows, we will only consider $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Setting $A := \mathcal{C}^\infty(M)$, $W := \Gamma(E^*)$ and $V := \Omega^n_c(M)$, (4) yields

$$\begin{aligned}(\Gamma(E^*)\otimes_{\mathcal{C}^\infty(M)}\Omega^n_c(M))^* &\cong \mathrm{B}_{\mathcal{C}^\infty(M)}(\Gamma(E^*),\Omega^n_c(M))\\ &\cong \mathrm{L}_{\mathcal{C}^\infty(M)}(\Gamma(E^*),\Omega^n_c(M)^*)\\ &\cong \mathrm{L}_{\mathcal{C}^\infty(M)}(\Omega^n_c(M),\Gamma(E^*)^*).\end{aligned} \tag{5}$$

$A = \mathcal{C}^\infty(M)$ being commutative in the case at hand, there is no need to distinguish between left and right modules; therefore we only write $\mathrm{L}_{\mathcal{C}^\infty(M)}$.

It is obvious that (5) falls short of giving relevant information on distribution spaces, due to only the full algebraic duals occurring. Linear topologies on the section spaces resp. on A, V, W in the general case definitely have to enter the scene.

Before tackling this question, however, let us introduce another basic algebraic tool. From now on we generally assume, in addition to the conditions on A, V, W stated above, A to be a commutative algebra having an ideal B with the property

$$\text{for any given } b_1,\ b_2 \in B \text{ there exists } c \in B \text{ with } cb_1 = b_1 \text{ and } cb_2 = b.$$

Denoting the A-submodules BV and BW of V resp. W by V_0 resp. W_0 one can show that the obvious maps provide canonical A-module homomorphisms

$$V_0 \otimes_A W \cong V_0 \otimes_A W_0 \cong V \otimes_A W_0. \tag{6}$$

Detailed proofs are given in [3]. Combining relations (4) (applied to the pairs V_0, W, V_0, W_0, V, W_0, respectively) and (6), we obtain twelve isomorphic representations of $(V \otimes_A W_0)^*$. The latter turns into $(\Gamma(E^*) \otimes_{\mathcal{C}^\infty(M)} \Omega^n_c(M))^*$ by setting $A := \mathcal{C}^\infty(M)$, $B := \mathcal{D}(M)$, $V := \Gamma(E^*)$ and $W := \Omega^n(M)$. A corresponding isomorphism result is, of course, true for general section spaces $V = \Gamma(E)$ and $W = \Gamma(F)$, where E and F are arbitrary vector bundles over M. We may add $\Gamma_c(E \otimes F)^*$ (being isomorphic to $(\Gamma(E) \otimes_{\mathcal{C}^\infty(M)} \Gamma_c(F))^*$) as the 13th member to this family.

Finally, noting that $\Omega^n(M)$ possesses a non-vanishing section θ (M being assumed to be orientable) we may consider the special case where W is "faithfully A-generated" in the sense that it is linearly isomorphic to A via (the inverse of) the map $a \mapsto a\theta$ for some suitable $\theta \in W$. In [3] it is shown that $V_0 \otimes_A W$ is isomorphic to V_0 in this case, allowing to add $\Gamma_c(E^*)^*$ (resp. $\Gamma_c(E)^*$ in the more general setting of vector bundles E, F) as 14th isomorphic representation to the above list. Due to the purely algebraic character of the results obtained so far, we refrain from giving a complete scheme of these spaces, deferring this to the point where linear topologies have been introduced and successfully dealt with.

3. Linear topologies

We endow section spaces $\Gamma(E)$ and $\Gamma_c(E)$ with their usual linear Fréchet resp. (LF)-topologies (cf., e.g., Section 2 of [5]). $\Gamma(E) \otimes_{\mathcal{C}^\infty(M)} \Gamma(F)$ inherits the Fréchet topology of $\Gamma(E \otimes F)$ via linear isomorphism and similarly for $\Gamma(E) \otimes_{\mathcal{C}^\infty(M)} \Gamma_c(F)$ and $\Gamma_c(E \otimes F)$ with respect to the (LF)-topology. It is easy to see that the natural map $\tau : \Gamma(E) \times \Gamma(F) \to \Gamma(E \otimes F)$ sending (u, v) to $u \otimes v = [p \mapsto u(p) \otimes v(p)]$ is continuous for this choice of topologies. The corresponding map for compactly supported sections is separately continuous, at least. Topological duals X' always are endowed with their weak topologies $\sigma(X', X)$, and similarly for algebraic duals X^* and $\sigma(X^*, X)$.

