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ESTIMATION OF A CONDITION NUMBER RELATED TO THE WEIGHTED DRAZIN INVERSE

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Abstract. In this paper we get the formula for the condition number of the W-weighted Drazin inverse solution of a linear system $WAWx = b$, where A is a bounded linear operator between Hilbert spaces X and Y, W is a bounded linear operator between Hilbert spaces Y and X, x is an unknown vector in the range of $(AW)^D$ and b is a vector in the range of $(WA)^D$.

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1. Introduction

In this paper X and Y denote arbitrary Hilbert spaces. We use $\mathcal{B}(X,Y)$ to denote the set of all linear bounded operators from X to Y. Set $\mathcal{B}(X) = B(X,X)$.

Let $A \in \mathcal{B}(X,Y)$, $W \in \mathcal{B}(Y,X)$ be nonzero operators. If there exists $S \in \mathcal{B}(X,Y)$ satisfying

$$(AW)^{k+1}SW = (AW)^k,$$
$$SWAWS = S,$$
$$AWS = SWA,$$

for some nonnegative integer k, then S is called the W-weighted Drazin inverse of A and denoted by $S = A_{d,W}$ [5]. If there exists $A_{d,W}$, then we say that A is W-Drazin invertible and $A_{d,W}$ must be unique [5]. If $X = Y$, $A \in \mathcal{B}(X)$ and $W = I$, then $S = A^D$, the ordinary Drazin inverse of A [1]. We use $i(S)$ to denote the Drazin index of $S \in \mathcal{B}(X)$. If S has a Drazin inverse, then $i(S) = \inf\{k \in N : S^k = S^{k+1}S^D\}$.

Let us recall that if $A \in \mathcal{B}(X,Y)$ and $W \in \mathcal{B}(Y,X)$ then the following conditions are equivalent [4]:

(1) A is W-Drazin invertible,

(2) AW is Drazin invertible,

(3) WA is Drazin invertible.

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Let $A \in \mathcal{B}(X,Y)$, $W \in \mathcal{B}(Y,X)$ and let A be W-Drazin invertible. Then AW and WA are Drazin invertible and

$$X = N((WA)^D) \oplus R((WA)^D), \qquad Y = N((AW)^D) \oplus R((AW)^D).$$

Let X and Y be equipped with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. The Q-norm for a vector $x \in X$, the P-norm for a vector $y \in Y$ and the PQ-norm for an operator $A \in \mathcal{B}(X,Y)$ are defined by

$$\|x\|_Q = \sqrt{\|x_1\|_X^2 + \|x_2\|_X^2},$$

$$\|y\|_P = \sqrt{\|y_1\|_Y^2 + \|y_2\|_Y^2},$$

$$\|A\|_{PQ} = \sup_{\|x\|_Q \leq 1} \|Ax\|_P$$

where

$$x = x_1 + x_2,\ x_1 \in R((WA)^D),\ x_2 \in N((WA)^D),$$
$$y = y_1 + y_2,\ y_1 \in R((AW)^D),\ y_2 \in N((AW)^D).$$

Notice that we can also change the inner product in X in the following way:

$$\langle x, y\rangle_P = \langle x_1, y_1\rangle_X + \langle x_2, y_2\rangle_X$$

where

$$x = x_1 + x_2,\ y = y_1 + y_2,\ x_1,\ y_1 \in R((WA)^D),\ x_2,\ y_2 \in N((WA)^D).$$

Now, $\|\cdot\|_P$ is induced by $\langle\cdot,\cdot\rangle_P$. Similarly, for $\langle\cdot,\cdot\rangle_Q$ and $\|\cdot\|_Q$ in Y.

