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On Isometry Links between 4-Vectors of Velocity

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Abstract. We give an overview on the problem of finding isometry as Lorentz transformation between given vectors of 4-velocity and its solutions. These Lorentz transformations are usually referenced as Lorentz links. We show that the links by boost, obtained independently by different authors in different forms, are identical. Finally, we discuss briefly some implications of the non-uniqueness of boost linking given pair of 4-velocities.

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1. Introduction

In Special Relativity (SR) each object, vector or tensor, is strongly related to a reference frame. A link (morphism) among such objects is an orthogonal inhomogeneous Lorentz coordinate transformation. A relative velocity between two objects is an essential concept in attempt to provide an observer independent relation between them. This relation is standardly represented by a symmetric matrix (boost) usually parameterized by a vector of velocity representing relative velocity between given objects. Unfortunately, as it was recently shown [7], the boost linking two given 4-vectors of velocity is not unique (it depends on an observer).

The problem of determining the Lorentz transformations that link two given 4-velocities, or more generally, 4-vectors with an equal norm, has been addressed by several authors using different approaches. The well-known solution based on boosts has been proposed in [2], and rewritten in [4, 5]. This isometry link, provided by straightforward calculations, is an algebraically simple solution presented as a tensor of rank 2. However, the question about the uniqueness of this solution has not been clarified. The difficulties appearing in attempts to determine most general transformation link, i.e. all boosts linking given 4-vectors, are evident from van Wyk attempts [13, 14].

In [10], the problem of finding isometry links between given 4-vectors is solved by gyrogroup formalism. However, the conclusion in [10] that the proposed boost is unique is incorrect as was pointed out in [7], where more general solution is obtained using standard covariant formalism.

An interesting approach was undertaken in [11], where the problem is solved by representing boost via two reflections. However, as we shall prove, this solution is the same with the known one proposed in [2].

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The most extensive analysis and discussion as well as the most general solution (under some conditions) to the problem of determining isometry links between given 4-vectors are given in [7]. The proposed solution is parameterized by an arbitrary 4-vector (this solution is given by the formula (4)). This solution shows that the link by boost is not unique, i.e. it depends on an additional 4-vector that could be considered as a 4-velocity of an observer.

Our aim is to show equivalence between some different link solutions proposed in the literature. We shall briefly discuss the most important properties of this solution.

2. Isometry Links Between Given 4-Velocities

Let us consider the 4-velocities in the Minkowski space with the metric $\eta = diag(-1,-1,-1,1)$. Relativistic 4-velocity V is defined as the rate of change of the event coordinates $q = (x, y, z, ct)$ with respect to proper time τ. Since $\frac{dt}{d\tau} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, the 4-velocity V is

$$\frac{dq}{d\tau}=\left(\frac{dx}{dt}\frac{dt}{d\tau},\frac{dy}{dt}\frac{dt}{d\tau},\frac{dz}{dt}\frac{dt}{d\tau},c\frac{dt}{d\tau}\right)=\left(\frac{v_x}{\sqrt{1-\frac{v^2}{c^2}}},\frac{v_y}{\sqrt{1-\frac{v^2}{c^2}}},\frac{v_z}{\sqrt{1-\frac{v^2}{c^2}}},\frac{c}{\sqrt{1-\frac{v^2}{c^2}}}\right)$$

where v_x, v_y and v_z are the components of the corresponding 3-velocity vector $\vec{v}$ and c is the speed of light. The magnitude of 4-velocity V is c, $VV = \eta_{ab}V^aV^b = c^2$, so it is independent of the magnitude of the 3-velocity $\vec{v}$. The space of relativistic 3-velocities is $\Lambda_3 = \{\vec{v} \in \mathbb{R}^3 \,|\, \|\vec{v}\| < c\}$ and the space of 4-velocities is $\Lambda_4 = \{V \in \mathbb{R}^4 \,|\, \eta_{ij}V^iV^j = c^2, \|\vec{v}\| < c\}$, where $\|\cdot\|$ is the usual Euclidean norm. The map between the open 3-ball in $(\mathbb{R}^3, \|\cdot\|)$ of 3-velocities and the 3-dimensional hyperbola in $(\mathbb{R}^4, \eta)$ of 4-velocities $f : \Lambda_3 \to \Lambda_4$ defined by $f(\vec{v}) = V$ is obviously, bijective.

