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# APPLICATION OF THE NEKHOROSHEV THEOREM TO THE REAL DYNAMICAL SYSTEM

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**Abstract.** The Nekhoroshev theorem describes the exponential stability of motion in perturbed Hamiltonian systems. Its spectral formulation is used to assess the stability of motion in dynamically diverse regions of the asteroid Main Belt. The obtained spectra exhibit a clear distinction between line, continuous and irregular structure, thus indicating regular vs. chaotic motion. We briefly describe the theorem and its spectral formulation, and show different dynamics revealed. The procedure of checking the fulfillment of conditions of convexity, quasi-convexity or 3-jet non-degeneracy for application of the theorem is also described, as well as an example of its use in combination with the spectral formulation of the theorem given.

## 1. Introduction

The issue of the stability of motion of asteroids over the solar system's lifetime is very important in terms of the study of their origin and dynamical evolution, it is crucial for the calibration of collisional models and for the assessment of the rate of depletion of the asteroid belt, it provides an insight into the structure of the phase space in the belt, gives clues to establish the comparative importance of the mechanisms shaping it, etc. The problem has been studied in many ways, both analytically and numerically (see e.g. [2], and the references therein), but with a limited success. The long-term perturbation theories encounter difficulties because of the degeneracy of the problem and of the inefficiency of analytical tools in estimation whether the perturbing parameters, like masses of the perturbing planets and their orbital eccentricities and inclinations, are small enough to allow the application of the stability result. As for the numerical studies, the detection of the structure of the resonances and the study of the stability requires the integrations of the orbits over the long time spans and the computation of some kind of the indicator of dynamics on a grid of many initial conditions, which is a CPU time consuming process, thus necessarily limited in terms of the accuracy and the reliability of the results.

The first successful attempt to employ another approach, that of using the Nekhoroshev theorem [8], to study the stability of motion of real asteroids over the exponentially long times, is due to Guzzo et al. [2], who applied for this

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purpose the spectral formulation of the theorem. Their method is based on the theoretical result [1] that the Fourier spectra of the orbits in Nekhoroshev dynamical regime have a particular band structure which can be detected in the output of the comparatively short numerical integrations of the orbits. The presence in the spectrum of the so-called *secondary peak structure*, that is, of peaks separated in frequency by an amount of the order of the small parameter $\epsilon$, is shown to be a signature of the Nekhoroshev stability regime for degenerate systems. The continuous spectrum and the lack of the regularity and of the peak structure indicates the non-Nekhoroshev regime. This enabled Guzzo et al. [2] to propose the classification of asteroids into the four dynamically distinct categories: (i) ostensibly stable; (ii) chaotic, but exponentially stable; (iii) moderately-to-strongly chaotic diffusive objects; and (iv) escapers.

It is well known, however, that the Nekhoroshev theorem provides an exponential estimate of the stability time for quasi-integrable non-degenerate Hamiltonian systems. Nekhoroshev himself [8] proved this stability result for the Hamiltonian functions that are analytic perturbations of steep functions. It is thus necessary, in terms of the practical application of the theorem, to verify that this is indeed so in the regions of the phase space of interest (in this case, in the main asteroid belt). This has been recently done by Pavlović and Guzzo [9] who used a simple integrable approximation of the asteroid Hamiltonian, the so-called *Kozai's Hamiltonian*, to show that it is indeed steep and that the conditions for the application of Nekhoroshev theorem are in this case fulfilled for most of the selected asteroids in the regions of the asteroid families Koronis and Veritas. The application of the theorem to the real asteroids, such as that by Guzzo et al. [2], has thus been justified.

## 2. Nekhoroshev theorem and its spectral formulation

The Nekhoroshev theorem [8] can be introduced as follows. Given the Hamiltonian:

$$H(I,\phi) = h(I) + \epsilon f(I,\phi)$$

where $I \in R^n$ and $\phi \in T^n$, $h$ is quasi–convex (more generally steepness is sufficient) and $f$ is analytic, there exist positive constants $a, b, \epsilon_0, I_0, t_0$ such that if $\epsilon < \epsilon_0$ then for any motion $I(t), \phi(t)$ it is:

$$|I(t) - I(0)| \le I_0 \epsilon^a$$

for any time $t$ satisfying:

$$|t| \le t_0 \exp\left(\frac{\epsilon_0}{\epsilon}\right)^b .$$

The theorem in this form cannot be directly applied to the real dynamical systems. Therefore, a spectral formulation has been developed in [1], to be later extended to degenerate systems in [3].

