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ON A SCALAR CONSERVATION LAW WITH NONLINEAR DIFFUSION AND LINEAR DISPERSION IN HETEROGENEOUS MEDIA

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Abstract. We obtain the strong precompactness of a family of solutions to a suitable regularization of multidimensional scalar conservation law via vanishing nonlinear diffusion and linear dispersion. We consider the flux which depends on the time and space variables, and obtain condition $\delta = O(\varepsilon^2)$, $\varepsilon \to 0$, for the existence of a weak entropy solution. In comparison to known results for heterogeneous media (cf. [4]), our condition is weaker, thus more general.

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1. Introduction

The subject of the paper is the following Cauchy problem for multidimensional scalar conservation law:

$$\partial_t u(t,x) + \mathrm{div}_x f(t,x,u) = 0, \quad u(0,x) = u_0(x), \qquad x \in \mathbb{R}^d, \ t \in \mathbb{R}^+. \tag{1}$$

We study the behavior of a family of solutions to a regularization of (1),

$$\partial_t u^{\varepsilon,\delta} + \mathrm{div}_x f(t,x,u^{\varepsilon,\delta}) = \varepsilon \sum_{j=1}^{d} \partial_{x_j} b_j(\nabla_x u^{\varepsilon,\delta}) + \delta \sum_{j=1}^{d} \partial_{x_j x_j x_j} u^{\varepsilon,\delta}, \tag{2}$$

$$u(0,x) = u_0^{\varepsilon}(x), \quad x \in \mathbb{R}^d, \ \varepsilon, \delta \in (0,1), \ \delta = \delta(\varepsilon), \tag{3}$$

where initial data u_0^ε from (3) converge to u_0 from (1) strongly in $L^2(\mathbb{R}^d)$ as $\varepsilon \to 0$ (all terms from (1), (2) and (3) are precisely described in the next section). In order to obtain a weak entropy solution to (1) as a limit of a subsequence of the solutions to (2)-(3), we study the precompactness properties of the family $(u^{\varepsilon,\delta})_{\varepsilon,\delta}$. The goal is to obtain optimal balance of the two parameters ε and δ.

Let us briefly recall already obtained results concerning diffusion-dispersion limits for (1). In [10], using compensated compactness arguments, the author proved that a family of solutions to KdV-Burgers equation converges to a weak

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solution to Burgers equation if diffusion and dispersion parameter are balanced in the sense that $\delta = \mathcal{O}(\varepsilon^2)$, as $\varepsilon \to 0$. Using the same methodology (in one dimensional case) with the same balance result, in [7] the diffusion-dispersion problem is solved in the case when the flux has a general homogeneous form, $f = f(u)$. Multidimensional case ($x \in \mathbb{R}^d$) is solved in [6], but with the stronger balance, $\delta = o(\varepsilon^2)$, as $\varepsilon \to 0$. More general result concerning the relative size of diffusion and dispersion parameters is made in [5]. Using the kinetic approach [9] and the averaging lemma [8, 9, 11], the author obtained in [5] the diffusion-dispersion limit with a weaker balance, $\delta = \mathcal{O}(\varepsilon^2)$, as $\varepsilon \to 0$. It is important to point out that in every of the previously mentioned works the flux which appears in the considered conservation law *does not depend on space and time explicitly.* Authors of [4], dealing with heterogeneous media for discontinuous flux in one-dimensional case, obtained stronger balance between two parameters, $\delta = o(\varepsilon^2)$, as $\varepsilon \to 0$. In order to obtain the results analogous to those from [5] (multidimensional case and optimal diffusion-dispersion ratio), similarly as in [5], we shall use the kinetic formulation of the conservation law under consideration. We will improve the balance result from [4], but with the stronger assumptions on the flux (see (H3) in the next chapter). Further improvements will be done in [1].

The paper is organized as follows. We give in Section 2 basic notations, assumptions and the statement of the main theorem. In Section 3, we prove a priori inequalities for the family $(u^{\varepsilon,\delta})_{\varepsilon,\delta}$. In Section 4 we prove the main theorem which is based on the Theorem 2.5 from [3].

