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ON SOME CONVERGENCES FOR NETS OF FUNCTIONS WITH VALUES IN GENERALIZED UNIFORM SPACES

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Abstract. In this paper, we consider the almost uniform convergence and quasi-uniform and almost quasi-uniform ones for nets of functions with values in generalized uniform spaces. For such nets, the continuity and quasi-continuity of pointwise limit are studied.

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1. Introduction

In [4] the quasi-uniform convergence in the sense of P.S. Alexandrov [1] was defined for nets of functions with values in uniform spaces and some characterizations of the continuity of pointwise limit for such nets were given.

The purpose of the present paper is to extend these results and the quasi-continuity property in the sense of Kempisty [8] to the nets of functions with values in generalized uniform spaces.

2. Preliminaries

Let X be a nonempty set and α, β coverings of X. Following J. W. Tuckey [15] and K. Morita [10], α is called a refinement of the covering β (denoted $\alpha \prec \beta$) iff for each $A \in \alpha$ there exists $B \in \beta$ such that $A \subset B$. By $St(x, \alpha)$ is denoted the union $\bigcup\{A \in \alpha : x \in A\}$ and it is called the star of x with respect to α. For $M \subset X$ the star of M with respect to α is the set

$$St(M, \alpha) = St^1(M, \alpha) = \bigcup\{A \in \alpha : A \cap M \neq \emptyset\} = \bigcup_{x \in M} St(x, \alpha).$$

By recurrence we define $St^{n+1}(M, \alpha) = St^1(St^n(M, \alpha), \alpha)$ for $n = 1, 2, ...$

A family Σ of coverings of X is called a generalized uniform structure iff for any $\alpha, \beta \in \Sigma$ there exists $\gamma \in \Sigma$ such that $\gamma \prec \alpha$, $\gamma \prec \beta$.
Especially, Σ is called regular if:

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(a) For any $x \in X$ and for any finite family $\{A_i\}_{i=\overline{1,n}}$ with $A_i \in \alpha_i \in \Sigma$, and $x \in \bigcap_{i=1}^{n} A_i$, there exists $\beta \in \Sigma$ such that $St(x, \beta) \subset \bigcap_{i=1}^{n} A_i$;

(b) For any $\alpha \in \Sigma$ there exists $\beta \in \Sigma$ with the property that for any $B \in \beta$ there exist $\gamma_B \in \Sigma$ and $A_B \in \alpha$ such that $St(B, \gamma_B) \subset A_B$.

If Σ is a generalized uniform structure for X, then (X, Σ) is called a generalized uniform space.

For a generalized uniform space (X, Σ) we will use the topology τ_Σ [14] determined by the subbase $\{A \subset X : A \in \alpha, \alpha \in \Sigma\}$.

Let X and Y be topological spaces. Following S. Marcus [9] a function $f : X \to Y$ is said to be quasi-continuous at $x \in X$ if for each neighbourhood U of x and each neighbourhood V of $f(x)$ there exists a nonempty open set, $G \subset U$ such that $f(G) \subset V$. The function f is said to be quasi-continuous on X if it is quasi-continuous at each $x \in X$.

Lemma 2.1. *Fix a natural number, $n \geq 1$. Then a function f defined from a topological space (X, τ) into a regular generalized uniform space (Y, Σ) is quasi-continuous if and only if for any $(x_0, \alpha) \in X \times \Sigma$ and for each neighbourhood V of x_0, there exists a nonempty open set $G \subset V$ such that $f(G) \subset St^n(f(x_0), \alpha)$.*

Proof. Suppose that f is quasi-continuous on X. Let $(x_0, \alpha) \in X \times \Sigma$ and V an arbitrary neighbourhood of x_0. Then, since $f(x_0) \in St^n(f(x_0), \alpha) \in \tau_\Sigma$, there exists a nonempty open set $G \subset V$ such that $f(G) \subset St^n(f(x_0), \alpha)$.