From [3] we can write A, W in the form

$$A = A_1 \oplus A_2, \quad W = W_1 \oplus W_2,$$

with A_1, W_1 invertible and W_2A_2 and A_2W_2 quasinilpotent. Hence, the W–weighted Drazin inverse of A has the form

$$A_{d,W} = (W_1A_1W_1)^{-1} \oplus 0.$$

Let us consider the equation

$$WAWx = b, \qquad b \in R((WA)^D).$$

Then there exists a unique $x \in R((AW)^D)$ such that

$$x = A_{d,W}b.$$

We say that $B \in \mathcal{B}(X,Y)$ obeys the condition (W) at A if

$$B - A = AWA_{d,W}W(B-A)WAWA_{d,W} \quad \text{and} \quad \|\mathrm{A_{d,W}W(B-A)}\|\|\mathrm{W}\| < 1.$$

Set $E = B - A$.

If F is a continuously differentiable function

$$F : \mathcal{B}(X, Y) \times X \longrightarrow Y$$

$$(A, x) \longmapsto F(A, x),$$

the absolute condition number of F at x is the scalar $\|F'(x)\|$. The relative condition number of F at x is

$$\frac{\|F'(x)\|\|x\|_X}{\|y\|_Y}.$$

Following [2] we introduce the operator

$$F : \mathcal{B}(X, Y) \times X \longrightarrow Y$$

$$(A, b) \longmapsto F(A, b) = A_{d,W} b = x.$$

We know that the operator F is a differentiable function, if the perturbation E of A fulfils the following condition:

$$A_{d,W}(WAW)EW = EW, \quad WE(WAW)A_{d,W} = WE. \tag{1}$$

We need the following important theorem.

Theorem 1.1. *([4]) Let $A, B \in \mathcal{B}(X, Y), W \in \mathcal{B}(Y, X)$, let A be W–Drazin invertible and let B obey condition (W) at A. Then B is W–Drazin invertible, $(BW)(B_{d,W}W) = (AW)(A_{d,W}W), i(BW) = i(AW)$,*

$$B_{d,W} = (I + A_{d,W}WEW)^{-1}A_{d,W} = A_{d,W}(I + WEWA_{d,W})^{-1},$$

$$R(B_{d,W}) = R(A_{d,W}) \quad \textit{and} \quad N(B_{d,W}) = N(A_{d,W}).$$

The norm on the data is the norm in $\mathcal{B}(X, Y) \times X$ defined as

$$(A, b) \longmapsto \|[\alpha WAW, \beta b]\| = \sqrt{\alpha^2 \|WAW\|_{QP}^2 + \beta^2 \|b\|_Q^2}.$$

In [2], Cui and Diao investigated the condition number of the W-weighted Drazin inverse solution of a linear system $WAWx = b$, where A is an $m \times n$ rank deficient matrix, the index of AW is k_1, the index of WA is k_2, b is a real vector of the size n in the range of $(WA)^{k_1}$, x is a real vector of the size m in the range of $(AW)^{k_2}$. For two positive real numbers α and β, they considered the weighted Frobenius norm $\|[\alpha WAW, \beta b]\|_{Q,\tilde{P}}^{(F)}$ and gave the explicit formula of the condition number of the W-weighted Drazin inverse solution of a rectangular linear system. In this paper we extend the result obtained in [2] to linear bounded operators between Hilbert spaces.

2. Results

Now, we prove the following result.

Theorem 2.1. *If the perturbation E in A fulfills the condition* (1)*, then the absolute condition number of the W–weighted Drazin inverse solution of linear system, with the norm*

$$\|[\alpha WAW, \beta b]\| = \sqrt{\alpha^2\|WAW\|_{QP}^2 + \beta^2\|b\|_Q^2}$$

on the data (A, b) and the norm $\|x\|_P$ on the solution, satisfies

$$C \leq \|A_{d,W}\|_{PQ}\sqrt{\frac{1}{\beta^2} + \frac{\|x\|_P^2}{\alpha^2}}.$$

Let $(E_n)_n$ be a sequence of perturbations in A fulfilling the condition (1) *and let $(f_n)_n$ be a sequence of perturbations in b. If C_n is the corresponding absolute condition number, then*

$$C_n \to \|A_{d,W}\|_{PQ}\sqrt{\frac{1}{\beta^2} + \frac{\|x\|_P^2}{\alpha^2}}, \qquad n \to \infty.$$

Hence, $\|A_{d,W}\|_{PQ}\sqrt{\frac{1}{\beta^2} + \frac{\|x\|_P^2}{\alpha^2}}$ is a sharp bound.