2. Notations, assumptions and the main result

In the sequel, we put $|g|^2 = \sum_{i=1}^{d} |g_i|^2$, for a vector valued function $g = (g_1, ..., g_d)$ defined on $\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}$. The derivative D_{x_i} at the point (t, x, u), where u depends on (t, x), is defined by $D_{x_i} g(t, x, u) = (\partial_{x_i} g(t, x, \lambda))|_{\lambda = u(t,x)}$. Derivatives ∂_{x_i} and D_{x_i} are connected by the identity $\partial_{x_i} g(t, x, u) = D_{x_i} g(t, x, u)$ $+\partial_u g(t, x, u)\partial_{x_i} u$. For the simplicity, in the sequel we shall write u^ε instead of $u^{\varepsilon,\delta}$ and consider that $\varepsilon \in (0, 1)$. We assume that f, b and u_0 are enough regular, so that solutions u^ε, $\varepsilon \in (0, 1)$ of (2) have enough regularity, so that all formal computations below are correct, as well as that $u^\varepsilon, \partial_{x_i} u^\varepsilon$ and $\partial_{x_i x_j} u^\varepsilon$ vanish as $|x| \to \infty$. Now we list additional assumptions. For the initial data we assume that $u_0 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $u_0^\varepsilon, \nabla u_0^\varepsilon \in H^1(\mathbb{R}^d)$. Here $u_0^\varepsilon = u_0 \star \delta_\varepsilon$, $\varepsilon \in (0, 1)$ for a suitable $(\delta_\varepsilon)_\varepsilon$ net of smooth compactly supported functions. Notice that from the last assumption $(u_0^\varepsilon)_\varepsilon$ is uniformly bounded in $L^2(\mathbb{R}^d)$. We assume that the diffusion term $b = (b_1, ..., b_d) : \mathbb{R}^d \to \mathbb{R}^d$ fulfils the following hypotheses:

(H1) There exist positive constants C_1, C_2 such that $C_1|\lambda|^2 \le \lambda \cdot b(\lambda) \le C_2|\lambda|^2$, for all $\lambda \in \mathbb{R}^d$.

(H2) The gradient matrix $Db(\lambda)$ is a positively definite matrix uniformly in $\lambda \in \mathbb{R}^d$, i.e. there exists positive constant C_3 such that $\xi^T Db(\lambda)\xi \ge C_3|\xi|^2$, for all $\lambda, \xi \in \mathbb{R}^d$.

We assume for the flux $f = (f_1, ..., f_d) : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ that $f = f(t, x, u)$ and $f_u(t, x, u)$ are continuous and that they have locally integrable derivatives with respect to t and x. These assumptions enable us to make calculations in a priori estimates of the next section. Moreover, we assume that f fulfils the following hypotheses:

(H3) $\partial_u f \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R})$, $D_{x_i} f_i \in L^2 \cap L^1(\mathbb{R}^+ \cap \times \mathbb{R}^d \times \mathbb{R})$ and $|D_{x_i} f_j(t,x,v)| \leq |\zeta_{i,j}(t,x)||v|$, for some $\zeta_{i,j} \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$.

The final assumption is the following nonlinearity condition **(NLC)**: For almost every $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$ and every $\xi \in S^d$ the mapping $\lambda \mapsto \xi_0 + \sum_{k=1}^d \partial_\lambda f_i(t,x,\lambda)\xi_k$ is not identically equal to zero on any set of positive Lebesgue measure.

Our main result which will be proved in Section 4 is the following theorem.

Theorem 2.1. *Under afore listed assumptions, a family of smooth solutions* $(u^{\varepsilon,\delta})_{\varepsilon,\delta}$ *to Cauchy problem* (2)-(3) *is strongly precompact in* $L^1_{\rm loc}(\mathbb{R}^+ \times \mathbb{R}^d)$ *if* ε *and* δ *from* (2) *are balanced in the sense that* $\delta = \mathcal{O}(\varepsilon^2)$ *as* $\varepsilon \to 0$.

3. A priori inequalities

In this section, we give necessary a priori inequalities.

