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A NOTE ON DISTRIBUTIONAL SEMI-RIEMANNIAN GEOMETRY[1]

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Abstract. We discuss some basic concepts of semi-Riemannian geometry in low-regularity situations. In particular, we compare the settings of (linear) distributional geometry in the sense of L. Schwartz and nonlinear distributional geometry in the sense of J.F. Colombeau.

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1. Introduction

In this note we discuss some foundational concepts of semi-Riemannian geometry in case of low regularity. While semi-Riemannian geometry is usually formulated for $\mathcal{C}^\infty$-metrics, most of the results still hold true in case the metric is locally $\mathcal{C}^{1,1}$, i.e., its first derivatives being locally Lipschitz continuous: Indeed, this condition guarantees (local) unique solvability of the geodesic equation and implies locally uniform boundedness of the curvature. In particular, the Riemann tensor can be interpreted as a distribution.

However, there is a strong motivation from physics to lower the regularity assumptions on the metric. In particular, in the context of weakly singular space-times in general relativity such as thin shells of matter or radiation, cosmic strings, impulsive pp-waves, and shell crossing singularities, one has to deal with Lorentz metrics of regularity below $\mathcal{C}^{1,1}$.

In this contribution we will mainly be concerned with the following two issues
(1) Defining the Levi-Civita connection of a metric of low regularity, and
(2) Defining the curvature from a connection or metric of low regularity,
in the context of two different mathematical frameworks, namely
(A) distributional geometry, i.e., the setting of tensor distributions in the sense of classical Schwartzian distribution theory, and
(B) nonlinear distributional geometry in the sense of Colombeau.

Approach (A) was pursued in [18, 7, 19] and more recently in [16], building on global accounts to distribution theory, e.g. provided in [3], while approach (B) is due to [15] and is based on global analysis ([4, 14]) in (special) Colombeau

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algebras ([5]). Applications of (B) in general relativity can, e.g. be found in [1, 2, 13, 8], and an overview of applications of (A) and (B) in relativity is provided by [21].

While this contribution does not provide any new mathematics, we collect the respective results for both settings (A) and (B) and present them in parallel, allowing for a direct comparison. Finally, we present results on the compatibility of (A) and (B), recently obtained in [22].

In some more detail, the plan of this paper is as follows. After collecting the necessary prerequisites to make our presentation self-contained in Sec. 2, in Sec. 3 we define the notions of semi-Riemannian metrics and linear connections for each of the above frameworks (A) and (B). In Sec. 4 we deal with issue (1) and provide a version of the fundamental lemma of semi-Riemannian geometry for each of our settings, while in Sec. 5 we discuss issue (2), again for each of the frameworks (A) and (B). Finally, in Sec. 6 we answer the question of compatibility of the two approaches in the affirmative. Our main references for approaches (A) and (B) will be [7, 16] and [15, 22], respectively.

2. Linear and nonlinear distributional geometry

We recall that distributions on a smooth (paracompact, Hausdorff) manifold M of dimension n are defined to be linear, continuous (w.r.t. the usual (LF)-topology) functionals on the space of compactly supported n-forms, $\mathcal{D}'(M) = (\Omega^n_c(M))'$. We will denote the action of a distribution on a test n-form by $\langle v, \omega\rangle$. Distributional tensor fields and more generally distributional sections of vector bundles can also be defined as elements of the dual space of appropriate spaces of sections. But for our purpose it will be sufficient (see, however, [9]) to view them as tensor fields with distributional coefficients, or as $\mathcal{C}^\infty(M)$-multilinear maps of vector fields and one-forms to scalar distributions, i.e., we denote with r, s the tensor character

$$\mathcal{D}'^r_s(M) = \mathcal{D}'(M) \otimes_{\mathcal{C}^\infty(M)} \mathcal{T}^r_s(M) \cong L_{\mathcal{C}^\infty(M)}\big(\Omega^1(M)^r, \mathfrak{X}(M)^s; \mathcal{D}'(M)\big). \tag{1}$$

Here $\mathcal{T}^r_s(M)$ denotes the space of (smooth) (r, s)-tensor fields, and we have set $\mathcal{T}^1_0(M) = \mathfrak{X}(M)$ and $\mathcal{T}^0_1(M) = \Omega^1(M)$. There is a well-developed theory of tensor distributions ([3, 18, 17, 19]), which parallels the smooth case but suffers from the natural limitations of distribution theory. In particular, in all multilinear operations only one factor may be distributional, while all others have to be smooth. For a pedagogical account we refer to [10, Ch. 3.1].