Conversely, suppose that for any $(x_0, \alpha) \in X \times \Sigma$, and for every neighbourhood V of x_0 there exists $G \subset V$ such that $f(G) \subset St^n(f(x_0), \alpha)$. Let $x_0 \in X$ and $U \in \tau_\Sigma$ with $f(x_0) \in U$. Then there exists $\beta \in \Sigma$ such that $f(x_0) \in St^n(f(x_0), \beta) \subset U$ (by Corollary 1.4 [5]). If V is an arbitrary neighbourhood of x_0, then there exists a nonempty open set $G \subset V$ such that $f(x) \in St^n(f(x_0), \beta) \subset U$ for every $x \in G$, that is f is quasi-continuous at x_0, whence f is quasi-continuous on X. □

Remark 2.1. Analogously, it can be proved that if $n \in \mathbb{N}$, $n \geq 1$, then a function $f : X \to Y$ is continuous on X iff for any $x_0 \in X$ and $\alpha \in \Sigma$ there exists a neighbourhood V of x_0 such that $f(V) \subset St^n(f(x_0), \alpha)$.

Let $(I, \geq)$, $(J, \geq')$ be two directed sets. Following J.L. Kelley [7], a function $h : J \to I$ is said to be a "K-application" iff for each $i \in I$ there exists $j \in J$ such that for every $j' \in J$, $j' \geq' j$ it follows $h(j') \geq i$.

3. Almost uniform convergence

Let (X, τ) be a topological space, (Y, Σ) a generalized uniform space, and $(I, \geq)$ a directed set.

Definition 3.1. *[5]* A net $(f_i)_{i\in I}$ of functions defined on X into Y converges almost uniformly to a function $f : X \to Y$ if for each $x_0 \in X$ and $\alpha \in \Sigma$ there exist $i_0 \in I$ and a neighbourhood V of x_0 such that

$$\{\{f(x), f_i(x)\} : x \in V, i \in I, i \geq i_0\} \prec \alpha. \tag{1}$$

This definition is equivalent to the following one:

Definition 3.2. A net $(f_i)_{i\in I}$, $f_i : X \to Y$, converges almost uniformly to $f : X \to Y$ if for each $x_0 \in X$ and $\alpha \in \Sigma$ there exist $i_0 \in I$ and a neighbourhood V of x_0 such that

$$f_i(x) \in St(f(x), \alpha), \text{ for every } x \in V \text{ and } i \in I,\ i \geq i_0. \tag{2}$$

Indeed, if relation (1) holds, then for $i \in I$, $i \geq i_0$ and $x \in V$ it results that there exists $A \in \alpha$ such that $\{f(x), f_i(x)\} \subset A$, that is $f_i(x) \in St(f(x), \alpha)$.

Conversely, if (2) holds, then for $i \in I$, $i \geq i_0$ and $x \in V$, there exists $A \in \alpha$ such that $f(x) \in A$ and $f_i(x) \in A$, that is $\{f(x), f_i(x)\} \subset A$. Therefore (1) holds.

It is known that the limit of an almost uniform convergent net of continuous functions is continuous too.

Now, we give a similar result for the preservation of the quasi-continuity property, as follows:

Theorem 3.1. *Let (Y, Σ) be a regular generalized uniform space. If a net $(f_i)_{i\in I}$ of quasi-continuous functions defined from X into Y converges almost uniformly to a function $f : X \to Y$, then f is quasi-continuous.*

Proof. Suppose that $(f_i)_{i\in I}$ converges almost uniformly to $f : X \to Y$ and that f_i is quasi-continuous on X for each $i \in I$ and let $(x_0, \alpha) \in X \times \Sigma$. Then there exist $i_0 \in I$ and V_1 a neighbourhood of x_0 such that

$$f_i(x) \in St(f(x), \alpha), \text{ for } i \in I,\ i \geq i_0 \text{ and } x \in V_1. \tag{3}$$

Let $i \in I$, $i \geq i_0$, fixed and V an arbitrary neighbourhood of x_0. Since f_i is quasi-continuous there exists a nonempty open set $G \subset V' = V_1 \cap V$ such that

$$f_i(x) \in St(f_i(x_0), \alpha), \text{ for } x \in G. \tag{4}$$

Hence, using successively relations (3), (4), (3), we obtain:

$$f(x) \in St(f_i(x), \alpha) \subset St^2(f_i(x_0), \alpha) \subset St^3(f(x_0), \alpha)$$

for every $x \in G$. Therefore, by Lemma 2.1, the quasi-continuity of f results. □

In general, the converse of this theorem is not true. Thus, it is known that there exist examples of sequences of continuous real functions defined on a compact set that converge pointwise to continuous functions but they do not converge uniformly [6]. Taking also into account that the almost uniform convergence is equivalent to the uniform one on compact spaces [5] and that the continuity of functions implies the quasi-continuity, the statement is justified.