Now a collection [TOP] of fairly general conditions on linear topologies given on V, V_0, W, W_0 and on $V_0 \otimes_A W$, $V_0 \otimes_A W_0$, $V_0 \otimes_A W_0$ can be formulated (with every $a \in A$ acting continuously on each of the first four of them, in particular) allowing, essentially, to replace the notions "linear" resp. "bilinear" in the isomorphism relations of Section 2 by their counterparts "linear contionuous" resp. "bilinear separately continuous". The key to applying this result to section spaces is provided by the fact that the natural topologies on spaces of sections satisfy conditions [TOP]. For a detailed account of [TOP], we refer to [3] once more. However, since we are focusing on the special case of section spaces in this note there is no need of stating [TOP] explicitly. Rather, we immeditately proceed to the main theorem given below for the setting of section spaces: Assuming F to be a line bundle possessing a non-vanishing section (hence, $\Gamma(F)$ to be faithfully $\mathcal{C}^\infty(M)$-generated) we may pass from the algebraic duals in the isomorphism results of Section 2 to the respective topological duals, at the same time replacing spaces $\mathrm{L}_{\mathcal{C}^\infty(M)}$ and $\mathrm{B}_{\mathcal{C}^\infty(M)}$ by the respective spaces $\mathfrak{L}_{\mathcal{C}^\infty(M)}$ of *continuous* linear $\mathcal{C}^\infty(M)$-module homomorphisms resp. $\mathfrak{B}^s_{\mathcal{C}^\infty(M)}$ of *separately continuous* bilinear $\mathcal{C}^\infty(M)$-balanced maps. Below, the subscript $\mathcal{C}^\infty(M)$ is abbreviated by the subscript $\mathcal{C}$.

Theorem 3.1. *For vector bundles E and F over M such that F is a line bundle possessing a non-vanishing section, the topological dual of the sections space $\Gamma_c(E\otimes F)$ with respect to the (LF)-topology has the following 14 isomorphic representations:*

$$
\begin{array}{ccccc}
\Gamma_c(E)' & = & \Gamma_c(E)' & = & \Gamma_c(E)' \\
\wr\| & & \wr\| & & \wr\| \\
\Gamma_c(E\otimes F)' & = & \Gamma_c(E\otimes F)' & = & \Gamma_c(E\otimes F)' \\
\wr\| & & \wr\| & & \wr\| \\
(\Gamma_c(E)\otimes_{\mathcal{C}}\Gamma(F))' & \cong & (\Gamma_c(E)\otimes_{\mathcal{C}}\Gamma_c(F))' & \cong & (\Gamma(E)\otimes_{\mathcal{C}}\Gamma_c(F))' \\
\wr\| & & \wr\| & & \wr\| \\
\mathfrak{B}^s_{\mathcal{C}}(\Gamma_c(E),\Gamma(F)) & \cong & \mathfrak{B}^s_{\mathcal{C}}(\Gamma_c(E),\Gamma_c(F)) & \cong & \mathfrak{B}^s_{\mathcal{C}}(\Gamma(E),\Gamma_c(F)) \\
\wr\| & & \wr\| & & \wr\| \\
\mathfrak{L}_{\mathcal{C}}(\Gamma_c(E),\Gamma(F)') & \cong & \mathfrak{L}_{\mathcal{C}}(\Gamma_c(E),\Gamma_c(F)') & \cong & \mathfrak{L}_{\mathcal{C}}(\Gamma(E),\Gamma_c(F)') \\
\wr\| & & \wr\| & & \wr\| \\
\mathfrak{L}_{\mathcal{C}}(\Gamma(F),\Gamma_c(E)') & \cong & \mathfrak{L}_{\mathcal{C}}(\Gamma_c(F),\Gamma_c(E)') & \cong & \mathfrak{L}_{\mathcal{C}}(\Gamma_c(F),\Gamma(E)')
\end{array}
\tag{7}
$$

The complete proof is to be found in [3].