Lemma 3.1. *Under afore listed assumptions, the family of solutions* $(u^\varepsilon)_\varepsilon$ *to* (2)-(3) *for every* $t \in [0,T]$ *satisfies the following inequality*

$$\int_{\mathbb{R}^d} |u^\varepsilon(t,x)|^2 dx + \varepsilon \int_0^t \int_{\mathbb{R}^d} |\nabla u^\varepsilon(t',x)|^2 dx dt' \leq c, \text{ for } \varepsilon < \varepsilon_0, \tag{4}$$

for suitable constant $c > 0$, *and some* $\varepsilon_0 \in (0,1)$.

Proof: Let $\eta = \eta(u)$, $u \in \mathbb{R}$, be a smooth function. We multiply (2) by $\eta'(u^\varepsilon)$ and define $q = (q_1, ..., q_n)$ as $q_i(t,x,u) = \int_0^u \eta'(v)\partial_v f_i(t,x,v)dv$, $i = 1, ..., d$. Thus, (2) becomes

$$\begin{aligned}
&\partial_t \eta(u^\varepsilon) + \mathrm{div}_x\, q_i(t,x,u^\varepsilon) - \sum_{i=1}^d \int_0^{u^\varepsilon} \partial_{x_i v} f_i(t,x,v)\eta'(v)dv + \\
&\sum_{i=1}^d \eta'(u^\varepsilon) D_{x_i} f_i(t,x,u^\varepsilon) = \varepsilon \sum_{i=1}^d \partial_{x_i}\big(\eta'(u^\varepsilon) b_i(\nabla u^\varepsilon)\big) - \\
&\varepsilon \eta''(u^\varepsilon) \sum_{i=1}^d b_i(\nabla u^\varepsilon)\partial_{x_i} u^\varepsilon + \delta \sum_{i=1}^d \partial_{x_i}\big(\eta'(u^\varepsilon)\partial_{x_i x_i} u^\varepsilon\big) - \\
&\frac{\delta}{2}\eta''(u^\varepsilon) \sum_{i=1}^d \partial_{x_i}(\partial_{x_i} u^\varepsilon)^2.
\end{aligned} \tag{5}$$

We choose here $\eta(u) = \frac{u^2}{2}$ and integrate over $[0,t) \times \mathbb{R}^d$. Taking into account

(H1) and partial integration, we obtain

$$\frac{1}{2}\int_{\mathbb{R}^d}|u^\varepsilon(t,x)|^2dx+\varepsilon C_1\int_0^t\int_{\mathbb{R}^d}|\nabla u^\varepsilon(t',x)|^2dxdt'$$
$$\leq\frac{1}{2}\int_{\mathbb{R}^d}|u_0^\varepsilon(x)|^2dx-\int_0^t\int_{\mathbb{R}^d}\int_0^{u^\varepsilon(t',x)}\sum_{i=1}^d D_{x_i}f_i(t',x,v)dvdxdt'.$$

This and (H3) imply (4). ❑

Lemma 3.2. *Under afore listed assumptions, for* $|D^2u|^2=\sum\limits_{i,k=1}^d|\partial_{x_ix_k}u|^2$, *a family of solutions* $(u^\varepsilon)_\varepsilon$ *to* (2)-(3) *satisfies the following inequality*

$$\varepsilon^2\int_{\mathbb{R}^d}|\nabla u^\varepsilon(t,x)|^2dx+\varepsilon^3\int_0^t\int_{\mathbb{R}^d}|D^2u^\varepsilon(t',x)|^2dxdt'\leq c, \tag{6}$$

for suitable $c>0$, *for every* $t\in[0,T]$ *and* $\varepsilon<\varepsilon_0$.

Proof: In the sequel we will use the existence of different constants which indexes will indicate that they are new ones. As well, we will not write $\varepsilon<\varepsilon_0$ for appropriate ε_0 which can be changed. We differentiate (2) with respect to x_k and multiply the obtained expression by $\partial_{x_k}u^\varepsilon$. Then, summing expressions for $k=1,...,d$, using partial integration, as well as (H2), we obtain

$$\frac{1}{2}\int_{\mathbb{R}^d}\partial_t|\nabla u^\varepsilon|^2dx-\sum_{k=1}^d\int_{\mathbb{R}^d}\nabla\partial_{x_k}u^\varepsilon\cdot\left(D_{x_k}f(t,x,u^\varepsilon)+\partial_uf\cdot\partial_{x_k}u^\varepsilon\right)dx$$
$$=-\varepsilon\sum_{k=1}^d\int_{\mathbb{R}^d}(\nabla\partial_{x_k}u^\varepsilon)^TDb(\nabla u^\varepsilon)\nabla\partial_{x_k}u^\varepsilon dx\leq^{(\mathrm{H2})}-\varepsilon C_3\sum_{k=1}^d\int_{\mathbb{R}^d}|\nabla\partial_{x_k}u^\varepsilon|^2dx.$$