In the next theorem we shall show how the almost uniform convergence characterizes the quasi-continuity of pointwise limit under some conditions which do not suppose by all means the quasi-continuity of net's terms.

Theorem 3.2. *If a net $(f_i)_{i\in I}$ of functions defined on X into a regular generalized uniform space (Y,Σ) converges almost uniformly to a function $f : X \to Y$, then f is quasi-continuous on X if and only if for any $(x_0,\alpha,i) \in X\times\Sigma\times I$ there exists $i_1 \in I$, $i_1 \geq i$ such that, for each neighbourhood V of x_0, there exists a nonempty open set $G \subset V$ with the property*

$$f_{i_1}(x) \in St(f_{i_1}(x_0),\alpha)\,,\ \textit{for every } x\in G. \tag{5}$$

Proof. Suppose that $(f_i)_{i\in I}$ converges almost uniformly on X to a quasi-continuous function $f : X \to Y$. Let $(x_0,\alpha,i) \in X\times\Sigma\times I$. Then, by Corollary 1.2 [5] there exist $\beta_0 \in \Sigma$ and $A_0 \in \alpha$ such that $St^2(f(x_0),\beta_0) \subset A_0$.

From the almost uniform convergence of the net $(f_i)_{i\in I}$ to f, it results that there exist $i_0 \in I$ and a neighbourhood V_1 of x_0 such that

$$f_{\bar\imath}(x) \in St(f(x),\beta_0),\ \text{for every } x\in V_1 \text{ and } \bar\imath\in I,\ \bar\imath\geq i_0. \tag{6}$$

Let V be an arbitrary neighbourhood of x_0. Then, since f is quasi-continuous, there exists a nonempty open set $G \subset V\cap V_1$ such that

$$f(x) \in St(f(x_0),\beta_0)\,,\ \text{for every } x\in G. \tag{7}$$

Now, let $i_1 \in I$, $i_1 \geq i$, $i_1 \geq i_0$. Then, for every $x \in G$, we have:

$$f_{i_1}(x) \in St(f(x),\beta_0) \subset St^2(f(x_0),\beta_0) \subset A_0 \tag{8}$$

and

$$f_{i_1}(x_0) \in St(f(x_0),\beta_0) \subset St^2(f(x_0),\beta_0) \subset A_0. \tag{9}$$

From (8) and (9) it follows

$$f_{i_1}(x) \in St(f_{i_1}(x_0),\alpha)\,,\ \text{for every } x\in G\subset V \tag{10}$$

that is relation (5) holds.

Conversely, suppose that the net $(f_i)_{i\in I}$ converges almost uniformly to f and that relation (5) holds. Let $(x_0,\alpha) \in X\times\Sigma$. From the almost uniform convergence, there exist $i_0 \in I$ and V_1 a neighbourhood of x_0 such that

$$f_i(x) \in St(f(x),\alpha)\,,\ \text{for every } i\in I,\ i\geq i_0 \text{ and } x\in V_1. \tag{11}$$

Let $i \in I$, $i \geq i_0$. Then, by supposition, there exists $i_1 \geq i$ such that for every neighbourhood V of x_0 there exists a nonempty open set $G \subset V\cap V_1$ with the property that relation (5) holds.

By using successively relations (11), (5), (11), we get

$$f(x) \in St(f_{i_1}(x), \alpha) \subset St^2(f_{i_1}(x_0), \alpha) \subset St^3(f(x_0), \alpha)$$

for every $x \in G$. Thus, by Lemma 2.1, the quasi-continuity of f at x_0, is proved. □

A similar characterization of the continuity, expressed as in the following theorem, holds.

Theorem 3.3. *The pointwise limit $f : X \to Y$ of an almost uniform convergent net of functions $(f_i)_{i \in I}$ defined from X to a regular generalized uniform space (Y, Σ) is continuous on X if and only if for any $(x_0, \alpha, i) \in X \times \Sigma \times I$ there exist $i_1 \in I$, $i_1 \geq i$ and a neighbourhood V of x_0 with the property*

$$f_{i_1}(V) \subset St(f_{i_1}(x_0), \alpha). \tag{12}$$

Proof. The technique of proof is like the previous one. □

The condition of theorem does not imply the continuity of net's terms, as can be seen from the following example:

