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A NOTE ON RADICALS OF SEMINEAR-RINGS

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Abstract. We generalize a few results of [2, 6, 8] for radical classes of rings for radical classes of seminear-rings by using the construction for radical classes of seminear-rings.

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1. Introduction and Preliminaries

V. G. Van Hoorn and B. Van Rootselaar [7] discussed general theory of seminear-rings. The theory was further enriched by many authors (see [1, 3, 4, 9, 10]).

The construction of upper radicals classes was investigate in [8] for radical classes of rings. Here we generalize a few results of (see [2,6,8]) in the framework of seminear-ring which is quite different from ring theoretical approach discussed in (see [2, 6, 8]). In the following we shall be working within the class of all seminear-rings.

A set S, together with two binary operations '+' and '.', is called a seminear-ring if

(i) $(S, +)$ is a semigroup;

(ii) $(S, .)$ is a semigroup;

(iii) $(s + s')s'' = ss'' + s's''$, $\forall s, s', s'' \in S$.

Upper radical classes for seminear-rings can be constructed similar to the construction of upper radicals for rings (see [2, 6, 8]). If S is a seminear-ring then $H(S)$, $K(S)$ denote the set of all homomorphic images of S and the set of all k-semi-ideals of S respectively. If I is an k-semi-ideals of S, then we denote I $\leq_k S$. Throughout this we used the k-semi-ideal, we generalized a many results of R. Wiegandt ([8], see also [2, 6]).

A radical class is defined as follows:

A non-empty subclass ρ of universal class of seminear-rings ω is called a radical class if;

(R_1) ρ is homomorphically closed.

(R_2) If $S \notin \rho$, then, there exists $K \in K(S)$ such that $K(S/K) \cap \rho = 0$ and $K \neq S$.

Henceforth, whenever ρ is used, it represents a radical class of seminear-rings, unless mentioned otherwise. For undefined terms of seminear-rings we may refer to [1, 3, 4, 5, 6, 7, 9, 10, 11].

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2. Radical and Semisimple Classes

The following definition is taken from D. M. Olson and T. L. Jenksins [5]. The maximal ρ-semi-ideal of a seminear-ring will be called the ρ-radical of S and designated $\rho(S)$. If $\rho(S)$ = 0, S is called semisimple.

The semisimple class $S\rho$ of a radical class ρ is defined as the class of all seminear-rings having zero ρ-radical.

Definition 2.1. Let $S \subseteq \omega$; S is said to be a semisimple class of seminear-ring, if the following two axioms are satisfied

(S_1) S $\in S \Rightarrow$ HI $\cap\, S \neq 0, \forall$ (0 $\neq$ I) $\in K(S)$

(S_2) Let S $\in \omega$ such that HI $\cap\, S \neq 0, \forall$ (I $\neq$ 0) $\leq_k$ S, then S $\in S$.

A subclass of seminear-rings ω satisfying the condition (S_1) is called a regular class.

Definition 2.2. A radical class ρ of seminear-ring is called hereditary if ρ is closed under semi-ideals, i.e. if $K(S) \subseteq \rho$, for all S $\in \rho$.

Observation 2.3. ρ is hereditary if and only if $K(S) \subseteq \rho$, for all $S \in \rho$.

In the case of seminear-rings one can easily prove all the standard results concerning radical classes, upper radicals, semisimple classes and semisimple closure. Here we mention only a few of them, which can be obtained on the lines of rings theoretical approach.

Proposition 2.4. *Let ρ be a radical class of seminear-ring and $S\rho$ = $\{S \in \omega : \rho(S)$ = 0$\}$. Then $S\rho$ is a semisimple class of seminear-ring.*

Theorem 2.5. *Let M be a regular class of seminear-ring. Define*

$$\bar{M} = \{S \in \omega : HI \cap M \neq 0, \forall (I \neq 0) \leq_k S\}$$

Then class $\bar{M}$ is a semisimple class.

Theorem 2.6. *Let M of seminear-rings be a regular class of seminear-ring, If*

$$UM = \{S \in \omega : H(S) \cap M = 0\},$$

then the class UM is a radical class and is called the upper radical class determined by the class M.