Replacing E by its dual bundle E^* and setting $F := \bigwedge^n \mathrm{T}^*M$ in the preceding theorem immediately produces 14 isomorphic representations for the distribution space $\mathcal{D}'(M, E)$. As a first example, we pick the isomorphism between $\Gamma_c(E\otimes F)'$ and $\mathfrak{L}_{\mathcal{C}^\infty(M)}(\Gamma(E), \Gamma_c(F)')$ which, for E replaced by E^* and

$F := \bigwedge^n \mathrm{T}^* M$, specializes to

$$\mathcal{D}'(M,E) \cong \mathfrak{L}_{\mathcal{C}^\infty(M)}(\Gamma(E^*), \Omega^n_c(M)')$$

which could be viewed as a "continuous" version of the well-known relation (3), due to the occurrence of $\mathfrak{L}_{\mathcal{C}^\infty(M)}$ rather than of $\mathrm{L}_{\mathcal{C}^\infty(M)}$. As a second example, from $\Gamma_c(E \otimes F)' \cong \mathfrak{L}_{\mathcal{C}^\infty(M)}(\Gamma_c(F), \Gamma(E)')$ we obtain

$$\mathcal{D}'(M,E) \cong \mathfrak{L}_{\mathcal{C}^\infty(M)}(\Omega^n_c(M), \Gamma(E^*)')$$

which, similarly, can be considered as a "continuous" version of the relation (stated in Theorem 4.1) $\mathcal{D}'(M,E) \cong \mathrm{L}_{\mathcal{C}^\infty(M)}(\Omega^n_c(M), \Gamma(E^*)')$. In fact, it was the special case $E := \mathrm{T}^r_s M$ of the latter which initiated the present study.

4. Automatic continuity

Within the two bottom lines of diagram (7), there are 12 (implicit) occurrences of the notion of continuity, altogether: Six in the form of the symbol $\mathfrak{L}_{\mathcal{C}^\infty(M)}$ denoting *continuous* module homomorphism and six in the form of *topological* duals occurring as the range spaces of the respective $\mathfrak{L}_{\mathcal{C}^\infty(M)}$-spaces. It is a remarkable fact that under the assumptions of Theorem 3.1, at all twelve places of occurrence, the continuity requirement can be omitted (one at a time, to be precise) without changing the respective space. In other words, for each of the twelve instances, an automatic continuity result holds.

Observe that it is certainly not legitimate to drop two continuity requirements simultaneously, say, by passing from $\mathfrak{L}_{\mathcal{C}^\infty(M)}(\Gamma(E^*)), \Gamma_c(F)')$ to $\mathrm{L}_{\mathcal{C}^\infty(M)}(\Gamma(E^*)), \Gamma_c(F)^*)$: Replacing E by E^* and setting $F := \bigwedge^n \mathrm{T}^* M$, the first of these two spaces, by Theorem 3.1, represents the distribution space $\mathcal{D}'(M,E)$, i.e., the topological dual of $\Gamma(E^*) \otimes_{\mathcal{C}^\infty(M)} \Omega^n_c(M)$, whereas the second turns into the *algebraic* dual of $\Gamma(E^*) \otimes_{\mathcal{C}^\infty(M)} \Omega^n_c(M)$, due to (4).

To put proofs of automatic continuity in a nutshell, consider some left R-module homomorphism T of a topological ring R possessing a right unit e into some left topological R-module X. Then, for $r \in R$, we have $T(r) = T(re) = rT(e)$, showing T to be continuous, due to the continuity (with respect to the first slot) of the action of R on X. Needless to say that, in order to handle the situation at hand, much more refined methods have to be employed.

It is not feasible in this note to present the proofs of the automatic continuity results obtained in [3]. As to the six cases of changing $\mathfrak{L}_{\mathcal{C}^\infty(M)}$ to $\mathrm{L}_{\mathcal{C}^\infty(M)}$, suffice it to say that for all three nodes in the last line of (7) and for the rightmost node in the last but one line, direct proofs are feasible, exploiting in some way or other the fact that F possesses a non-vanishing section. To cover the remaining positions of the two bottom lines of (7), a transfer principle is employed which holds under some mild assumptions even in the case of general A-modules V and W: By this principle, automatic continuity logically propagates from the right to the left in the last but one line, and from the left to the right in the last line.

By establishing automatic continuity in the six cases $\mathfrak{L}_{\mathcal{C}^\infty(M)}$ vs. $\mathrm{L}_{\mathcal{C}^\infty(M)}$, we obtain six more repressentations for $\mathcal{D}'(M,E)$. Yet there is more to it: Relation (4) strongly indicates that, in fact, V and W in $L_A(V,W^*)$ enjoy a symmetric position also with respect to automatic continuity. Indeed, a corresponding statement can be shown to hold for all six nodes of the two bottom lines of (7), yielding once more a set of six isomorphic representations for the space of E-valued distributions. In the following concluding Theorem 4.1, we collect the 14 respresentations obtained by Theorem 3.1 and the 12 more obtained from automatic continuity as outlined above. Recall that $\mathcal{C}$ stands for $\mathcal{C}^\infty(M)$.