One way to deal with products is to restrict oneself to subspaces of $\mathcal{D}'$. We will, in particular, be interested in Sobolev spaces. For $m \in \mathbb{N}_0$ and $1 \le p \le \infty$ we denote by $W^{m,p}_{\mathrm{loc}}(M)$ the space of distributions whose derivatives up to order m locally belong to L^p. Recall that $W^{m,p}_{\mathrm{loc}}(M)$ is a Fréchet space with its topology induced by the semi-norms $\|u \circ \varphi_\alpha^{-1}\|_{W^{m,p}(V)}$, where $(U_\alpha, \varphi_\alpha)$ denotes the charts of an atlas for M, V denotes any open, relatively compact subset of $\varphi_\alpha(U_\alpha)$, and $\|f\|^p_{W^{m,p}(V)} = \sum_{\alpha\le m} \int_V |\partial^\alpha f|^p$. Moreover, we write

$$(W^{m,p}_{\mathrm{loc}})^r_s(M) = W^{m,p}_{\mathrm{loc}}(M) \otimes_{\mathcal{C}^\infty(M)} \mathcal{T}^r_s(M)$$

for the spaces of $W^{m,p}_{\mathrm{loc}}$-tensor fields. In case $p = 2$ we use the usual convention and set $H^m_{\mathrm{loc}} = W^{m,2}_{\mathrm{loc}}$, and in case $m = 0$ we obtain the usual local Lebesgue spaces which we denote by L^p_{loc}.

In nonlinear distributional geometry ([10, Ch. 3]) in the sense of J.F. Colombeau ([5]) one replaces the vector space $\mathcal{D}'(M)$ by the algebra of generalised functions $\mathcal{G}(M)$ to overcome the problem of multiplication of distributions. Indeed, in the light of Schwartz' impossibility result ([20]), this setting provides a minimal framework within which tensor fields may be subjected to nonlinear operations, while maintaining consistency with smooth and distributional geometry: tensor products of smooth tensor fields are preserved as well as Lie derivatives of distributional ones. The basic idea of the construction is smoothing of distributions (via convolution) and the use of asymptotic estimates in terms of a regularisation parameter: these are employed in a quotient construction which, in particular, provides consistency with the product of smooth functions.

The (special) Colombeau algebra of generalised functions on M is defined as the quotient

$$\mathcal{G}(M) := \mathcal{E}_M(M)/\mathcal{N}(M)$$

of moderate nets of smooth functions modulo negligible ones, where the respective notions are defined by (P denoting linear differential operators on M)

$$\mathcal{E}_M(M) := \{(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(M) : \forall K \subset\subset M \,\forall P \,\exists N : \sup_{p\in K} |Pu_\varepsilon(p)| = O(\varepsilon^{-N})\}$$

$$\mathcal{N}(M) := \{(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(M) : \forall K \subset\subset M \,\forall P \,\forall m : \sup_{p\in K} |Pu_\varepsilon(p)| = O(\varepsilon^{m})\}.$$

Elements of $\mathcal{G}(M)$ are denoted by $u = [(u_\varepsilon)_\varepsilon] = (u_\varepsilon)_\varepsilon + \mathcal{N}(M)$. With componentwise operations, $\mathcal{G}(\_)$ is a fine sheaf of differential algebras where the derivations are Lie derivatives with respect to smooth vector fields defined by $L_X u := [(L_X u_\varepsilon)_\varepsilon]$, also denoted by $X(u)$.

The $\mathcal{G}(M)$-module $\mathcal{G}^r_s(M)$ of generalised tensor fields can be defined along the same lines using analogous asymptotic estimates. However, for our purpose it will suffice to set

$$\begin{aligned}\mathcal{G}^r_s(M) &:= \mathcal{G}(M) \otimes_{\mathcal{C}^\infty(M)} \mathcal{T}^r_s(M)\\ &\cong L_{\mathcal{C}^\infty(M)}(\Omega^1(M)^r, \mathfrak{X}(M)^s; \mathcal{G}(M)) \cong L_{\mathcal{G}(M)}(\mathcal{G}^0_1(M)^r, \mathcal{G}^1_0(M)^s; \mathcal{G}(M)).\end{aligned}$$

Note that in contrast to classical distributions (c.f. (1)), generalised tensor fields map generalised (and not merely smooth) fields and forms to generalised functions. It is precisely this property that allows one to raise and lower indices with the help of a generalised metric (see Sec. 3 below), just as in the smooth case.