Now we integrate this over $[0,t]$, use the Cauchy-Schwartz inequality, as well as the Young inequality (C_3 below is the same as above), $ab\leq\frac{C_3\varepsilon}{2}a^2+\frac{C_6}{\varepsilon}b^2$, $a,b\in\mathbb{R}$. Then we multiply obtained inequality by ε^2, use inequality $(a+b)^2\leq 2a^2+2b^2$ and the consequence from (4) that $\varepsilon\int_0^t\int_{\mathbb{R}^d}|\nabla u(s,x)|^2dxds\leq C$, $t\in[0,T]$ in order to obtain

$$\frac{\varepsilon^2}{2}\int_{\mathbb{R}^d}|\nabla u^\varepsilon(t,x)|^2dx+C_3\frac{\varepsilon^3}{2}\int_{\mathbb{R}^d}\int_0^t|D^2u^\varepsilon|^2dxdt\leq\varepsilon^2C_9\int_{\mathbb{R}^d}|\nabla u_0^\varepsilon|^2dxdt'$$
$$+\varepsilon C_{10}\int_0^t\int_{\mathbb{R}^d}\sum_{k=1}^d|D_{x_k}f(t',x,u^\varepsilon(t',x))|^2dxdt'+C_{11}\|\partial_uf\|^2_{L^\infty(\mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R})}$$

with appropriate constants. Taking into account (H3), (6) follows from the last inequality. ❑

4. The proof of the main theorem

To prove strong precompactness of the family $(u^\varepsilon)_\varepsilon$ of solutions to (2-3), we use the averaging lemma proved in [3, Theorem 2.5]. We give its variant which is adapted to our purposes.

Theorem 4.1. *Let $(h_n)_n \subset L^2_{\rm loc}(\mathbb{R}^N \times \mathbb{R})$ be a sequence of solutions to the following transport equation*

$$\operatorname{div}_y(F(y,\lambda)h_n(y,\lambda)) = \sum_{i=1}^{d} \partial_\lambda^{k_i} G_n^i(y,\lambda), \quad y \in \mathbb{R}^N, \quad \lambda \in \mathbb{R},$$

where flux $F = (F_1, ..., F_N) \in C(\mathbb{R}^d \times \mathbb{R})$, and families $(G_n^i)_n$, $i = 1, \dots, N$, are strongly precompact in $H^{-1}(\mathbb{R}^N \times \mathbb{R})$. Furthermore, assume that the following non-degeneracy condition is fulfilled: For almost every $y \in \mathbb{R}^N$ and every $\xi \in S^{N-1}$ the mapping

$$\lambda \mapsto \textstyle\sum_{k=1}^N F_k(y,\lambda)\xi_k$$

is not identically equal to zero on any set of positive Lebesgue measure.

Then for every $\rho \in C_0^\infty(\mathbb{R})$ the sequence $\left(\int_{\mathbb{R}} h_n(x,\lambda)\rho(\lambda)d\lambda\right)_n$ is strongly precompact in $L^1_{\rm loc}(\mathbb{R}^d \times \mathbb{R})$.

Proof of the Theorem 2.1: Let $\eta \in C_0^\infty(\mathbb{R})$ and

$$h_\varepsilon(t,x,\lambda) = \begin{cases} 1, & \text{for } \ 0 < \lambda \le u^\varepsilon(t,x), \quad (t,x) \in \Pi, \\ -1, & \text{for } \ 0 > \lambda \ge u^\varepsilon(t,x), \quad (t,x) \in \Pi, \\ 0, & \quad \text{otherwise,} \end{cases}$$

where $\Pi = (0,T) \times \mathbb{R}^d$. Let $\varphi \in C_0^\infty(\Pi)$. We rewrite (5) as