Example 3.1. Let $(g_n)_{n \in \mathbb{N}}$ be a sequence of real continuous functions defined on the interval $[0, 1] \subset \mathbb{R}$ that converges uniformly on $[0, 1]$ to a continuous function $g : [0, 1] \to \mathbb{R}$, and let the sequence $(f_n)_{n \geq 1}$ of real functions be defined on $[0, 1]$ as

$$f_n(x) = \begin{cases} g_n(x) + \dfrac{1}{n} & , x \in \mathbb{Q} \cap [0, 1] \\ g_n(x) & , x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1]. \end{cases}$$

The sequence $(f_n)_{n \geq 1}$ converges almost uniformly to the continuous function $f : [0, 1] \to \mathbb{R}$, $f(x) = g(x)$, but f_n is not continuous at $x \in [0, 1]$ for every $n \geq 1$.

4. Quasi-uniform and almost quasi-uniform convergence

In this section some characterization theorems of the continuity and quasi-continuity of pointwise limit of nets of functions with values in generalized uniform spaces are given.

For such characterizations an extension of the quasi-uniform convergence [4] is given and a new type of convergence is introduced.

Let (X, τ), (Y, Σ) and $(I, \geq)$ be as in the previous section.

Definition 4.1. A net $(f_i)_{i \in I}$, $f_i : X \to Y$ is called quasi-uniform convergent to a function $f : X \to Y$ if it converges pointwise to f on X and for each $\alpha \in \Sigma$ there exist a directed set $(J, \geq')$, a "K-application" $h : J \to I$ and a family $\{G_j : j \in J\} \subset \tau$ such that $\bigcup_{j \in J} G_j = X$ and $f_{h(j)}(x) \in St(f(x), \alpha)$ for every $x \in G_j$.

Theorem 4.1. *If (Y,Σ) is a generalized uniform space with a regular structure, then the limit function $f : X \to Y$ of a pointwise convergent net $(f_i)_{i\in I}$ of continuous functions $f_i : X \to Y$ is continuous if and only if the net converges quasi-uniformly on X.*

Proof. Suppose that the net $(f_i)_{i\in I}$ converges pointwise on X to f and that f_i, f are continuous functions on X for each $i \in I$.

Let $\alpha \in \Sigma$ and let $g_i : X \to (Y \times Y, \tau_\Sigma \times \tau_\Sigma)$ be the function defined by $g_i(x) = (f_i(x), f(x))$, $i \in I$. Since f_i and f are continuous on X, the function g_i is continuous on X. Set

$$G_i = \{x \in X : f_i(x) \in St(f(x), \alpha)\}, \; i \in I.$$

We show that $G_i = g_i^{-1}\left(\bigcup_{A\in\alpha}(A \times A)\right)$ and $\bigcup_{i\in I} G_i = X$.

If $x \in G_i$, then

$$f_i(x) \in St(f(x), \alpha) = \bigcup\{A : A \in \alpha, f(x) \in A\}.$$

Therefore there exists $A_x \in \alpha$ such that $f(x) \in A_x$ and $f_i(x) \in A_x$, hence $(f_i(x), f(x)) \in A_x \times A_x \subset \cup\{A \times A : A \in \alpha\}$, that is $x \in g_i^{-1}\left(\bigcup_{A\in\alpha}(A \times A)\right)$, whence $G_i \subset g_i^{-1}\left(\bigcup_{A\in\alpha}(A \times A)\right)$.

Conversely, if $x \in g_i^{-1}\left(\bigcup_{A\in\alpha}(A\times A)\right)$, then $(f_i(x), f(x)) \in \bigcup_{A\in\alpha}(A\times A)$ which implies that there exists $A_x \in \alpha$ such that $f(x) \in A_x$ and $f_i(x) \in A_x$, hence $f_i(x) \in St(f(x), \alpha)$, that is $x \in G_i$, whence $g_i^{-1}\left(\bigcup_{A\in\alpha}(A \times A)\right) \subset G_i$. It is evident that $G_i \in \tau$ since g_i is continuous.

Now we show that $\bigcup_{i\in I} G_i = X$.

Let $x_0 \in X$. From the pointwise convergence of $(f_i)_{i\in I}$ to f on X, it results that for each $\alpha \in \Sigma$ there exists $i_0 \in I$ such that $f_{i_0}(x_0) \in St(f(x_0), \alpha)$, that is $x_0 \in G_{i_0}$.