The basis of the method using the spectral formulation is the Fourier analysis of a test function $\mathcal{G}$ of the equinoctial elements defined on a numerically

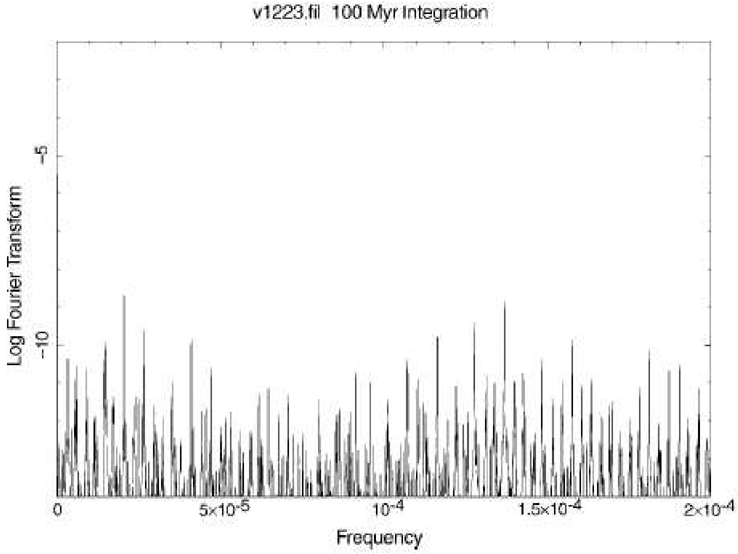


Figure 1: Fourier spectrum for asteroid 1223 Neckar of the Koronis family. The spectrum consists of many lines: the dynamics are quasi-periodic.

computed solution:

$$a,\ \ell,\ h = e\cos\varpi,\ k = e\sin\varpi,$$
$$p = \tan(i/2)\cos\Omega,\ q = \tan(i/2)\sin\Omega$$

where $a, e, i, \tilde{\omega}, \Omega, \ell$ are the usual Keplerian orbital elements. The role of the test function $\mathcal{G}$ is to mix in a convenient way the degrees of freedom of the problem. We choose the function:

$$\mathcal{G} = \Big([\cos(h) + \sin(h) + \cos(k) + \sin(k) + \cos(q) + \sin(q) + \cos(p) + \sin(p) + \cos(a) + \sin(a)]^{14} + 1\Big)^{-1}.$$

Then, we compute the Fast Fourier Transform of

$$g(t) = \Phi(t)\mathcal{G}(a(t), h(t), k(t), p(t), q(t)),$$

where $\Phi(t)$ is a suitable analytic window function described in [3].

On the basis of the shape of the spectrum of function $g(t)$, we can judge whether the object is in the Nekhoroshev regime or not. We can distinguish different cases, such as a line spectrum indicating regular, quasi-periodic motion (Figure 1), a continuous spectrum with peak structure typical of the Nekhoroshev (Figure 2), or perhaps a continuous irregular spectrum revealing the non-Nekhoroshev dynamical regime (Figure 3).

