NOVI SAD J. MATH.
VOL. 39, NO. 1, 2009, 57-64

# A NOTE ON THE INTERSECTION OF A RADICAL CLASS WITH THE SUM OF RADICAL CLASSES OF HEMIRINGS

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**Abstract.** We extend the notion of intersection of a radical class with the sum of radical classes of rings due to Y. Lee and R. E. Propes (see [3, 4]) to the intersection of a radical class with the sum of radical classes of hemirings. A few results of (see [1, 3, 4]) can be concluded from this paper.

*AMS Mathematics Subject Classification (2000)*: 16Y60, 16W50

*Key words and phrases:* hemiring, sum of radical classes,universal class, accessible sub-hemiring, Yu Lee construction, intersection of radical classes,lower radical, semisimple classes

## 1. Introduction

The notion of radical classes of hemirings was introduced by D. M. Olson and T. L. Jenkins [5], as an extension of radical classes of rings (see [3]). The theory was further enriched by many authors (see [5, 6]).

Y. Lee and R. E. Propes [3] introduced the concept of the sum of two radical classes of rings. They have shown that the 'sum' is not a radical class in general. In [6], M. Zulfiqar generalized a few results of [3]. In the present paper, we extend the notion of intersection of a radical class with the sum of radical classes of hemirings and generalize a few results of (see [1, 3, 4]) in the framework of hemirings. By this extension of radical classes of rings (see [1, 3, 4]), a few results of radical classes of rings can be generalized. In the following we shall be working within the class of all hemirings.

A semiring (A, +, .) is called a hemiring if
(i) '+' is commutative
(ii) there exists an element 0 $\varepsilon$ A such that 0 is the identity of (A, +) and the zero element of (A, .).

$$i.e. 0a = a0 = 0, \forall\ a \,\varepsilon\, A$$

Let $\rho_1$, $\rho_2$ be radical classes of hemirings, then we define their sum

$$\rho_1 + \rho_2 = \{A \,\varepsilon\, \mu : \rho_1(A) + \rho_2(A) = A\}.$$

Lower radical classes for hemiring can be constructed similarly to the construction of lower radicals for rings (see [2]).

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Let A, B $\varepsilon$ $\mu$, and B $\subseteq$ A, B is said to be an accessible sub-hemiring of A if there exists a chain $C_0$, $C_1$, ... ,$C_n$ such that

$$B = C_n \leq C_{n-1} \leq C_{n-2} \leq ... \leq C_1 \leq C_0 = A.$$

Let $D_1(A) =$ set of all ideals of $A$, inductively defined

$$D_{n+1}(A) = \{C \,\varepsilon\, A : C \leq B \ for \ B \,\varepsilon\, D_n(A)\}$$

Put $D(A) = \bigcup_{n \in N} D_n(A)$, then $D(A)$ is the collection of all accessible sub-hemirings of $A$.

The lower radical for hemirings can be constructed along the ring theoretical lines (see [2, 6]).

If $A$ is a homomorphically closed class of hemirings, then its lower radical class $LA$ can be constructed on the ring theoretical lines. If

$$YA = \{A\varepsilon\mu : \text{every non-zero homomorphic image of } A \text{ has a non-zero accessible } A - \text{sub-hemiring}\}$$

then it can be established, in a manner similar to that of rings, that $YA = LA$.

## 2. Results

**Definition 1.** [6] Let $\rho_1$ and $\rho_2$ be radical classes in $\mu$. We define

$$\rho_1 + \rho_2 = \{A \,\varepsilon\, \mu : \rho_1(A) + \rho_2(A) = A\}$$

We write ( $\rho_1$ + $\rho_2$)(A) = $\rho_1$(A) + $\rho_2$(A) for all A $\varepsilon$ $\mu$.

The following theorem can be obtained on the lines of direction in [3].

**Theorem 2.** $\rho_1 \cup \rho_2 \subseteq \rho_1 + \rho_2$

As $\rho_1 \cup \rho_2$ is a homomorphically closed class, therefore, we can consider its lower radical class L($\rho_1 \cup \rho_2$). The following theorem was proved by Yu-Lee Lee and R.E. Propes [3] and we generalize it in the framework of hemiring. Here we give a proof of this theorem, which is entirely different from [3].