The Lorentz group together with translations (Poincare group) is the group of symmetries of the metric tensor of the empty space-time, no matter.

Definition 2.1. *The Lorentz group is the linear group (subgroup of $GL(4,\mathbb{R})$) of orthogonal transformations of Minkowski space, also called $O(3,1)$ which preserves distances. So, $L \in O(3,1)$ iff $\eta(L(x), L(y)) = \eta(x, y)$ ($\eta_{ij}L^i_\alpha L^j_\beta = \eta_{\alpha\beta}$).*

We shall denote scalar product briefly by $x \cdot y \equiv \eta(x, y) \equiv \eta_{ij}x^iy^j$.

As a topological space, $O(3,1)$ decomposes into the disjoint union of four connected components $O(3,1) = O(3,1)^{\uparrow}_{+} \cup O(3,1)^{\downarrow}_{+} \cup O(3,1)^{\uparrow}_{-} \cup O(3,1)^{\downarrow}_{-}$ where $+/-$ stands for positive/negative determinant, i.e. proper/improper and $\uparrow / \downarrow$ for time orientation preserving/reversing, i.e. orthochronous/antichronous transformations respectively. Of these four components only $O(3,1)^{\uparrow}_{+}$, the component containing the group identity is a subgroup called the group of proper orthochronous Lorentz transformations.

From geometrical point of view, Lorentz transformations are classified as:

– Rotation, that is proper, orthochronous transformation (belongs to $O(3,1)^{\uparrow}_{+}$) that represents Euclidean rotation around spatial axes. Represented in matrix form, the rotation R is given by

$$\begin{bmatrix} R_3 & \vec{0}^T \\ \vec{0} & 1 \end{bmatrix}, \qquad R_3 \in SO(3).$$

– Boost, that is also proper, orthochronous transformation ($Boosts \in O(3,1)^{\uparrow}_{+}$, actually $Boosts \cup Rotations$ generates $O(3,1)^{\uparrow}_{+}$), representing hyperbolic rotation in a plane that includes a timelike direction. Boosts are sometimes called pure Lorentz transformations, because do not include spatial rotations. Parameterized by 3-velocity $\vec{v}$ the boost B is represented by the following symmetric 4×4 matrix

$$\begin{bmatrix} 1 - \dfrac{\gamma_v^2}{1+\gamma_v}\vec{v}\vec{v}^T & \gamma_v \vec{v}^T \\ & \\ \gamma_v \vec{v} & \gamma_v, \end{bmatrix} \tag{1}$$

where indeed $\|\vec{v}\| < c$ and $\gamma_v = \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. While the combination of rotations is always a rotation, the combination of boosts in general case is not a boost.

– Reflection, that is improper or/and antichronous transformation S, characterized by $S^2 = \delta$. The space or/and time reflections are elements of the other 3 connected components $O(3,1)^{\downarrow}_{+}$, $O(3,1)^{\uparrow}_{-}$ and $O(3,1)^{\downarrow}_{-}$.

As the 4-velocities form a set of 4-vectors with constant magnitude, there exists an isometry link that transforms one 4-velocity into another.

Let $U = (U_1, U_2, U_3, U_4)$ and $V = (V_1, V_2, V_3, V_4)$ be vectors of 4-velocity with Minkowski magnitude $U^2 = V^2 = c^2$. Each link $L(U,V)$ between 4-velocities U and V defines a set of Lorentz transformations parameterized by the 6 independent components given by the elements of $\vec{u}$ and $\vec{v}$. It is not difficult to see that if L transforms U into V, $LU = V$, then it is possible to obtain additional links by $(Id_V \circ L \circ Id_U)U = V$ for any two orthogonal transformations Id_U and Id_V such that $Id_U U = U$ and $Id_V V = V$. For example, simplest transformations of that kind are $Id_U = \dfrac{U \otimes \eta U}{U^2}$, $Id_V = \dfrac{V \otimes \eta V}{V^2}$.