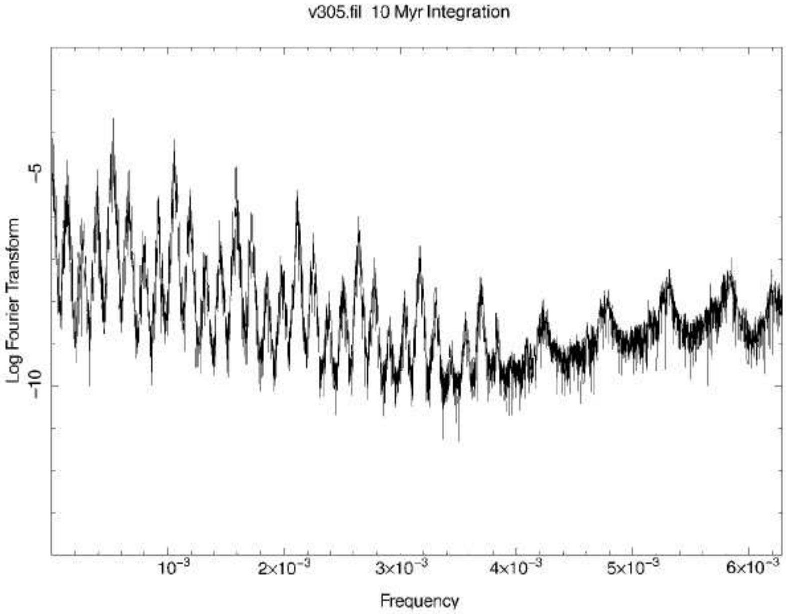


Figure 2: Fourier spectrum for asteroid 305 Gordonia. The spectrum is continuous and has a peak structure. The object is in Nekhoroshev regime.

## 3. Fulfillment of the conditions for application of Nekhoroshev theorem

It is well known that the condition of steepness involved with the proof of the Nekhoroshev theorem is originally written in an implicit form difficult to use in practical applications. Instead, however, one can use some stronger sufficient conditions, which can be explicitly written in terms of the derivatives of the Hamiltonian of the problem. These conditions are convexity, quasi-convexity and 3-jet non-degeneracy [8], which were used, for example, by Benettin et al. [1] to study the long-term stability of the Lagrangian points.

Let us also recall the definitions of convexity, quasi-convexity and 3-jet non-degeneracy. A function is:

• *convex* in $\xi_0 \in R^n$ if its Hessian is positive (or negative) definite:

$$\left( u \in R^n, \sum_{i,j} \frac{\partial^2 h}{\partial \xi_i \partial \xi_j}(\xi_0) u_i u_j = 0 \right) \Rightarrow u = 0$$

• *quasi-convex* in $\xi_0 \in R^n$ if the restriction of its Hessian to the plane orthogonal to the frequency vector $\omega = \nabla h(\xi_0)$ is either positive or negative definite:

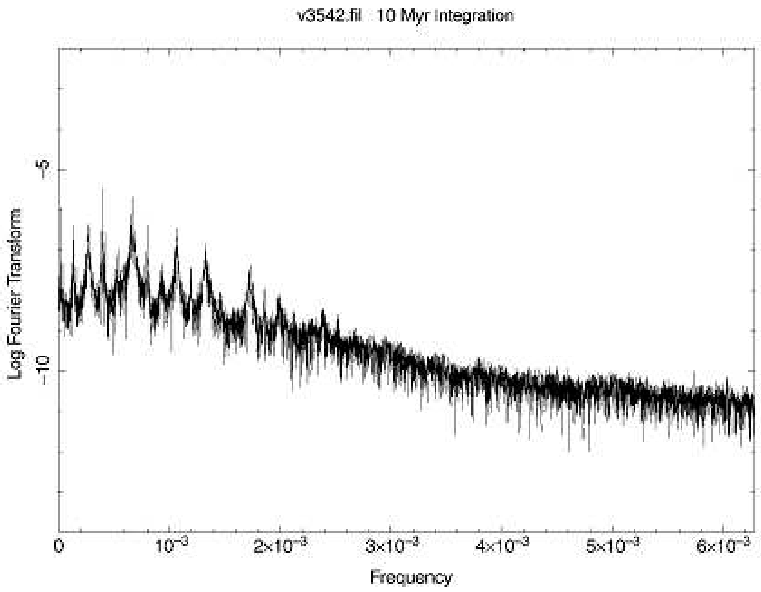


Figure 3: Fourier spectrum for asteroid 3542 Tanjiazhen. The spectrum is continuous and has no peak structure. The object is not in the Nekhoroshev regime.

$$\left( u \in R^n,\, \omega \cdot u = 0,\, \sum_{i,j} \frac{\partial^2 h}{\partial \xi_i \partial \xi_j}(\xi_0) u_i u_j = 0 \right) \Rightarrow u = 0$$

- *3-jet non degenerate* in $\xi_0 \in R^n$ if

$$\left( u \in R^n,\, \omega \cdot u = 0,\, \sum_{i,j} \frac{\partial^2 h}{\partial \xi_i \partial \xi_j}(\xi_0) u_i u_j = 0, \right.$$
$$\left. \sum_{i,j,k} \frac{\partial^3 h}{\partial \xi_i \partial \xi_j \partial \xi_k}(\xi_0) u_i u_j u_k = 0 \right) \Rightarrow u = 0.$$