Smooth functions are embedded into $\mathcal{G}(M)$ simply by the "constant" embedding σ, i.e., $\sigma(f) := [(f)_\varepsilon]$. In case $M \subseteq \mathbb{R}^n$ open, compactly supported distributions are embedded into $\mathcal{G}$ via convolution with a mollifier $\rho \in \mathcal{S}(\mathbb{R}^n)$ with unit integral satisfying $\int \rho(x)x^\alpha dx = 0$ for all $|\alpha| \geq 1$; more precisely setting $\rho_\varepsilon(x) = (1/\varepsilon^n)\rho(x/\varepsilon)$, we define $\iota(w) := [(w * \rho_\varepsilon)_\varepsilon]$. In case $\mathrm{supp}(w)$ is not compact, one uses a sheaf-theoretical construction which can be lifted to an arbitrary manifold using a partition of unity subordinate to the charts of some atlas ([10, Thm. 3.2.10]). From the explicit formula, it is clear that the embedding commutes with differentiation. It is, however, not canonical since it depends on the mollifier as well as the partition of unity. (A canonical embedding of distributions *is* provided by the so-called full version of the construction (see [11, 12]), however, at the price of a technical machinery, which we have chosen to avoid here.)

The interplay between generalised functions and distributions is most conveniently formalised in terms of the notion of association. We call a distribution $v \in \mathcal{D}'(M)$ associated with $u \in \mathcal{G}(M)$ and write $u \approx v$ if, for all compactly supported n-forms ω and one (hence any) representative $(u_\varepsilon)_\varepsilon$, we have $\lim_{\varepsilon\to 0} \int_M u_\varepsilon \omega = \langle w, \omega\rangle$.

3. Semi-Riemannian metrics and connections

Here we discuss Semi-Riemannian metrics and linear connections in the distributional and the generalised setting. To begin with following Marsden ([18, Def. 10.6]), we define:

Definition 3.1. A distributional $(0,2)$-tensor field $\mathbf{g} \in {\mathcal{D}'}^0_2(M)$ is called a distributional metric if it is symmetric and nondegenerate in the sense that $g(X,Y) = 0$ for all $Y \in \mathfrak{X}(M)$ implies $X = 0 \in \mathfrak{X}(M)$.

Observe that due to its non-locality this condition of nondegeneracy is rather weak. For example, the classically singular line element $ds^2 = x^2\, dx^2$ is nondegenerate in the above sense. Therefore it is appropriate to additionally ask for Parker's condition ([19]), demanding that $\mathbf{g}$ is nondegenerate in the usual sense of its singular support, see also the discussion in [22, Sec. 3].

By the above natural limitations of distribution theory it is not possible to insert $\mathcal{D}'$-vector fields into $\mathbf{g}$, hence it does not induce a map ${\mathcal{D}'}^1_0 \to {\mathcal{D}'}^0_1$ and cannot be used to pull indices of distributional tensor fields. Moreover, the map induced by $\mathbf{g} : \mathfrak{X}(M) \ni X \mapsto X^\flat := \mathbf{g}(X, .) \in {\mathcal{D}'}^0_1(M)$ is injective, but clearly not surjective, and, in general, there is no way to define the inverse metric. Also, the notions like the index or geodesics of a distributional metric are not (easily) defined.

Let us now turn to the generalised setting. Following [15, Def. 3.4] we define in this case (omitting some technicalities concerning the index).