Definition 2.2. *All elements of the Lorentz group L that fix the 4-vector V, form a subgroup Id_V of L, called the little Lorentz group at V [12].*

It has been shown that every little Lorentz group Id_V is isomorphic to the proper rotation group $SO(3)$. This isomorphism, which is not easily seen by traditional methods [3], emerges naturally in the gyrogroup formalism [10]. Now if we put an additional requirement on L in the form $L(U,U) = \delta$, we ensure that $L \in O(3,1)^{\uparrow}_{+}$ (the connected component with δ) and we eliminate the rotations from L providing L to be a pure Lorentz transformation, i.e. a boost. Conversely, let $L(U,V)$ be a boost which links two 4-velocities U and V,

$L(U,V)U = V$. Replacing V by U we obtain $L(U,U)U = U$, where $L(U,U)$ is a boost implying that it must be $L(U,U) = \delta$.

So, we proved the following proposition.

Proposition 2.1. *Let $L(U,V)$ be a Lorentz transformation which links two 4-velocities U and V. Then, $L(U,V)$ is a boost iff $L(U,U) = \delta$.*

This proposition enables an easy check whether a given link is boost.

Reflections S are involutory Lorentz transformations, that is $S^2 = \delta$. Parameterized by 4-velocity U, a reflection is given by the following tensor

$$S(U) = \delta - 2\frac{U \otimes \eta U}{U^2}.$$

Let us note that $S(U)$ is improper, $det(S(U)) = 1$.

Link by reflection between given 4-velocities U and V has a simple solution up to non-zero scalar $\mu \neq 0$. The solution (see e.g. [7]) is obtained by $\mu(UV)$ as an argument in S, where $U - V$ is the standard subtraction.

Proposition 2.2. *Let U and V be vectors of 4-velocity. The link by reflection parameterized only by the initial 4-velocity U and the resulting 4-velocity V is given by*

$$S(U,V) = S(\mu(U-V)) = \delta - \frac{(U-V)\otimes\eta(U-V)}{c^2 - U\cdot V}, \qquad (U-V)^2 \neq 0.$$

Note that a link by reflection does not satisfy $S(U,U) = \delta$, i.e. it is not a boost. The proof that this transformation is orthogonal and that $S(U,V)U = V$ is straightforward.

The reflections are the only links that posses a desired property of symmetry, i.e. $S(U,V)U = V$ and $S(U,V)V = U$ that is a direct consequence of $S^2 = \delta$.

The links by rotations are not so interesting because they are reduced to Euclidean 3-dimensional transformations. Lorentz transformation is possible only if $U_4 = V_4$, i.e. $\|\vec{u}\| = \|\vec{v}\|$.

3. Boost Links

Transformation link between 4-velocities by boost is the most important link because it determines relative velocity between two moving objects.

Proposition 3.1. *Let U and V be vectors of 4-velocity. The boost link parameterized only by the initial 4-velocity U and the resulting 4-velocity V is given by [2, 4]*

$$B(U,V) = \delta - \frac{(U+V)\otimes\eta(U+V)}{c^2 + U\cdot V} + 2\frac{V\otimes\eta U}{c^2}, \qquad (U+V)^2 \neq 0. \tag{2}$$

It is easy to see that $B(U,V)^{-1} = \eta B(U,V)^T\eta = B(V,U)$. The proof that the tensor $B(U,V)$ is an orthogonal transformation, i.e. that $B(U,V)B(V,U) =$

δ and that $B(U,V)U = V$ are straightforward. It is obvious that $B(U,U) = \delta$, and that implies that Lorentz transformation given by (2) is pure, i.e. a boost.

By simple calculation one can obtain that $B(U,V)V = 2\dfrac{V \otimes \eta V}{c^2}U - U$. If $U = (0,0,0,c)$, then $B(U,V)$ is a boost which corresponds to a motion with 3-velocity $-\vec{v}$. In this special case, the tensor B becomes

$$B(0,V) = \begin{bmatrix} 1+\frac{1}{\zeta}V_1^2 & \frac{1}{\zeta}V_1V_2 & \frac{1}{\zeta}V_1V_3 & \frac{1}{c}V_1 \\ \frac{1}{\zeta}V_2V_1 & 1+\frac{1}{\zeta}V_2^2 & \frac{1}{\zeta}V_2V_3 & \frac{1}{c}V_2 \\ \frac{1}{\zeta}V_3V_1 & \frac{1}{\zeta}V_3V_2 & 1+\frac{1}{\zeta}V_3^2 & \frac{1}{c}V_3 \\ \frac{1}{c}V_1 & \frac{1}{c}V_2 & \frac{1}{c}V_3 & \frac{1}{c}V_4 \end{bmatrix}, \qquad \zeta = c^2(1+\gamma_v).$$