Thus, we infer that $X = \bigcup_{i\in I} G_i$. Moreover, from the definition of G_i, it results that for each $x \in G_i$ the relation $f_i(x) \in St(f(x), \alpha)$ holds. Therefore the quasi-uniform convergence of net $(f_i)_{i\in I}$ to f is proved.

Conversely. Suppose that f_i is continuous for each $i \in I$ and that the net $(f_i)_{i\in I}$ converges quasi-uniformly to f on X. We will prove that f is continuous on X.

Let $\alpha \in \Sigma$. Then, from the quasi-uniform convergence of net $(f_i)_{i\in I}$ to f, there exist a directed set $(J, \geq')$, a "K-application" $h : J \to I$ and a family of open sets $\{G_j : j \in I\}$ such that $\bigcup_{j\in J} G_j = X$ and $f_{h(j)}(x) \in St(f(x), \alpha)$ for every $x \in G_j$.

Let $x_0 \in X$. Then there exists $j_0 \in J$ such that $x_0 \in G_{j_0}$. For each $x \in G_{j_0}$ we have

$$f_{h(j_0)}(x) \in St(f(x), \alpha). \tag{13}$$

From the continuity of $f_{h(j_0)}$ it follows that there exists a neighbourhood $U_{h(j_0)}$ of x_0 such that

$$f_{h(j_0)}(x) \in St(f_{h(j_0)}(x_0), \alpha)\,, \text{ for every } x \in U_{h(j_0)}. \tag{14}$$

Let $V = G_{j_0} \cap U_{h(j_0)}$. By using successively relations (13), (14), (13), we obtain:

$$f(x) \in St(f_{h(j_0)}(x), \alpha) \subset St^2(f_{h(j_0)}(x_0), \alpha) \subset St^3(f(x_0), \alpha)$$

for every $x \in V$. Therefore, for each $(x_0, \alpha) \in X \times \Sigma$ there exists a neighbourhood V of x_0 such that $f(x) \in St^3(f(x_0), \alpha)$, whence, according to Remark 2.1, f is continuous at x_0, which finishes the proof. □

Now we shall prove that the quasi-uniform convergence also ensures the transfer of quasi-continuity from the terms of a net of functions to its limit.

Theorem 4.2. *If a net $(f_i)_{i\in I}$, $f_i : X \to Y$ of quasi-continuous functions with values in a regular generalized uniform space (Y, Σ) converges quasi-uniformly to a function $f : X \to Y$, then f is quasi-continuous on X.*

Proof. Let $(x_0, \alpha) \in X \times \Sigma$ and suppose that the net $(f_i)_{i\in I}$ of quasi-continuous functions converges quasi-uniformly to f. Then there exist a directed set $(J, \geq')$, a "K-application" $h : J \to I$ and a family of open sets $\{G_j : j \in J\} \subset \tau$ such that $\bigcup_{j\in J} G_j = X$ and

$$f_{h(j)}(x) \in St(f(x), \alpha)\,, \text{ for every } x \in G_j. \tag{15}$$

Let G_{j_0}, $j_0 \in J$, such that $x_0 \in G_{j_0}$ and let V be an arbitrary neighbourhood of x_0. Because $f_{h(j_0)}$ is quasi-continuous, it results that there exists an open set $G \subset V \cap G_{j_0}$ such that

$$f_{h(j_0)}(x) \in St(f_{h(j_0)}(x_0), \alpha)\,, \text{ for every } x \in G. \tag{16}$$

Then it follows

$$f(x) \in St(f_{h(j_0)}(x), \alpha) \subset St^2(f_{h(j_0)}(x_0), \alpha) \subset St^3(f(x_0), \alpha)$$

for every $x \in G$. According to Lemma 2.1, it results that f is quasi-continuous at x_0, which finishes the proof. □

To characterize the quasi-continuity of a pointwise limit of a net of continuous functions with values in a generalized uniform space we are in need of a weaker quasi-uniform convergence, as follows:

Definition 4.2. A net $(f_i)_{i\in I}$, $f_i : X \to Y$ is called almost quasi-uniform convergent to $f : X \to Y$ if it is pointwise convergent to f on X and for each $\alpha \in \Sigma$ there exist a directed set $(J, \geq')$, a "K-application" $h : J \to I$ and a family of subsets of X, $\{A_j : A_j \subset X, j \in J\}$, such that:

(a) $A_j \subset \overline{\operatorname{int} A_j}$ for every $j \in J$;

(b) $\bigcup_{j\in J} A_j = X$;

(c) $f_{h(j)}(x) \in St(f(x), \alpha)$ for every $x \in A_j$.