Definition 3.2. A symmetric section $\mathbf{g} \in \mathcal{G}^0_2(M)$ is called a generalised semi-Riemannian metric if $\det \mathbf{g}$ is invertible in the generalised sense, i.e., for any representative $\left(\det(\mathbf{g}_\varepsilon)\right)_\varepsilon$ of $\det \mathbf{g}$ we have

$$\forall K \subset\subset M\ \exists m \in \mathbb{N} : \inf_{p\in K} |\det(\mathbf{g}_\varepsilon)(p)| \geq \varepsilon^m.$$

This notion of nondegeneracy can be characterised pointwise (using generalised points, see [15, Sec. 2]) and the following characterisation of generalised metrics captures the intuitive idea of a generalised metric as a net of classical metrics approaching a singular limit: $\mathbf{g}$ is a generalised metric iff on every relatively compact open subset $V \subseteq M$ there exists a representative $(\mathbf{g}_\varepsilon)_\varepsilon$ of $\mathbf{g}$ such that, for fixed ε, $\mathbf{g}_\varepsilon$ is a classical metric and its determinant, $\det \mathbf{g}$, is invertible in the generalised sense. The latter condition basically means that the determinant is not too singular.

A generalised metric induces a $\mathcal{G}(M)$-linear isomorphism from $\mathcal{G}^1_0(M)$ to $\mathcal{G}^0_1(M)$. The inverse of this isomorphism gives a well-defined element of $\mathcal{G}^2_0(M)$, the inverse metric, which we denote by $\mathbf{g}^{-1}$, with representative $\big(\mathbf{g}_\varepsilon^{-1}\big)_\varepsilon$ ([15, Props. 3.6, 3.9]).

Next we turn to connections. To fix notations we recall that, classically, a connection is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfying $(X, X', Y, Y' \in \mathfrak{X}(M)$, $f \in \mathcal{C}^\infty(M))$

$$(\nabla_1) \qquad \nabla_{fX+X'}Y = f\nabla_X Y + \nabla_{X'} Y$$

$$(\nabla_2) \qquad \nabla_X(fY + Y') = f\nabla_X Y + X(f)Y + \nabla_X Y'.$$

We now define.

Definition 3.3.

(i) A distributional connection ([18, p. 358][3],[16, Def. 3.1]) is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{D}'^1_0(M)$ satisfying (∇_1), (∇_2) for all $X, X', Y, Y' \in \mathfrak{X}$, $f \in \mathcal{C}^\infty$.

(ii) A generalised connection ([15, Def. 5.1]) is a map $\nabla :\ \mathcal{G}^1_0(M) \times \mathcal{G}^1_0(M) \to \mathcal{G}^1_0(M)$ satisfying (∇_1), (∇_2) for all $X, X', Y, Y' \in \mathcal{G}^1_0$, $f \in \mathcal{G}$.

Both versions extend to the full smooth resp. generalised tensor algebra by using the Leibniz rule and defining $\nabla_X u := X(u)$ for scalars. Also, in both cases the standard coordinate formulae hold.

4. Versions of the fundamental lemma

In this section we discuss the question in which sense a distributional resp. generalised metric defines a Levi-Civita connection. Recall that the Levi-Civita connection ∇ of a smooth metric $\mathbf{g}$ is classically given as the unique connection which is metric and torsion free, i.e., satisfies

$$(\nabla_3) \qquad \nabla\mathbf{g} = 0 \ \Big(\Longleftrightarrow X\big(\mathbf{g}(V,W)\big) = \mathbf{g}(\nabla_X V, W) + \mathbf{g}(V, \nabla_X W)\Big)$$

$$(\nabla_4) \qquad T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] = 0,$$

and is characterised by the Koszul formula

$$\begin{aligned} 2\mathbf{g}(\nabla_X Y, Z) \ &= \ X\big(\mathbf{g}(Y,Z)\big) + Y\big(\mathbf{g}(Z,X)\big) - Z\big(\mathbf{g}(X,Y)\big) \\ &\quad -\mathbf{g}(X,[Y,Z]) + \mathbf{g}(Y,[Z,X]) + \mathbf{g}(Z,[X,Y]) \ =: \ F(X,Y,Z). \end{aligned}$$

[3] Note, however, the typo in the very definition.

Observe that in the distributional framework (∇_3) cannot be formulated: a distributional connection can only act on smooth tensor fields but not on the distributional metric and likewise, in the terms on the r.h.s. of the condition equivalent to (∇_3) the distributional metric cannot act on the distributional vector fields $\nabla_X V$ and $\nabla_X W$.