The flow chart of an algorithm to check these conditions is given in Figure 4. Once the integrable Hamiltonian is known and the corresponding Hessian derived, one proceeds with the check of the signs of its eigen values. If all the three signs are the same (more precisely, if the Hessian is either positive or negative definite), the Hamiltonian is *convex*. If one of the signs differ, the restriction of the Hessian to the plane orthogonal to the frequency vector $\omega$ is considered; if it is either positive or negative definite, that is, if the signs of its two eigen values

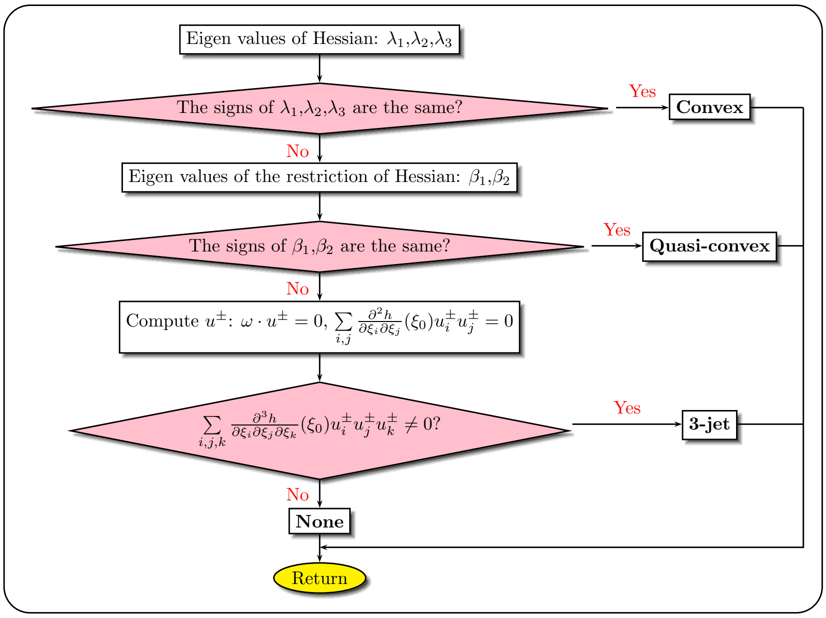


Figure 4: Algorithm to check the fulfillment of conditions for the application of Nekhoroshev theorem.

are the same, the Hamiltonian is *quasi-convex*. If the signs differ one must first compute vectors $u^{\pm}$ in the plane orthogonal to the frequency vector $\omega$ defined by an additional condition, and then check the fulfillment of the *3-jet* condition itself. This involves computation of the third derivatives of the Hamiltonian, which in the case of an asteroid Hamiltonian must be done by means of the semi-numerical techniques, extending the techniques introduced by Henrard [4], Henrard and Lemaitre [5] and Lemaitre and Morbidelli [6]. If none of the above conditions are fulfilled, we conclude that the Nekhoroshev theorem cannot be applied for a given asteroid.

## 4. Application to asteroids

As an example of the check of fulfillment of the conditions for application of the Nekhoroshev theorem we have considered a simplified asteroid Hamiltonian, consisting of the Keplerian $h_0$ and Kozai's $\mathcal{K}_0$ parts [9], extended by the terms $\mathcal{K}_1$ linear in eccentricities $e'$ and inclinations $i'$ of perturbing planets [7, 10].

This extended Hamiltonian is thus given by:

$$h = h_0 + \varepsilon\mathcal{K}_0 + \varepsilon\mathcal{K}_1. \tag{1}$$

Hamiltonian (1) depends on angles which must be removed by means of a suitable canonical transformation. This can be achieved in two steps: i) by using