It is easy to see that a quasi-uniform convergent net converges almost quasi-uniformly.

In general, the converse of this statement is not true, as it results from the following example:

Example 4.1. Let τ be the usual topology on the set of real numbers and let there be the sequence $(f_n)_{n\in\mathbb{N}^*}$ of real functions:

$$f_n(x) = \begin{cases} 0 & \text{, for } x < -\dfrac{1}{n} \\ 1 + nx & \text{, for } -\dfrac{1}{n} \leq x < 0 \\ 1 & \text{, for } 0 \leq x \end{cases}$$

This sequence converges pointwise to the function

$$f : \mathbb{R} \to \mathbb{R}\,,\; f(x) = \begin{cases} 0 & \text{, for } x < 0 \\ 1 & \text{, for } x \geq 0 \end{cases}\;,$$

but it does not converge quasi-uniformly to f. Indeed, let there be the open covering

$$\alpha = \left\{(-\infty, 0),\; (-\frac{2}{3}, \frac{2}{3}),\; (\frac{1}{3}, +\infty)\right\}$$

of $\mathbb{R}$ and let $(f_{n_i})_{i\in\mathbb{N}^*}$ be an arbitrary subsequence of $(f_n)_{n\in\mathbb{N}^*}$.

If $\{G_i : i \in \mathbb{N}^*\} \subset \tau$ is an arbitrary covering of $\mathbb{R}$, that is $\mathbb{R} = \bigcup_{i\in\mathbb{N}^*} G_i$, then there exists $i_0 \in \mathbb{N}^*$ such that $(-\frac{1}{3n_{i_0}}, \frac{1}{3n_{i_0}}) \subset G_{i_0}$. Then, for $x_{i_0} \in (-\frac{1}{3n_{i_0}}, 0) \subset G_{i_0}$, we have $f(x_{i_0}) = 0$ and $f_{n_{i_0}}(x_{i_0}) = 1 + n_{i_0}x_{i_0} > 1 - \frac{1}{3} = \frac{2}{3}$, whence $f_{n_{i_0}}(x_{i_0}) \notin St(f(x_{i_0}), \alpha) = (-\frac{2}{3}, \frac{2}{3})$. Hence $(f_n)_{n\in\mathbb{N}^*}$ does not converge quasi-uniformly to f. Nevertheless, it converges almost quasi-uniformly to the quasi-continuous function f, as it results from the following characterization theorem of the quasi-continuity:

Theorem 4.3. *A pointwise convergent net* $(f_i)_{i\in I}$, $f_i : X \to Y$ *of continuous functions on* X *with values in a regular generalized uniform space* (Y, Σ) *converges to a quasi-continuous function* $f : X \to Y$ *if and only if the net converges almost quasi-uniformly to* f *on* X.

Proof. Suppose that f is quasi-continuous on X. First we will show that the function $g_i : X \to Y \times Y$, $g_i(x) = (f_i(x), f(x))$, where the space $Y \times Y$ is endowed with the product topology $\tau_\Sigma \times \tau_\Sigma$, is quasi-continuous on X.

Let $x_0 \in X$ and $U_1(f_i(x_0)) \times U_2(f(x_0))$ be an arbitrary neighbourhood of the point $(f_i(x_0), f(x_0)) \in Y \times Y$. Then there exists a neighbourhood V_1 of x_0 such that, for each $x \in V_1$, $f_i(x) \in U_1(f_i(x_0))$ and for any neighbourhood V of x_0 there exists a nonempty open set $G \subset V_1 \cap V$ such that $f(x) \in U_2(f(x_0))$ for every $x \in G$. Hence $(f_i(x), f(x)) \in U_1(f_i(x_0)) \times U_2(f(x_0))$for every $x \in G$, that is g_i is quasi-continuous on X. Then, since $\bigcup\limits_{A\in\alpha}(A \times A)$ is open in the product topology $\tau_\Sigma \times \tau_\Sigma$, it results that

$$g_i^{-1}\left(\bigcup_{A\in\alpha}(A \times A)\right) \subset \overline{\operatorname{int} g_i^{-1}\left(\bigcup_{A\in\alpha}(A \times A)\right)}$$

(see Corollary 1 [3]).