One way to circumvent this obstacle is (following [16, Sec. 4]) to primarily use the Koszul formula. Observe that its r.h.s. $F(X,Y,Z)$ is defined for an arbitrary distributional metric and $X,Y,Z \in \mathfrak{X}(M)$, and the standard calculation shows that $Z \mapsto F(X,Y,Z)$ is $\mathcal{C}^\infty(M)$-linear. Hence

$$\nabla^\flat_X Y: \ Z \mapsto \frac{1}{2}\, F(X,Y,Z)$$

defines a distributional one-form. But recall that we cannot use the metric to turn it into a distributional vector field, as is done in the smooth case. On the other hand, it is readily shown that $\nabla^\flat$ satisfies the properties $(X,Y,Z \in \mathfrak{X}(M))$

$$\begin{array}{ll} (\nabla_3)^\flat & \nabla^\flat_X Y - \nabla^\flat_Y X - [X,Y]^\flat = 0 \\ (\nabla_4)^\flat & X\big(g(Y,Z)\big) - \nabla^\flat_X Y(Z) - \nabla^\flat_X Z(Y) = 0, \end{array}$$

which lead LeFloch and Mardare to define.

Definition 4.1. The distributional Levi-Civita connection of a distributional metric $\mathbf{g}$ is defined as the mapping $\nabla^\flat: \ \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{D}'^0_1(M)$ given by

$$\nabla^\flat_X Y(Z) := \frac{1}{2}\, F(X,Y,Z).$$

Note, however, that $\nabla^\flat$ is *not* a distributional connection in the sense of definition 3.3(i): only if $\mathbf{g}$ possesses additional regularity we may set $\nabla_X Y := \mathbf{g}^{-1}(\nabla^\flat_X Y, .)$, which implies (∇_3) and (∇_4). This, of course, holds true if $\mathbf{g}$ is smooth but also if the conditions

$$\nabla^\flat_X Y \in (L^2_{\mathrm{loc}})^0_1(M) \text{ and } \mathbf{g}^{-1} \in (L^\infty_{\mathrm{loc}})^0_2(M) \tag{2}$$

hold: we then have that $\nabla_X Y \in (L^2_{\mathrm{loc}})^1_0(M) \subseteq \mathcal{D}'^1_0(M)$.

Turning now to the generalised setting, we observe that we may follow the classical proof of the fundamental lemma and use the properties of the inverse of the generalised metric to obtain (cf. [15, Thm. 5.2]).

Theorem 4.1. *For any generalised metric $\mathbf{g} \in \mathcal{G}^0_2(M)$ there exists a unique generalised connection ∇ that is metric and torsion free, i.e., satisfies (∇_3) and (∇_4) for all $X,Y,Z \in \mathcal{G}^1_0(M)$. It is called the generalised Levi-Civita connection of $\mathbf{g}$ and is characterised by the Koszul formula.*

5. Curvature

Again we start by recalling the standard formula to fix our notation. In the smooth setting, the Riemann tensor is given by $(X,Y,Z \in \mathfrak{X}(M))$

$$\mathrm{Riem}(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{3}$$

Beginning with the distributional case, we immediately observe that the terms involving second derivatives cannot be defined: ∇_X does not act on a general $\nabla_Y Z \in {\mathcal{D}'}^1_0$. To answer the question for which restricted class of distributional connections we *can* define the curvature we consider (following [16, Sec. 3.2]) distributional connections which take values in a subspace of the distributional vector fields,

$$\nabla: \ \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{A}(M) \subseteq {\mathcal{D}'}^1_0(M),$$

where $\mathcal{A}(M)$ is to be chosen in such a way that ∇ can be extended to it, i.e.,

$$\nabla: \ \mathfrak{X}(M) \times \mathcal{A}(M) \to {\mathcal{D}'}^1_0(M) \text{ via } \nabla_X Y(\Theta) := X\big(Y(\Theta)\big) - \nabla_X \Theta(Y),$$

where $X \in \mathfrak{X}$, $Y \in \mathcal{A}$, and $\Theta \in \Omega^1$. Now the term $X(Y(\Theta)) \in {\mathcal{D}'}^1_0$, and the obvious choice to make the action of $\nabla_X \Theta \in \mathcal{A}(M)$ on $Y \in \mathcal{A}(M)$ well-defined is to set $\mathcal{A}(M) = (L^2_{\mathrm{loc}})^1_0(M)$. Indeed, then $\nabla_X \Theta(Y) \in (L^1_{\mathrm{loc}})^1_0(M)$ can be interpreted as a distributional vector field and we may define.

Definition 5.1.