Theorem 4.1. *For vector bundles E and F over M such that F is a line bundle possessing a non-vanishing section, the topological dual of the sections space $\Gamma_c(E\otimes F)$ with respect to the (LF)-topology has the following 26 isomorphic representations:*

$$
(8)\qquad
\begin{array}{ccccc}
\Gamma_c(E)' & = & \Gamma_c(E)' & = & \Gamma_c(E)' \\
\wr\| & & \wr\| & & \wr\| \\
\Gamma_c(E\otimes F)' & = & \Gamma_c(E\otimes F)' & = & \Gamma_c(E\otimes F)' \\
\wr\| & & \wr\| & & \wr\| \\
(\Gamma_c(E)\otimes_{\mathcal{C}}\Gamma(F))' & \cong & (\Gamma_c(E)\otimes_{\mathcal{C}}\Gamma_c(F))' & \cong & (\Gamma(E)\otimes_{\mathcal{C}}\Gamma_c(F))' \\
\wr\| & & \wr\| & & \wr\| \\
\mathfrak{B}^s_{\mathcal{C}}(\Gamma_c(E),\Gamma(F)) & \cong & \mathfrak{B}^s_{\mathcal{C}}(\Gamma_c(E),\Gamma_c(F)) & \cong & \mathfrak{B}^s_{\mathcal{C}}(\Gamma(E),\Gamma_c(F)) \\
\wr\| & & \wr\| & & \wr\| \\
\mathfrak{L}_{\mathcal{C}}(\Gamma_c(E),\Gamma(F)') & \cong & \mathfrak{L}_{\mathcal{C}}(\Gamma_c(E),\Gamma_c(F)') & \cong & \mathfrak{L}_{\mathcal{C}}(\Gamma(E),\Gamma_c(F)') \\
\wr\| & & \wr\| & & \wr\| \\
\mathfrak{L}_{\mathcal{C}}(\Gamma(F),\Gamma_c(E)') & \cong & \mathfrak{L}_{\mathcal{C}}(\Gamma_c(F),\Gamma_c(E)') & \cong & \mathfrak{L}_{\mathcal{C}}(\Gamma_c(F),\Gamma(E)') \\
\wr\| & & \wr\| & & \wr\| \\
\mathrm{L}_{\mathcal{C}}(\Gamma_c(E),\Gamma(F)') & \cong & \mathrm{L}_{\mathcal{C}}(\Gamma_c(E),\Gamma_c(F)') & \cong & \mathrm{L}_{\mathcal{C}}(\Gamma(E),\Gamma_c(F)') \\
\wr\| & & \wr\| & & \wr\| \\
\mathrm{L}_{\mathcal{C}}(\Gamma(F),\Gamma_c(E)') & \cong & \mathrm{L}_{\mathcal{C}}(\Gamma_c(F),\Gamma_c(E)') & \cong & \mathrm{L}_{\mathcal{C}}(\Gamma_c(F),\Gamma(E)') \\
\wr\| & & \wr\| & & \wr\| \\
\mathfrak{L}_{\mathcal{C}}(\Gamma_c(E),\Gamma(F)^*) & \cong & \mathfrak{L}_{\mathcal{C}}(\Gamma_c(E),\Gamma_c(F)^*) & \cong & \mathfrak{L}_{\mathcal{C}}(\Gamma(E),\Gamma_c(F)^*) \\
\wr\| & & \wr\| & & \wr\| \\
\mathfrak{L}_{\mathcal{C}}(\Gamma(F),\Gamma_c(E)^*) & \cong & \mathfrak{L}_{\mathcal{C}}(\Gamma_c(F),\Gamma_c(E)^*) & \cong & \mathfrak{L}_{\mathcal{C}}(\Gamma_c(F),\Gamma(E)^*)
\end{array}
$$

Replacing E by E^* and setting $F := \bigwedge^n \mathrm{T}^*M$ once more we obtain, from row 7, column 3 of diagram (8), a new independent proof of

$$\mathcal{D}'(M,E) \cong \mathrm{L}_{\mathcal{C}^\infty(M)}(\Gamma(E^*),\mathcal{D}'(M))$$

which is (3) presented in Section 1 (cf. 3.1.12 of [4]). On the other hand, from row 8, column 3 of (8) we obtain, as a (general) affirmative answer to the question lying at the origin of this work, the relation

$$\mathcal{D}'(M,E) \cong \mathrm{L}_{\mathcal{C}^\infty(M)}(\Omega^n_c(M), \Gamma(E^*)').$$

By Theorem 4.1, in either of the preceding relations the respective spaces of linear module homomorphisms can be replaced by the corresponding space of *continuous* linear module homomorphisms.

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