$$\begin{aligned}
(7) \qquad & -\int_{\Pi\times\mathbb{R}} h_\varepsilon(t,x,\lambda)\eta'(\lambda)\varphi_t(t,x)\, d\lambda dx dt \\
& - \sum_{i=1}^d \int_{\Pi\times\mathbb{R}} h_\varepsilon(t,x,\lambda)\partial_\lambda f_i(t,x,\lambda)\eta'(\lambda)\varphi_{x_i}(t,x)\, d\lambda dx dt \\
& + \sum_{i=1}^d \int_{\Pi\times\mathbb{R}} h_\varepsilon(t,x,\lambda)\partial_{x_i\lambda} f_i(t,x,\lambda)\eta''(\lambda)\varphi(t,x) d\lambda dx dt \\
& = -\int_\Pi \sum_{i=1}^d (\varepsilon b_i(\nabla u^\varepsilon) + \delta\partial_{x_i x_i} u^\varepsilon)\, \eta'(u^\varepsilon)\varphi_{x_i}(t,x) dx dt \\
& - \sum_{i=1}^d \int_\Pi \left(\varepsilon b_i(\nabla u^\varepsilon) u^\varepsilon_{x_i} + \delta u^\varepsilon_{x_i}\partial_{x_i x_i} u^\varepsilon\right) \eta''(u^\varepsilon)\varphi(t,x) dx dt.
\end{aligned}$$

As in [5], we represent equation (7) as an equation in $\mathcal{D}'(\Pi\times\mathbb{R})$. With an abuse of notation (see notation in Section 2) we put $H_i^\varepsilon(t,x) = \varepsilon b_i(\nabla u^\varepsilon)$, $\bar{H}_i^\varepsilon(t,x) =$

$\delta\partial_{x_i x_i} u^\varepsilon$, $G_i^\varepsilon(t,x) = \varepsilon b_i(\nabla u^\varepsilon) u^\varepsilon_{x_i}$, $\bar{G}_i^\varepsilon(t,x) = \delta u^\varepsilon_{x_i} \partial_{x_i x_i} u^\varepsilon$, and note that the nets $H_i^\varepsilon, \bar{H}_i^\varepsilon, G_i^\varepsilon(t,x), \bar{G}_i^\varepsilon$ are uniformly bounded in $L^1_{\rm loc}(\Pi \times \mathbb{R})$ (cf. (H1)-(H3) and Lemmas 3.1-3.2). Let $\delta(\lambda - u)$ be a Dirac delta function defined by $\langle \delta(\lambda - u), \eta(\lambda)\rangle = \eta(u)$. Then $m_i^\varepsilon = \delta(\lambda - u^\varepsilon) G_i^\varepsilon$, $k_i^\varepsilon = \delta(\lambda - u^\varepsilon)\bar{G}_i^\varepsilon$, $\pi_i^\varepsilon = \delta(\lambda - u^\varepsilon) H_i^\varepsilon$ and $\bar{\pi}_i^\varepsilon = \delta(\lambda - u^\varepsilon)\bar{H}_i^\varepsilon$, $i = 1, ..., d$, $\varepsilon < 1$, are defined as distributions in $\mathcal{D}'(\Pi \times \mathbb{R})$ via the following tensor products:

$$
\begin{aligned}
(8) \qquad \langle m_i^\varepsilon, \varphi \otimes \eta'\rangle &= \int_\Pi G_i^\varepsilon(t,x)\varphi(t,x)\eta'(u^\varepsilon(t,x))dxdt,\\
\langle k_i^\varepsilon, \varphi \otimes \eta'\rangle &= \int_\Pi \bar{G}_i^\varepsilon(t,x)\varphi(t,x)\eta'(u^\varepsilon(t,x))dxdt,\\
\langle \pi_i^\varepsilon, \varphi \otimes \eta'\rangle &= \int_\Pi H_i^\varepsilon(t,x)\varphi(t,x)\eta'(u^\varepsilon(t,x))dxdt,\\
\langle \bar{\pi}_i^\varepsilon, \varphi \otimes \eta'\rangle &= \int_\Pi \bar{H}_i^\varepsilon(t,x)\varphi(t,x)\eta'(u^\varepsilon(t,x))dxdt.
\end{aligned}
$$