Henrard's seminumerical method [4] we remove angles from $\mathcal{K}_0$ and get new variables (actions $\Lambda, J, Z$, and angles $\lambda, \psi, z$) introduced in $\mathcal{K}_1$; ii) we look for the generating function $W_1$ such that $\{W_1, \mathcal{K}_0\} + \mathcal{K}_1 = 0$, where $\{.,.\}$ denotes Poisson bracket. Function $W_1$ generates another canonical transformation resulting in new variables $(\overline{\Lambda}, \overline{J}, \overline{Z}, \overline{\lambda}, \overline{\psi}, \overline{z})$. This transformation is implicitly defined by

$$
\begin{aligned}
\Lambda &= \overline{\Lambda}, & \lambda &= \overline{\lambda} - \frac{\partial W_1}{\partial \overline{\Lambda}}(\overline{\Lambda}, \overline{J}, \overline{Z}, \overline{\psi}, \overline{z}),\\
J &= \overline{J} + \frac{\partial W_1}{\partial \tilde{\psi}}(\overline{\Lambda}, \overline{J}, \overline{Z}, \overline{\psi}, \overline{z}), & \psi &= \overline{\psi} - \frac{\partial W_1}{\partial \tilde{J}}(\overline{\Lambda}, \overline{J}, \overline{Z}, \overline{\psi}, \overline{z}),\\
Z &= \overline{Z} + \frac{\partial W_1}{\partial \tilde{z}}(\overline{\Lambda}, \overline{J}, \overline{Z}, \overline{\psi}, \overline{z}), & z &= \overline{z} - \frac{\partial W_1}{\partial \tilde{Z}}(\overline{\Lambda}, \overline{J}, \overline{Z}, \overline{\psi}, \overline{z}),
\end{aligned}
$$

which can be iteratively solved [6].

The resulting Hamiltonian does not depend on angles and can be used for computation of derivatives over the actions $(\overline{\Lambda}, \overline{J}, \overline{Z})$. Making use of the algorithm of Figure 4 we can now straightforwardly check the fulfillment of the convexity, quasi-convexity or 3-jet conditions.

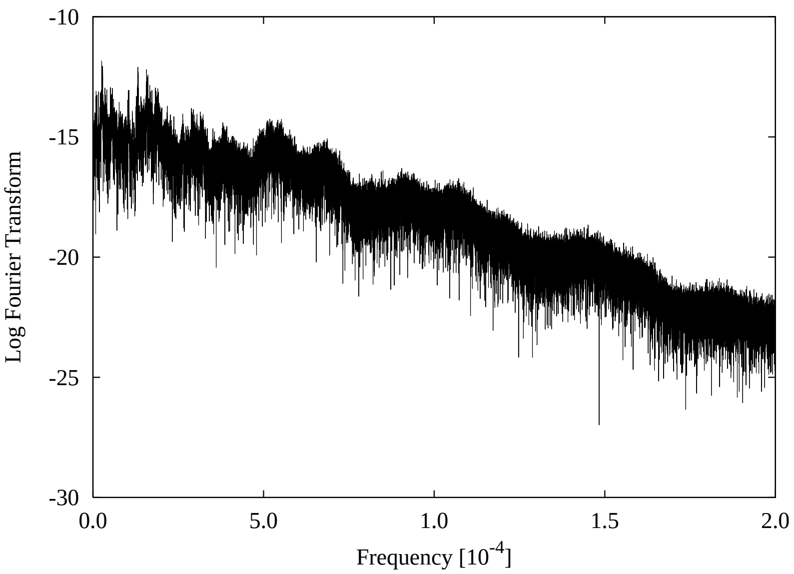


Figure 5: Fourier spectrum for asteroid 582 Olympia, computed from the output of the 100 Myr numerical integration.

As an example we show the results for asteroid 582 Olympia, located very close to the strong $\nu_5$ secular resonance [7]. By applying the algorithm described in Section 3 we found that the condition of convexity is fulfilled for this asteroid and that we can apply the Nekhoroshev theorem to assess the character of its motion. Figure 5 shows the spectrum of this asteroid as determined from the numerical integration covering 100 Myr. Since the secondary peak structure does not show up in the plot we conclude that this asteroid is not in the Nekhoroshev regime. Analyzing in addition the time variations of the orbital elements we conclude that this object most probably can be classified in the category of "moderately-to-strongly chaotic diffusive objects" proposed by Guzzo et al. [2].

We conclude that the procedure of checking the conditions for the application of Nekhoroshev theorem in combination with spectral formulation of the theorem represents an efficient and powerful tool to establish the character of asteroid motion and classify asteroids into different dynamical categories.

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