(i) A distributional connection ∇ is called an L^2_{loc}-connection if $\nabla_X Y \in (L^2_{\mathrm{loc}})^1_0(M)$ for all $X, Y \in \mathfrak{X}(M)$.

(ii) The distributional Riemann tensor Riem of an L^2_{loc}-connection is defined by the usual formula (3).

Note that for any L^2_{loc}-connection also the Ricci tensor and the scalar curvature can be defined.

We now turn to the question of assigning a curvature to a distributional metric. Guided by the above consideration we aim at an L^2_{loc}-Levi-Civita connection. By (2) we see that a sufficient condition is $\mathbf{g} \in (H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}})^0_2(M)$ and $|\det \mathbf{g}| \geq C > 0$ almost everywhere on compact sets. In fact, the latter condition together with the L^∞_{loc}-bound on $\mathbf{g}$ implies local boundedness of $\mathbf{g}^{-1}$ by the cofactor formula. Hereby we have essentially rediscovered the key-notion of R. Geroch and J. Traschen's paper [7] (however, see [16] and [22] for the nondegeneracy condition) and may define.

Definition 5.2. We call a distributional metric $\mathbf{g} \in (H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}})^0_2(M)$ gt-regular if it is a semi-Riemannian metric (of fixed index) almost everywhere. A gt-regular metric is called nondegenerate if its determinant is locally uniformly bounded away from zero, i.e.,

$$\forall K \subset\subset M \ \exists C: \ |\det \mathbf{g}(x)| \geq C > 0 \text{ almost everywhere on } K.$$

Observe that $H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(M)$ is an algebra and that the invertible elements are precisely those which are locally uniformly bounded away from zero. In particular, the inverse of a nondegenerate gt-regular metric is again gt-regular and nondegenerate in the sense that $\det(\mathbf{g}^{-1})$ is locally uniformly bounded away from zero. Also note the similarity of this notion of nondegeneracy with the nondegeneracy condition employed for generalised metrics in Definition 3.2.

Moreover, observe that by (2) the distributional Levi-Civita connection of a nondegenerate gt-regular metric really is a distributional connection in the sense of Definition 3.3(i). Finally, our above discussion indicates how to prove the folowing result which was first obtained in [7] by analysing local coordinate expressions and rederived in [16] in a coordinate invariant way.

Theorem 5.1. *For a nondegenerate gt-regular metric the Riemann and Ricci tensor and the scalar curvature are defined as distributions.*

Summing up Definition 5.2 provides sufficient conditions on a distributional metric that allow to perform the most basic operations of semi-Riemannian geometry. On the other hand, the question of necessity is hard to tackle in a precise sense, but there are strong indications that we have indeed found the most general "reasonable" distributional framework providing the geometric foundations of general relativity. First of all, the Bianchi identities, which provide conservation of energy, cannot be formulated in the gt-setting. Moreover, the following consideration is essential when modelling singular scenarios in relativity: since a distributional metric does not directly make sense as a physical model we have to interpret it as an idealisation obtained as the limit of some approximating sequence of "physically realistic" metrics. It is now vital to have at hand a notion of convergence for these sequences that also implies convergence of the respective curvature quantities. While such stability properties have been derived for gt-regular metrics (see also Section 6 below) it is known that such results are not available for a slightly wider class of metrics considered in [6].

On the other hand, already in [7, Thm. 1] it was observed that the gt-setting only allows for a limited range of applications: The support of the Riemann tensor of a nondegenerate gt-regular metric can only be concentrated to a submanifold of codimension of at most one. Hence thin shells of matter can be described in the gt-setting while cosmic strings, and point particles cannot be covered. This fact provides a strong motivation for a "generalised curvature framework" whose basis we recall now.

Again, due to the fact that the generalised framework allows to proceed componentwise we may define without any obstacle (see [15, Def. 6.1]).

Definition 5.3. Let $\mathbf{g} \in \mathcal{G}^0_2(M)$ be a generalised metric, then the Riemann and Ricci tensors as well as the generalised scalar curvature are defined by the usual formulae.

Moreover, we have the following basic consistency with the smooth theory: If one (hence any) representative $\mathbf{g}_\varepsilon$ of a generalised metric $\mathbf{g}$ converges locally uniformly together with its derivatives up to order 2 to a vacuum solution of Einstein's equations (which then necessarily is a $\mathcal{C}^2$-metric), then the Ricci tensor of $\mathbf{g}$ is associated to 0. For details see [15, Sec. 6].