The mapping $\eta(\lambda) \mapsto \langle \delta(\lambda - u^\varepsilon) G_i^\varepsilon(t,x), \eta(\lambda)\rangle = \langle \delta(\lambda - u^\varepsilon)\varepsilon b_i(\nabla u^\varepsilon) u^\varepsilon_{x_i}, \eta(\lambda)\rangle = \varepsilon u^\varepsilon b_i(\nabla u^\varepsilon)\partial_x \eta(u^\varepsilon(t,x))$, with values in $\mathcal{D}'(\Pi)$, is continuous, so (8) holds and we can prove other identities in the same way. Thus, (7) can be rewritten as equation in $\mathcal{D}'(\Pi \times \mathbb{R})$ as follows

$$
\begin{aligned}
(9) \qquad & \partial_t h_\varepsilon(t,x,\lambda) + \sum_{i=1}^d \partial_{x_i}(h_\varepsilon(t,x,\lambda)\partial_\lambda f_i(t,x,\lambda)) = \\
& \sum_{i=1}^d \partial_\lambda(h_\varepsilon(t,x,\lambda)\partial_{x_i} f_i(t,x,\lambda)) + \sum_{i=1}^d \left(\partial_{x_i}(\pi_i^\varepsilon + \bar{\pi}_i^\varepsilon) + \partial_\lambda(m_i^\varepsilon + k_i^\varepsilon)\right).
\end{aligned}
$$

Now we estimate terms on the right-hand side of (9). By Lemmas 3.1-3.2 and (H1) (as in [5]), we obtain the following results ((10) and (11)):

$$
(10) \qquad \pi_i^\varepsilon = \bar{g}_i^\varepsilon + \partial_\lambda g_i^\varepsilon, \quad \bar{\pi}_i^\varepsilon = \bar{p}_i^\varepsilon + \partial_\lambda p_i^\varepsilon, \quad i = 1, ..., d,
$$

with $\bar{g}_i^\varepsilon, g_i^\varepsilon \to 0$ and $\bar{p}_i^\varepsilon, p_i^\varepsilon \to 0$ as $\varepsilon \to 0$ in $L^2(\Pi \times \mathbb{R})$;

$$
(11) \quad (m_i^\varepsilon)_\varepsilon, (k_i^\varepsilon)_\varepsilon \ \text{ lie in a bounded set of } \mathcal{M}(\Pi \times \mathbb{R}), \ \text{ for every } i = 1, ..., d,
$$

where $\mathcal{M}(\Pi \times \mathbb{R})$ stands for the space of bounded measures. The proof is technical(cf. [5]) and will be omitted. Consider now the remaining term on the right hand side of (9). Denote by $\Pi_i^\varepsilon(t,x,\lambda) = \partial_\lambda(h_\varepsilon(t,x,\lambda)\partial_{x_i} f_i(t,x,\lambda))$, $(t,x,\lambda) \in \Pi \times \mathbb{R}$, $i = 1, ..., d$. Let $\theta(t,x,\lambda) \in C_0^\infty((0,T) \times \mathbb{R}^d \times \mathbb{R})$ and $i = 1, ..., d$. Then,

$$
\begin{aligned}
\langle \Pi_i^\varepsilon, \theta\rangle &= \Big| \int_{\Pi \times \mathbb{R}} h_\varepsilon(t,x,\lambda)\partial_{x_i} f_i(t,x,\lambda)\theta_\lambda(t,x,\lambda) dtdxd\lambda \Big| \\
&\leq \|\theta_\lambda\|_{C^0(\Pi \times \mathbb{R})} \int_{\mathrm{supp}\theta} |\partial_\lambda f_i(t,x,\lambda)| dtdxd\lambda \leq C\|\theta_\lambda\|_{C^0(\Pi \times \mathbb{R})},
\end{aligned}
$$

where C is a constant depending only on the support of a test function θ. Thus, for every $i=1,...,d$ the family $(\Pi_i^\varepsilon)_\varepsilon$ lies in a locally bounded subset of the space of bounded measures $\mathcal{M}(\Pi\times\mathbb{R})$. Knowing that every sequence of measures bounded in $\mathcal{M}(\Pi\times\mathbb{R})$ is precompact in $H^{-1}(\Pi\times\mathbb{R})$(cf. [2] Theorem 5), we can apply Theorem 4.1 for the net $(h_\varepsilon)_\varepsilon$ and conclude that a subsequence $(h_k)_k$ of $(h_\varepsilon)_\varepsilon$ satisfies