Proof. We know that $F(A, b) = A_{d,W}b$. Under the condition (1), F is a differentiable function and F' is defined as follows

$$F'(A, b)|_{(E,f)} = \lim_{\epsilon\to 0}\frac{(A + \epsilon E)_{d,W}(b + \epsilon f) - A_{d,W}b}{\epsilon},$$

where E is the perturbation of A and f is the perturbation of b.

Since E satisfies the condition (1), we have ([4])

$$(A + \epsilon E)_{d,W} = A_{d,W} - \epsilon A_{d,W}WEWA_{d,W} + O(\epsilon^2),$$

and then we can easily get that

$$F'(A, b)|_{(E,f)} = -A_{d,W}WEWx + A_{d,W}f.$$

Then

$$\begin{aligned} \|F'(A, b)|_{(E,f)}\|_P &= \|A_{d,W}(WEWx - f)\|_P \\ &\leq \|A_{d,W}\|_{PQ}(\|WEW\|_{QP}\|x\|_P + \|f\|_Q). \end{aligned}$$

The norm of a linear map $F'(A, b)$ is the supermum of $\|F'(A, b)|_{(E,f)}\|_P$ on the unit ball of $\mathcal{B}(X, Y) \times X$. Since

$$\|[\alpha WEW, \beta f]\|^2 = \alpha^2\|WEW\|_{QP}^2 + \beta^2\|f\|_Q^2$$

we get

$$\begin{aligned}
&\|F'(A,b)\| \\
&= \sup_{\alpha^2\|WEW\|_{QP}^2+\beta^2\|f\|_Q^2\le 1} \|A_{d,W}(WEWx-f)\|_P \\
&\le \sup_{\alpha^2\|WEW\|_{QP}^2+\beta^2\|f\|_Q^2\le 1} \|A_{d,W}\|_{PQ}(\|WEW\|_{QP}\|x\|_P+\|f\|_Q) \\
&= \sup_{\alpha^2\|WEW\|_{QP}^2+\beta^2\|f\|_Q^2\le 1} \|A_{d,W}\|_{PQ}\left(\alpha\|WEW\|_{QP}\frac{\|x\|_P}{\alpha}+\beta\|f\|_Q\frac{1}{\beta}\right) \\
&= \|A_{d,W}\|_{PQ} \sup_{\alpha^2\|WEW\|_{QP}^2+\beta^2\|f\|_Q^2\le 1} (\alpha\|WEW\|_{QP},\beta\|f\|_Q)\cdot\left(\frac{\|x\|_P}{\alpha},\frac{1}{\beta}\right)
\end{aligned}$$

where $(\alpha\|WEW\|_{QP},\beta\|f\|_Q)$ and $\left(\frac{\|x\|_P}{\alpha},\frac{1}{\beta}\right)$ can be considered as vectors in R^2.

Therefore, from the Cauchy–Schwarz inequality we get:

$$\|F'(A,b)\| \le \|A_{d,W}\|_{PQ}\sqrt{\frac{\|x\|_P^2}{\alpha^2}+\frac{1}{\beta^2}}.$$

Next, we show the other part of the theorem. For a sequence $(u_n)_n$ in $R((WA)^D)$, $\|u_n\|=1$, there exists a sequence $(v_n)_n$ in $R((AW)^D)$, $\|v_n\|\le 1$ and $\lim_{n\to\infty}\|v_n\| = 1$, such that, for all $n\in N$,

$$(W_1A_1W_1)^{-1}u_n = \|(W_1A_1W_1)^{-1}\|v_n = \|A_{d,W}\|_{PQ}v_n.$$

Taking, for all $n\in N$,

$$\hat{u}_n = \begin{bmatrix} u_n \\ 0 \end{bmatrix}, \quad \hat{v}_n = \begin{bmatrix} v_n \\ 0 \end{bmatrix},$$

we obtain

$$\begin{aligned}
A_{d,W}\hat{u}_n &= \begin{bmatrix} (W_1A_1W_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_n \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} (W_1A_1W_1)^{-1}u_n \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \|(W_1A_1W_1)^{-1}\|v_n \\ 0 \end{bmatrix} \\
&= \|(W_1A_1W_1)^{-1}\|\begin{bmatrix} v_n \\ 0 \end{bmatrix} \\
&= \|A_{d,W}\|_{PQ}\hat{v}_n.
\end{aligned}$$