**Theorem 3.** *$\rho_1 + \rho_2 \subseteq L(\rho_1 \cup \rho_2)$*

*Proof.* Let A $\varepsilon$ $\rho_1$ + $\rho_2$. We claim that A $\varepsilon$ L($\rho_1 \cup \rho_2$), on the contrary suppose that A $\notin$ L($\rho_1 \cup \rho_2$). Observe that $\rho_1 \cup \rho_2$ is homomorphically closed. Therefore L($\rho_1 \cup \rho_2$) exists and

$$L(\rho_1 \cup \rho_2) = Y(\rho_1 \cup \rho_2)$$

Let A $\notin$ Y($\rho_1 \cup \rho_2$). Since $\rho_1 \cup \rho_2$ is homomorphically closed, L($\rho_1 \cup \rho_2$) = Y($\rho_1 \cup \rho_2$) and

$$Y(\rho_1 \cup \rho_2) = \{A \,\varepsilon\, \mu : D(A/I) \cap (\rho_1 \cup \rho_2) \neq 0, \forall (0 \neq A/I) \,\varepsilon\, HA\}$$

This implies that there exists I $\leq$ A such that I $\neq$ A.

$$\Rightarrow D(A/I) \cap (\rho_1 \cup \rho_2) = 0$$

$$\Rightarrow D(A/I) \cap \rho_1 = 0 \;\; and \;\; D(A/I) \cap \rho_2 = 0$$

$$\Rightarrow D_1(A/I) \cap \rho_1 = 0, D_1(A/I) \cap \rho_2 = 0 \; (\therefore D_1(A/I) \subseteq D(A/I))$$

$$\Rightarrow \rho_1(A/I) = 0, \rho_2(A/I) = 0$$

Let $\varphi$(A) = A / I, $\rho_1(\varphi$(A)) = 0, then we have

$$\varphi(\rho_1(A) + \rho_2(A)) \subseteq \rho_1(\varphi(A)) + \rho_2(\varphi(A)) \;\; \text{(see [5, Lemma 5] )}$$

$$\varphi(A) \subseteq \rho_1(A/I) + \rho_2(A/I) = 0(:\,.\rho_1(A/I) = 0, \rho_2(A/I) = 0)$$

$$\varphi(A) = 0$$

This implies that A / I = 0 and hence a contradiction. Consequently, we have A $\varepsilon$ L($\rho_1$ $\cup$ $\rho_2$). Therefore

$$\rho_1 + \rho_2 \subseteq L(\rho_1 \cup \rho_2).$$

□

**Remark 4.** Since L($\rho_1 \cup \rho_2$) is the smallest radical class containing both $\rho_1$ and $\rho_2$, it follows that $\rho_1$ + $\rho_2$ is a radical class if and only if

$$\rho_1 + \rho_2 = L(\rho_1 \cup \rho_2) \;\;\; \text{(by Theorem 3)}$$

**Theorem 5.** *[6] The class $\rho_1$ + $\rho_2$ is homomorphically closed.*

As $\rho_1$ + $\rho_2$ is a homomorphically closed class, we can define its lower radical class L($\rho_1$ + $\rho_2$).

**Theorem 6.** *L($\rho_1$ + $\rho_2$) = L($\rho_1 \cup \rho_2$).*

*Proof.* Since L($\rho_1$ + $\rho_2$) is the smallest radical class containing both $\rho_1$ + $\rho_2$. But $\rho_1$ + $\rho_2$ $\subseteq$ L($\rho_1 \cup \rho_2$) (by theorem 3) and hence we have

(1) $$L(\rho_1 + \rho_2) \subseteq L(\rho_1 \cup \rho_2)$$

For reverse inclusion, observe that

(2) $$\begin{gathered} \rho_1 \cup \rho_2 \subseteq \rho_1 + \rho_2 \;\;\; \text{(by Theorem 2)} \\ \Rightarrow L(\rho_1 \cup \rho_2) \subseteq L(\rho_1 + \rho_2) \end{gathered}$$