6. Compatibility

So far we have described the distributional and the generalised setting in parallel. A major question, however, is the compatibility between these frameworks, which we are going to discuss now: Given a nondegenerate gt-regular metric $\mathbf{g}$, we have at our hands two ways to compute its curvature. The first one is to proceed within the gt-setting to compute $\mathrm{Riem}[\mathbf{g}] \in \mathcal{D}'^{1}_{3}(M)$, while the second one consist of embedding $\mathbf{g}$ into $\mathcal{G}^0_2(M)$ and to calculate the curvature $\mathrm{Riem}[\iota(\mathbf{g})] \in \mathcal{G}^1_3(M)$ of the smoothed metric $\iota(\mathbf{g})$ within the generalised framework. Naturally, the question arises of whether these two procedures lead to the same result. A more precise formulation of this question is whether the generalised curvature tensor $\mathrm{Riem}[\iota(\mathbf{g})]$ of the embedded metric is associated to the distributional curvature tensor $\mathrm{Riem}[\mathbf{g}]$ of the original metric, i.e., whether the folowing diagram commutes.

$$\begin{tikzcd} H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}} \ni \mathbf{g} \arrow[r, "\iota"] \arrow[d, "\text{gt-setting}"'] & {[\iota(\mathbf{g})] \in \mathcal{G}} \arrow[d, "\mathcal{G}\text{-setting}"] \\ \mathrm{Riem}[\mathbf{g}] & \mathrm{Riem}[\iota(\mathbf{g})] \arrow[l, "\approx"'] \end{tikzcd}$$

Since we are only interested in convergence results, it suffices to work locally: denote by g_{ij} the local components of $\mathbf{g}$ and write g^ε_{ij} for their smoothings, i.e., $g^\varepsilon_{ij} = g_{ij} * \rho_\varepsilon$, where ρ is a mollifier as in Sec. 2, and denote the resulting metric by $\mathbf{g}_\varepsilon$. Now the question of compatibility may be rephrased as a question of stability: does the convergence of $\mathbf{g}_\varepsilon \to \mathbf{g} \in H^1_{\mathrm{loc}} \cap L^p_{\mathrm{loc}}$ for all $p < \infty$, which follows from the standard properties of convolution, imply the $\mathcal{D}'$-convergence of the curvature?

Indeed, in [7] and in [16], several stability results have been provided. For our purpose it will be sufficient to recall the following one[4].

Theorem 6.1. *([16, Thm. 4.6(2)]) If a sequence of nondegenerate gt-regular metrics converges in H^1_{loc} and the sequence of its inverses convergences in L^∞_{loc}, then their distributional Riemann and Ricci tensors converge in distributions.*

We now see that this result is not strong enough for our purpose. Indeed ([22, Prop. 4.8]) the inverse $\mathbf{g}_\varepsilon^{-1}$ again converges in $H^1_{\mathrm{loc}} \cap L^p_{\mathrm{loc}}$ for all $p < \infty$, which falls short of implying the assumptions of the theorem. A positive answer to our question is, however, provided by [22, Thm. 5.1] under an additional assumption (called stability) which guarantees that the smoothing $\mathbf{g}_\varepsilon$ of $\mathbf{g}$ obtained via convolution with mollifiers from as suitable class (called admissible) indeed provides a generalised metric in the sense of Definition 3.2 (see [22, Sec. 4] for details). An analogous statement is provided for the Ricci tensor and the scalar curvature ([22, Cor. 5.2]), so that we may give the following precise statement.

Theorem 6.2. *Let $\mathbf{g}$ be a nondegenerate, stable (see [22, Def. 4.5]), gt-regular metric and $\mathbf{g}_\varepsilon$ be a smoothing of $\mathbf{g}$ obtained via convolution with an admissible (see [22, Lem. 4.3]) mollifier. Then we have*

$$\mathrm{Riem}[\mathbf{g}_\varepsilon] \approx \mathrm{Riem}[\mathbf{g}],\ \mathrm{Ric}[\mathbf{g}_\varepsilon] \approx \mathrm{Ric}[\mathbf{g}],\ \textit{and } R[\mathbf{g}_\varepsilon] \approx R[\mathbf{g}].$$

[4]But see the discussion at the end of Sec. 5 in [22].

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