The statement that the link given by (2) is a boost deserves separate considerations. Namely, considering the fact that the 3×3 submatrix $B(U,V)_{ij}$ for $i,j \in \{1,2,3\}$ is not a symmetric matrix one could incorrectly conclude that the $B(U,V)$ is not a boost. Even more, one could try to calculate exactly the "involved space rotation" via the following 3-vector:

$$\frac{1}{2}(B_{32}-B_{23}, B_{13}-B_{31}, B_{21}-B_{12}) = (U_2V_3-U_3V_2, U_3V_1-U_1V_3, U_1V_2-U_2V_1) =$$

$$= \frac{\gamma_u\gamma_v}{c^2}(v_yu_z - v_zu_y, v_zu_x - v_xu_z, v_xu_y - v_yu_x) = \gamma_u\gamma_v\frac{\vec{v}\times\vec{u}}{c^2}$$

as it was made in [1]. However, it is obvious that $B(U,U) = \delta$, and that implies that Lorentz transformation given by (2) is pure, i.e. a boost.

Different authors arrived at the same conclusion using different reasoning. Thus, Urbantke [11], who, as we shall show, achieved the boost (2) by two reflections, concluded that the transformation is a boost using geometrical observations. We shall give analogous prove that (2) is a boost. Namely, if W is a 4-velocity in the timelike 2-plane spanned by U and V, then $B(U,V)W$ is also in the same plane as follows from $B(U,V)U = V$ and $B(U,V)V = \kappa V - U \quad (\kappa = 2UV/c^2)$ and so, $B(U,V)W$ is also a linear combination of U and V. If W is orthogonal to the same 2-plane, it follows straightforwardly from (2) that $B(U,V)W = W$. Thus, W is left fixed by B and hence B is a boost.

Matolcsi [5] has arrived at the same conclusion by explicitly proposing single relative velocity $\vec{v}$ from which this boost is parameterized. The formula (2) is the usual boost although it depends on two velocities. The explicit matrix form of a textbook boost (1) depends on a single relative velocity but, in fact, it also refers to two inertial observers (one of which is the "rest frame", not appearing explicitly in the formulae).

Ungar [10] has proposed a unique solution for boost link between given 4-velocities using gyro formalism. His solution is not given in a tensorial form and so, it is difficult to be directly compared with (2). However, since his boost B refers to a relative velocity which is spanned by the velocities $\vec{u}$ and $\vec{v}$, it belongs to the group of "planar" solutions (we explain them later) which are identical to (2).

The boost link between given 4-velocities based on two consecutive reflections is given in [11] in the form $B(U,V) = S(U+V)S(U)$. We shall straightforwardly prove that Urbantke's solution is the same with (2). Namely,

$$S(U+V)S(U) = (\delta - \frac{(U+V)\otimes\eta(U+V)}{c^2+U\cdot V})(\delta - 2\frac{U\otimes\eta U}{c^2}) =$$

$$= \delta - 2\frac{U\otimes\eta U}{c^2} - \frac{(U+V)\otimes\eta(U+V)}{c^2+U\cdot V} + 2\frac{((U+V)\otimes\eta(U+V))(U\otimes\eta U)}{c^2(c^2+U\cdot V)} =$$

$$= \delta - 2\frac{U\otimes\eta U}{c^2} - \frac{(U+V)\otimes\eta(U+V)}{c^2+U\cdot V} + 2\frac{(U+V)\otimes\eta U}{c^2} =$$

$$= \delta - \frac{(U+V)\otimes\eta(U+V)}{c^2+U\cdot V} + 2\frac{V\otimes\eta U}{c^2}.$$

Generally, boosts refer to two observers given by two 4-velocities rather than to a single velocity as in the usual boost representation in matrix form (1). Thus, it is more suitable to consider the usual boost matrix referring to relative velocity of two observers. One observer is hidden in the coordinate axes and the velocity in the boost matrix is the relative velocity of another observer taken with respect to the hidden observer.