From equation (1) and (2), we get

$$L(\rho_1 + \rho_2) = L(\rho_1 \cup \rho_2)$$

**Definition 7.** [6] Let $\rho_1 + \rho_2$ be a radical class. Then

$$S(\rho_1 + \rho_2) = \{A \ \varepsilon \ \omega : (\rho_1 + \rho_2)(A) = 0\}$$

We now investigate conditions under which $\rho_1 + \rho_2$ will be a radical class.

**Theorem 8.** *[6] If $\rho_1$ and $\rho_2$ are radical classes and $S\rho_1 \cap \rho_2 = 0$, then $\rho_1 + \rho_2$ is a radical class.*

The above result can be extended in the following form :

**Theorem 9.** *If $S\rho_i \cap \sum_{i=1}^{n} \rho_i = 0$, then $\sum_{i=1}^{n} \rho_i$ is a radical class.*

*Proof.* Since

$$\sum_{i=1}^{n} \rho_i \subseteq L(\sum_{i=1}^{n} \rho_i) \tag{3}$$

For reverse inclusion, we proceed as follows. Let A $\varepsilon$ $\omega$ such that

$$\begin{aligned} A \notin \sum_{i=1}^{n} \rho_i \quad &\Rightarrow \quad \sum_{i=1}^{n} \rho_i(A) \neq A \\ &\Rightarrow \quad A \notin \rho_1 \\ &\Rightarrow \quad 0 \neq A/\rho_1(A). \end{aligned}$$

Now consider

$$D(A/\rho_1(A)) \cap \sum_{i=1}^{n} \rho_i$$

From the proof of Theorem 8, it follows that

$$D(A/\rho_1(A)) \cap \sum_{i=1}^{n} \rho_i = 0$$

Hence

$$A \notin L(\sum_{i=1}^{n} \rho_i)$$

Thus A $\notin \sum_{i=1}^{n} \rho_i$ implies that A $\notin$ L($\sum_{i=1}^{n} \rho_i$)

Hence

$$L(\sum_{i=1}^{n} \rho_i) \subseteq \sum_{i=1}^{n} \rho_i \tag{4}$$

From equations (3) and (4), we conclude that

$$L(\sum_{i=1}^{n} \rho_i) = \sum_{i=1}^{n} \rho_i.$$

Hence $\sum_{i=1}^{n} \rho_i$ is a radical class. □

The following theorem was proved by Yu-Lee Lee and R.E. Propes [4] and we generalize it in the framework of hemiring. Here we give a proof of this theorem which is entirely different from [4].

**Theorem 10.** *Let $\rho_1$ and $\rho_2$ be radical classes in some universal class $\mu$ of hemirings and define $\rho$(A) = $\rho_1$(A) ∩ $\rho_1$(A), and set*

$$\rho = \{A \,\varepsilon\, \mu : \rho(A) = A\}.$$

*Then $\rho$ = $\rho_1 \cap \rho_2$ and $\rho$ is a radical class of hemirings.*

*Proof.* i) Let A $\varepsilon$ $\rho$ and let $\bar{A}$ $\varepsilon$ HA. Then

$$A \,\varepsilon\, \rho_1 \cap \rho_2$$

$$\Rightarrow A \,\varepsilon\, \rho_1 \text{ and } A \,\varepsilon\, \rho_2$$

Since $\rho_1$ and $\rho_2$ are radical classes, by [5], we have

$$\bar{A} \,\varepsilon\, \rho_1 \text{ and } \bar{A} \,\varepsilon\, \rho_2$$

$$\Rightarrow \bar{A} \;\varepsilon\; \rho_1 \cap \rho_2 = \rho$$

$$\Rightarrow HA \subseteq \rho$$

Thus $\rho$ is homomorphically closed.
ii) Let $\{I_a\}_{\alpha \,\varepsilon \Lambda}$ be a family of $\rho$-semi-ideals of the hemiring A.