It is easy to check that $\|\hat{u}_n\|_Q = 1$ and $\|\hat{v}_n\|_P\le 1$, for all $n\in N$.

Let $u \in R((WA)^D)$ and $v \in R((AW)^D)$. Define $S_{u,v} \in \mathcal{B}(R((AW)^D), R((WA)^D))$ as follows: if $x \in R((AW)^D)$, then

$$S_{u,v}(x) \stackrel{\text{def}}{=} \langle x, v\rangle u.$$

For all $T \in \mathcal{B}(R((WA)^D), R((AW)^D))$ we have

$$TS_{u,v}(x) = T(u)\langle x, v\rangle.$$

Now we choos, for $n = 1, 2, 3, \dots,$

$$\eta = \sqrt{\frac{\|x\|_P^2}{\alpha^2} + \frac{1}{\beta^2}}, \quad f_n = \frac{1}{\beta^2\eta}\hat{u}_n,$$

$$E_n = -\frac{1}{\alpha^2\eta}\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} S_{u_n,x} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Then we have, for a fixed n,

$$\begin{aligned} E_nW &= -\frac{1}{\alpha^2\eta}\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} S_{u_n,x} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} \\ &= -\frac{1}{\alpha^2\eta}\begin{bmatrix} W_1^{-1}S_{u_n,x} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \\ &= -\frac{1}{\alpha^2\eta}\begin{bmatrix} W_1^{-1}S_{u_n,x} & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Since

$$A_{d,W}(WAW) = I \oplus 0,$$

we can verify that E_n fulfills the first equation of condition (1):

$$\begin{aligned} A_{d,W}(WAW)E_nW &= -\frac{1}{\alpha^2\eta}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} W_1^{-1}S_{u_n,x} & 0 \\ 0 & 0 \end{bmatrix} \\ &= -\frac{1}{\alpha^2\eta}\begin{bmatrix} W_1^{-1}S_{u_n,x} & 0 \\ 0 & 0 \end{bmatrix} \\ &= E_nW. \end{aligned}$$

In the same way we have

$$\begin{aligned} WE_n &= -\frac{1}{\alpha^2\eta}\begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} S_{u_n,x} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\ &= -\frac{1}{\alpha^2\eta}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} S_{u_n,x}W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\ &= -\frac{1}{\alpha^2\eta}\begin{bmatrix} S_{u_n,x}W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Since

$$(WAW)A_{d,W} = I \oplus 0,$$

we know

$$\begin{aligned} WE_n(WAW)A_{d,W} &= -\frac{1}{\alpha^2\eta}\begin{bmatrix} S_{u_n,x}W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \\ &= -\frac{1}{\alpha^2\eta}\begin{bmatrix} S_{u_n,x}W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\ &= WE_n. \end{aligned}$$

Thus E_n fulfills the condition (1), for all $n \in N$. Now we want to verify that the perturbation (E_n, f_n) satisfies $\alpha^2\|WE_nW\|_{QP}^2 + \beta^2\|f_n\|_Q^2 \leq 1$.