$$\text{(12)} \qquad \Big(\int_{-R}^{R} h_k(t,x,\lambda)d\lambda\Big)_{k\in\mathbf{N}} \ \text{ is convergent in } \ L^1_{\text{loc}}(\mathbb{R}^+\times\mathbb{R}^d),$$

for every $R\in\mathbf{N}$. Furthermore,

$$\begin{aligned}\text{(13)} \qquad \Big|u^\varepsilon-\int_{-R}^{R} h_\varepsilon(t,x,\lambda)d\lambda\Big| &= \Big|\int_\lambda h_\varepsilon(t,x,\lambda)d\lambda-\int_{-R}^{R} h_\varepsilon(t,x,\lambda)d\lambda\Big|\\ &= \Big|\int_R^\infty h_\varepsilon(t,x,\lambda)d\lambda+\int_{-\infty}^{-R} h_\varepsilon(t,x,\lambda)d\lambda\Big|\\ &= H(u^\varepsilon-R)(u^\varepsilon-R)+H(-u^\varepsilon-R)(-u^\varepsilon-R).\end{aligned}$$

Thus by Lemma 3.1, we have that there exists constant $K_1>0$, so that

$$\begin{aligned}\text{(14)} \qquad &\int_0^t\int_{\mathbb{R}}[H(u^\varepsilon-R)(u^\varepsilon-R)+H(-u^\varepsilon-R)(-u^\varepsilon-R)]dxdt\\ &\leq \int_{|u^\varepsilon|>R}|u^\varepsilon|dxdt\leq\frac{1}{R}\int_0^t\int_x|u^\varepsilon|^2dxdt\leq\frac{K_1}{R},\end{aligned}$$

since $\int_{|u^\varepsilon|>R}R|u^\varepsilon|dxdt\leq\int_{|u^\varepsilon|>R}|u^\varepsilon|^2dxdt<\tilde{K_1}$. Therefore, from (13) and (14) it follows

$$\text{(15)} \qquad \int_0^t\int_{\mathbb{R}}\Big|u^\varepsilon-\int_{-R}^{R} h_\varepsilon(t,x,\lambda)d\lambda\Big|dtdx\leq\frac{K_1}{R}$$

Now by (15) it is easy to prove that $(u^k)_k$ (where the indexing is taken from (12)) is a Cauchy sequence in $L^1_{\text{loc}}(\Pi)$. Indeed, for every compact set $K\subset\subset\Pi$, we have

$$\begin{aligned}&\int_K|u^{k_1}-u^{k_2}|dxdt\\ &\leq\int_K\Big|u^{k_1}-\int_{-R}^{R} h_{k_1}(t,x,\lambda)d\lambda\Big|dxdt+\int_K\Big|u^{k_2}-\int_{-R}^{R} h_{k_2}(t,x,\lambda)d\lambda\Big|dxdt\\ &+\int_K\Big|\int_{-R}^{R} h_{k_1}(t,x,\lambda)d\lambda-\int_{-R}^{R} h_{k_2}(t,x,\lambda)d\lambda\Big|dxdt\leq\frac{2K_1}{R}+\gamma(k_1,k_2),\end{aligned}$$

where $\frac{2K_1}{R}$ appears due to (15), and γ is a function tending to zero as $k_i\to\infty$, $i=1,2$, because $(h_k)_k$ is convergent in $L^1_{\text{loc}}(\Pi\times\mathbb{R})$. Thus, we see that the subsequence $(u^k)_k$ of $(u^\varepsilon)_\varepsilon$ is the Cauchy sequence in $L^1_{\text{loc}}(\Pi)$. This implies that the family $(u^\varepsilon)_\varepsilon$ is precompact in $L^1_{\text{loc}}(\Pi)$. ❑

Remark 4.1. *Notice that if* $\delta=o(\varepsilon^2)$, $\varepsilon\to0$, *then* $(u^k)_k$ *tends to a unique entropy solution to (1). The proof is analogous to the one from [6].*

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