$$I_a \,\varepsilon\, \rho = \rho_1 \cap \rho_2 \;\forall \alpha \,\varepsilon\, \Lambda$$

$$\Rightarrow I_a \,\varepsilon\, \rho_1 \text{ and } I_a \,\varepsilon\, \rho_2 \;\; \forall\, \alpha \,\varepsilon\, \Lambda$$

Since $\rho_1$ and $\rho_2$ are radical classes, then

$$\sum_{\alpha \,\varepsilon\, \Lambda} I_a \,\varepsilon\, \rho_1 \text{ and } \sum_{\alpha \,\varepsilon \Lambda} I_a \,\varepsilon\, \rho_2$$

$$\Rightarrow \sum_{\alpha \,\varepsilon\, \Lambda} I_a \,\varepsilon\, \rho_1 \cap \rho_2$$

$$\Rightarrow \sum_{\alpha \,\varepsilon\, \Lambda} I_a \,\varepsilon\, \rho$$

Thus maximal $\rho$-semi-ideal, namely $\rho$(A) exists.
iii) Let A be a hemiring and I $\leq$ A such that A / I $\varepsilon$ $\rho$, I $\varepsilon$ $\rho$. Then we have

$$\begin{aligned} A/I \,\varepsilon\, \rho_1 \cap \rho_2, I \,\varepsilon\, \rho_1 \cap \rho_2 \;\; &\Rightarrow \;\; A/I \,\varepsilon\, \rho_1, I \,\varepsilon\, \rho_1 \text{ and } A/I \,\varepsilon\, \rho_2, I \,\varepsilon\, \rho_2 \\ &\Rightarrow \;\; A \,\varepsilon\, \rho_1 \text{ and } A \,\varepsilon\, \rho_2 \\ &\Rightarrow \;\; A \,\varepsilon\, \rho_1 \cap \rho_2 = \rho \\ &\Rightarrow \;\; A \,\varepsilon\, \rho. \end{aligned}$$

By [5] we can conclude that $\rho$ is a radical class.

Next we shall show that

$$\rho(A) = \rho_1(A) \cap \rho_2(A)$$

Let

$$\rho = \{A \,\varepsilon\, \mu : \rho(A) = A\}.$$

Then

$$\begin{aligned} A \,\varepsilon\, \rho \quad &\Leftrightarrow \quad \rho_1(A) \cap \rho_2(A) = \rho \\ &\Leftrightarrow \quad \rho_1(A) = A \text{ and } \rho_2(A) = A \\ &\Leftrightarrow \quad A \,\varepsilon\, \rho_1 \text{ and } A \,\varepsilon\, \rho_2 \\ &\Leftrightarrow \quad A \,\varepsilon\, \rho_1 \cap \rho_2 \\ &\Leftrightarrow \quad A \,\varepsilon\, \rho. \end{aligned}$$

Hence

$$\rho = \{A \,\varepsilon\, \mu : \rho(A) = A\}$$

Thus

$$\rho(A) = \rho_1(A) \cap \rho_2(A)$$

Hence $\rho = \rho_1 \cap \rho_2$ , clearly $\rho_1 \cap \rho_2$ is a radical class and this completes the proof. □

The following theorem was proved by David M. Burton [1] and we generalize it in the framework of hemiring.

**Theorem 11.** *Let $\rho_1$, $\rho_2$ and $\rho_3$ be radical classes of hemiring, then*

$$\rho_1 \cap (\rho_2 + \rho_3) = \rho_1 \cap \rho_2 + \rho_1 \cap \rho_3$$

*Proof.* Let

$$\begin{aligned} A\varepsilon\rho_1 \cap (\rho_2 + \rho_3) \quad &\Rightarrow \quad A\varepsilon\rho_1 \text{ and } A\varepsilon\rho_2 + \rho_3 \\ &\Rightarrow \quad \rho_1(A) = A \text{ and } \rho_2(A) + \rho_3(A) = A \\ &\Rightarrow \quad \rho_2(A) \subseteq \rho_1(A). \end{aligned}$$