$$\begin{aligned} &\alpha^2\|WE_nW\|_{QP}^2 + \beta^2\|f_n\|_Q^2 \\ &= \frac{1}{\alpha^2\eta^2}\left\|\begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} S_{u_n,x} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\right. \\ &\left.\cdot\begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}\right\|_{QP}^2 + \frac{1}{\beta^2\eta^2}\|\hat{u}_n\|_Q^2 \\ &= \frac{1}{\alpha^2\eta^2}\left\|\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} S_{u_n,x} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right\|_{QP}^2 + \frac{1}{\beta^2\eta^2}\|\hat{u}_n\|_Q^2 \\ &= \frac{1}{\alpha^2\eta^2}\left\|\begin{bmatrix} S_{u_n,x} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right\|_{QP}^2 + \frac{1}{\beta^2\eta^2} \\ &= \frac{1}{\alpha^2\eta^2}\left\|\begin{bmatrix} S_{u_n,x} & 0 \\ 0 & 0 \end{bmatrix}\right\|_{QP}^2 + \frac{1}{\beta^2\eta^2} \\ &= \frac{1}{\alpha^2\eta^2}\|S_{u_n,x}\|^2 + \frac{1}{\beta^2\eta^2} \\ &\leq \frac{1}{\alpha^2\eta^2}\|u_n\|^2\|x\|_P^2 + \frac{1}{\beta^2\eta^2} \\ &= \frac{1}{\eta^2}\left(\frac{\|x\|_P^2}{\alpha^2} + \frac{1}{\beta^2}\right) \\ &= 1. \end{aligned}$$

Thus, we have

$$\begin{aligned}
F'(A,b)|_{(E_n,f_n)} &= -A_{d,W}WE_nWx + A_{d,W}f_n \\
&= \frac{1}{\alpha^2\eta}((W_1A_1W_1)^{-1}\oplus 0)(S_{u_n,x}\oplus 0)x + \frac{1}{\beta^2\eta}A_{d,W}\hat{u}_n \\
&= \frac{1}{\alpha^2\eta}((W_1A_1W_1)^{-1}S_{u_n,x}\oplus 0)x + \frac{1}{\beta^2\eta}A_{d,W}\hat{u}_n \\
&= \frac{1}{\alpha^2\eta}\begin{bmatrix} (W_1A_1W_1)^{-1}\langle x,x\rangle u_n \\ 0 \end{bmatrix} + \frac{1}{\beta^2\eta}A_{d,W}\hat{u}_n \\
&= \frac{1}{\alpha^2\eta}\begin{bmatrix} \|x\|_P^2(W_1A_1W_1)^{-1}u_n \\ 0 \end{bmatrix} + \frac{1}{\beta^2\eta}A_{d,W}\hat{u}_n \\
&= \frac{1}{\alpha^2\eta}\|x\|_P^2\begin{bmatrix} \|(W_1A_1W_1)^{-1}\|v_n \\ 0 \end{bmatrix} + \frac{1}{\beta^2\eta}A_{d,W}\hat{u}_n
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\alpha^2\eta}\|x\|_P^2\|(W_1A_1W_1)^{-1}\|\begin{bmatrix} v_n \\ 0 \end{bmatrix} + \frac{1}{\beta^2\eta}\|A_{d,W}\|_{PQ}\hat{v}_n \\
&= \frac{1}{\alpha^2\eta}\|x\|_P^2\|A_{d,W}\|_{PQ}\hat{v}_n + \frac{1}{\beta^2\eta}\|A_{d,W}\|_{PQ}\hat{v}_n \\
&= \frac{\|A_{d,W}\|_{PQ}}{\eta}\left(\frac{\|x\|_P^2}{\alpha^2} + \frac{1}{\beta^2}\right)\hat{v}_n \\
&= \|A_{d,W}\|_{PQ}\eta\hat{v}_n.
\end{aligned}$$

So

$$\|F'(A,b)|_{(E_n,f_n)}\|_P \to \|A_{d,W}\|_{PQ}\sqrt{\frac{\|x\|_P^2}{\alpha^2} + \frac{1}{\beta^2}} \qquad (n\to\infty),$$

with $\alpha^2\|WE_nW\|_{QP}^2 + \beta^2\|f_n\|_Q^2 \le 1$, we get

$$\|F'(A,b)\| \to \|A_{d,W}\|_{PQ}\sqrt{\frac{\|x\|_P^2}{\alpha^2} + \frac{1}{\beta^2}}, \qquad (n\to\infty)$$

and we complete the proof. □

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