Theorem 2.7. *Every semisimple class S is the semisimple class of its upper radical class, i.e. S = $S(US)$.*

Colorallary 2.8. *Every semisimple class of seminear-rings is hereditary.*

Colorallary 2.9. *Let M be a regular class of seminear-rings, then SUM = $SU\bar{M}$ = $\bar{M}$.*

Colorallary 2.10. *Let M be a regular class of seminear-ring. Then*

$$UM = \{S \in \omega : H(S) \cap M = 0\}$$

is hereditary if and only if M satisfies the condition:

$(**)$ *Let $(I \neq 0) \in K(S)$ such that $HI \cap M \neq 0,$ then $H(S) \cap M \neq 0.$*

The ring theoretical version of the following theorem was proved by N. Divinsky [2]. Here we generalize it in the framework of seminear-rings. The proof of the following theorem is quite different from [2].

Theorem 2.11. *A class S is a semisimple class of seminear-ring if and only if it satisfies*
(S) If $S \in S$, then HI $\cap\, S \neq 0, \forall$ $(I \neq 0) \in K(S)$*
*(S**) Let $S \in \omega$, if HI $\cap\, S \neq 0$, $\forall (I \neq 0) \in K(S)$, then $S \in S$.*

Proof. Assume that S is a semisimple class of seminear-ring, we have to establish (S*) and (S**).

(S*) Let S $\in S$. As S is hereditary, therefore $K(S) \subseteq S$ (by definition 2.2). Let (I $\neq$ 0) $\in K(S)$, then I $\in S$. Since I $\in$ HI (by identity mapping). This shows that (0 $\neq$ I) $\in$ HI $\cap\, S$ and hence HI $\cap\, S \neq 0$, for all (0 $\neq$ I) $\in K(S)$. This established the (S*).

(S**) Let $S \in \omega$, HI $\cap\, S \neq 0$, $\forall$ (I $\neq$ 0) $\in K(S)$. Since $K_1(S) \subseteq K(S)$. This shows that HI $\cap\, S \neq 0$, $\forall$ (I $\neq$ 0) $\in K(S)$. As S is semisimple, therefore by S_2, we have S $\in S$.

Conversely, assume that (S*) and (S**) are valid for S.

(S_1) Let S $\in S$, (I $\neq$ 0) $\in K_1(S)$. Since $K_1(S) \subseteq K(S)$. By (S*), we have HI $\cap\, S \neq 0$, $\forall$ (I $\neq$ 0) $\in K(S)$. Therefore S_1 is satisfied.

(S_2) Let S $\in \omega$, HI $\cap\, S \neq 0$,$\forall$ (I $\neq$ 0) $\in K(S)$ (*)

We will show that S $\in S$. Suppose on the contrary S $\notin S = S(US) = S\rho$ where ρ = US and hence S $\notin S\rho$. Therefore, $\rho(S) \neq 0$. Let I = $\rho(S)$. Now I $\in K(S)$, I $\in \rho$ imply that HI $\subseteq \rho$ (ρ is homorphically closed) HI $\cap\, S \subseteq \rho \cap S\rho$ = 0 ($\therefore S = S(US)$ = $S\rho$ (by Theorem 2.7)). This implies that H($\rho(S)$) $\cap\, S$ = 0 ($\rho(S) \neq 0$) $\in K(S)$. This contradicts (*) and hence S $\in S$. Thus (S_2) is satisfied. Consequently, S is a semisimple class (c.f. Definition 2.1). This completes the proof. □

The following theorem provides a necessary and sufficient condition for the given radical class ρ of seminear-ring to satisfy the upper radicals associated to a certain class of seminear-rings, which is indeed an extension of [8].

Theorem 2.12. *Let M be a regular class of seminear-ring. The radical class UM hereditary if and only if the following condition is satisfied.*

$(***)$ $$S \in UM \Leftrightarrow HI \cap \bar{M} = 0, \forall (I \neq 0) \in K(S).$$

Proof. Suppose the upper radical class UM is hereditary. Let $S \in UM$, (I $\neq$ 0) $\subseteq K(S)$. Since $U\bar{M}$ is hereditary, therefore $K(S) \subseteq UM$ (by Observation 2.3). This implies that (I $\neq$ 0) $\in K(S)$ and hence I $\in UM = U\bar{M}$. Consequently, we have HI $\cap$ $\bar{M}$ = 0. Conversely, assume that UM satisfies the condition (***), we will show that UM is hereditary. Let I $\leq_k S$. If I = 0, then I $\in UM$. Suppose I $\neq$ 0, I $\in K_1(S) \subseteq$ $K(S)$, then by condition (***), we have HI $\cap$ $\bar{M}$ = 0. This implies that I $\in U\bar{M}$ and hence I $\in UM$. This shows that UM is hereditary.

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