Thus we have

$$\begin{aligned} \rho_1(A) \cap (\rho_2(A) + \rho_3(A)) \quad &= \quad \rho_1(A) \cap \rho_2(A) + \rho_1(A) \cap \rho_3(A) \\ &= \quad A \cap \rho_2(A) + A \cap \rho_3(A) (\text{by } \rho_1(A) = A) \\ &= \quad \rho_2(A) + \rho_3(A) \\ &= \quad A. \end{aligned}$$

Now

$$\begin{aligned} &\rho_1(A) \cap \rho_2(A) + \rho_1(A) \cap \rho_3(A) = A \\ &\Rightarrow \quad (\rho_1 \cap \rho_2)(A) + (\rho_1 \cap \rho_3)(A) = A (\text{by Theorem 10}) \\ &\Rightarrow \quad (\rho_1 \cap \rho_2 + \rho_1 \cap \rho_3)(A) = A \\ &\Rightarrow \quad A\varepsilon(\rho_1 \cap \rho_2 + \rho_1 \cap \rho_3). \end{aligned}$$

Hence

(5) $$\rho_1 \cap (\rho_2 + \rho_3) \subseteq \rho_1 \cap \rho_2 + \rho_1 \cap \rho_3$$

Conversely, assume that

$$\begin{aligned} & A\varepsilon(\rho_1 \cap \rho_2 + \rho_1 \cap \rho_3) \\ & \Rightarrow \quad (\rho_1 \cap \rho_2 + \rho_1 \cap \rho_3)(A) = A \\ & \Rightarrow \quad (\rho_1 \cap \rho_2)(A) + (\rho_1 \cap \rho_3)(A) = A \\ & \Rightarrow \quad \rho_1(A) \cap \rho_2(A) + \rho_1(A) \cap \rho_3(A) = A \quad (6) \\ & \Rightarrow \quad \rho_1(A) \cap [\rho_1(A) \cap \rho_2(A) + \rho_1(A) \cap \rho_3(A)] = \rho_1(A) \cap A = \rho_1(A) \end{aligned}$$

Since

$$\rho_1(A) \cap \rho_2(A) \subseteq \rho_1(A)$$

So we have

$$\begin{aligned} & \rho_1(A) \cap \rho_1(A) \cap \rho_2(A) + \rho_1(A) \cap \rho_1(A) \cap \rho_3(A) = \rho_1(A) \\ & \Rightarrow \quad \rho_1(A) \cap \rho_2(A) + \rho_1(A) \cap \rho_3(A) = \rho_1(A) \\ & \Rightarrow \quad (\rho_1 \cap \rho_2)A + (\rho_1 \cap \rho_3)A = \rho_1(A) \\ & \Rightarrow \quad A = \rho_1(A) \text{ (Using equation (6))} \quad (7) \\ & \Rightarrow \quad A\varepsilon\rho_1. \end{aligned}$$

By equation (6) and (7), we have

$$\begin{aligned} & \rho_1(A) \cap \rho_2(A) + \rho_1(A) \cap \rho_3(A) = A \\ & \Rightarrow \quad A \cap \rho_2(A) + A \cap \rho_3(A) = A \\ & \Rightarrow \quad \rho_2(A) + \rho_3(A) = A \\ & \Rightarrow \quad (\rho_2 + \rho_3)(A) = A \\ & \Rightarrow \quad A\varepsilon\rho_2 + \rho_3. \end{aligned}$$

Hence

$$A \,\varepsilon\, \rho_1 \cap (\rho_2 + \rho_3)$$

or

(8) $$\rho_1 \cap \rho_2 + \rho_1 \cap \rho_3 \subseteq \rho_1 \cap (\rho_2 + \rho_3)$$

By equation (5) and (8), we have

$$\rho_1 \cap (\rho_2 + \rho_3) = \rho_1 \cap \rho_2 + \rho_1 \cap \rho_3$$

This completes the proof. □

## Acknowledgement

The author thank the referee for his useful comments and suggestions for the improvement of the paper.

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*Received by the editors July 12, 2008*