The solution (2) is not a unique solution for boost link parameterized by 4-velocities alone. However, its importance is in the fact that it has been achieved by different researchers using different approaches. A simple way to obtain this solution is by straightforward parametrization of the required transformation by U and V in the form

$$L = \delta - (aU\otimes\eta U + bU\otimes\eta V + cV\otimes\eta U + dV\otimes\eta V)$$

and solving the necessary equations

$$\eta LU\cdot LV = U\cdot V, \qquad L^{-1}\eta L = \eta \tag{3}$$

by a, b, c, d. Other solutions could be obtained by parametrization of the Lorentz transformation using: single vector, pair of the vectors, triple of the vectors, etc. Parametrization by a single vector X takes the form $L = \delta - X\otimes\alpha$ for an unknown covector α. Inserting this into (3), one can obtain the unique solution for the covector α, given by $L = \delta - 2\frac{X\otimes\eta X}{X^2}$, with the property $L^{-1} = L$ that gives link by reflection for $X = \mu(U-V)$ and μ is an arbitrary scalar.

Parameterizing by pair of vectors, the Lorentz transformation takes the form $L = \delta - X\otimes\alpha - Y\otimes\beta$, where $\alpha = a\eta X + b\eta Y, \beta = c\eta X + d\eta Y$ for unknown scalars a, b, c, d, as it was considered in [13]. Using this parametrization for $Y = U - V$, Oziewicz [7] has given general solution for boost link of two 4-vectors with equal norms, parameterized additionally by an arbitrary 4-vector X. Slightly adapted to the 4-velocities U and V ($U^2 = V^2 = c^2$) his formula is given in the following proposition.

Proposition 3.2. *Let U and V be vectors of 4-velocity and X an arbitrary 4-vector. The boost links parameterized by the initial 4-velocity U, the resulting 4-velocity V and the 4-vector X are given by*

$$B(U,V) = \delta - \frac{2X \otimes [(c^2 - U\cdot V)\eta X - (X\cdot U)\mu(U-V)]}{X^2(c^2 - U\cdot V) + 2(X\cdot U)(X\cdot V)} - \tag{4}$$

$$-\frac{(U-V)\otimes [X^2\eta(U-V) + 2(X\cdot V)\eta X]}{X^2(c^2 - U\cdot V) + 2(X\cdot U)(X\cdot V)},$$

$$X(U+V) \neq 0, \quad X\wedge(U-V)\neq 0, \quad X^2(c^2 - U\cdot V) + 2(X\cdot U)(X\cdot V) \neq 0$$

The proof that the tensor $B(U,V)$ is an orthogonal transformation and that $B(U,V)U = V$ is straightforward.

The solution (2) is obtained from (4) when the 4-vectors U, V and X are coplanar. The set of solutions given by (4) shows that the boost link between given 4-velocities is not unique. When the parametrization is made only by linear combination of the initial and the final velocity, i.e if we replace the vector X by a linear combination of U and V, we obtain 3 coplanar vectors and then, the link becomes unique, given by (2). However, by choosing a preferred 4-vector X to be non-coplanar with $U\wedge V$, one can obtain arbitrary many boosts. In 2 dimensions (with pseudometric $diag(-1,1)$) all vectors are coplanar and uniqueness of the solution in case of the corresponding 2-velocities is provided. In ≥ 3 dimensions, it is always possible to choose a vector X not in $U\wedge V$ providing arbitrary many solutions. The 4-velocity X could be considered as a 4-velocity of an independent observer.

The non-uniqueness of the Lorentz transformation linking given 4-velocities could have a significant influence on the theory of relativity as was pointed out in [7]. Intuitively, this non-uniqueness is equivalent to the group embedding $O(3) \to O(3,1)$. This situation can be considered as a source of non-commutativity and non-associativity of relativistic velocity addition, i.e. a primary source of the Thomas precession [9] and a source of various SR paradoxes e.g. [6, 8].

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