Now we show that the family $\{A_i : i \in I\}$ of subsets of X, where $A_i = g_i^{-1}\left(\bigcup\limits_{A\in\alpha}(A \times A)\right)$, satisfies conditions *(a)*, *(b)*, *(c)* of the almost quasi-uniform convergence of the net $(f_i)_{i\in I}$.

From the definition of A_i, obviously, condition *(a)* is fulfilled, that is $A_i \subset \overline{\operatorname{int} A_i}$.

Now let $(x_0, \alpha) \in X \times \Sigma$. In virtue of the pointwise convergence of the net $(f_i)_{i\in I}$, there exists $i_0 \in I$ such that $f_{i_0}(x_0) \in St(f(x_0), \alpha)$. Then, it results $f_{i_0}(x_0) \in \cup\{A : f(x_0) \in A, A \in \alpha\}$, whence $(f_{i_0}(x_0), f(x_0)) \in \bigcup\limits_{A\in\alpha}(A \times A)$, which implies $x_0 \in g_{i_0}^{-1}\left(\bigcup\limits_{A\in\alpha}(A \times A)\right)$, that is $x_0 \in A_{i_0}$. Hence $\bigcup\limits_{i\in I} A_i = X$ and therefore condition *(b)* is fulfilled.

Finally, by definition, for each $(x, i) \in A_i \times I$, $g_i(x) \in \bigcup\limits_{A\in\alpha}(A \times A)$, that is $f_i(x) \in St(f(x), \alpha)$, hence condition *(c)* holds.

In conclusion the net $(f_i)_{i\in I}$ converges almost quasi-uniformly to f on X.

Conversely, suppose that the net $(f_i)_{i\in I}$ of continuous functions converges almost quasi-uniformly to f on X, that is for each $\alpha \in \Sigma$ there exist a directed set $(J, \geq')$, a "K-application" $h : J \to I$ and a family $\{A_j : j \in J\}$ of subsets of X satisfying conditions *(a)*, *(b)*, *(c)*.

Let $x_0 \in X$. Then, from condition *(b)*, there exists $j_0 \in J$ such that $x_0 \in A_{j_0}$ and by *(a)* $x_0 \in \overline{\operatorname{int} A_{j_0}}$.

Since $f_{h(j_0)}$ is continuous at x_0, there exists a neighbourhood V_{j_0} of x_0 such that

(17) $$f_{h(j_0)}(x) \in St(f_{h(j_0)}(x_0), \alpha)\,, \text{ for every } x \in V_{j_0}.$$

Let V be an arbitrary neighbourhood of x_0 and let $G = \text{int}\,(V \cap V_{j_0} \cap \text{int}\, A_{j_0})$. It is evident that $G \in \tau$, $G \neq \emptyset$, $G \subset V$, $G \subset V_{j_0}$, $G \subset \text{int}\, A_{j_0}$ and for every $x \in G$, the relations

(18) $$f_{h(j_0)}(x) \in St(f_{h(j_0)}(x_0), \alpha)$$

(19) $$f(x) \in St(f_{h(j_0)}(x), \alpha)$$

hold.
Also, by condition *(c)*, we have

(20) $$f_{h(j_0)}(x_0) \in St(f(x_0), \alpha)$$

By using successively relations (19), (18), (20) we obtain

$$f(x) \in St(f_{h(j_0)}(x), \alpha) \subset St^2(f_{h(j_0)}(x_0), \alpha) \subset St^3(f(x_0), \alpha).$$

That means, by virtue of Lemma 2.1, that f is quasi-continuous at x_0, which completes the proof. □

Now, by using the quasi-uniform convergence, we will prove a characterization theorem for the continuity of pointwise limit of a net whose terms are not, by all means, continuous functions.

Theorem 4.4. *Let (Y, Σ) be a generalized uniform space with a regular structure. If a net $(f_i)_{i\in I}$, $f_i : X \to Y$, is quasi-uniform convergent to a function $f : X \to Y$, then f is continuous if and only if for any $(x_0, \alpha) \in X \times \Sigma$ there exist $i_1 \in I$ and a neighbourhood V of x_0 with the property that for each $i \in I$ there exists $i_2 \in I$, $i_2 \geq i$, such that:*

(21) $$f_{i_1}(x) \in St(f(x), \alpha) \text{ and } f_{i_2}(x_0) \in St(f_{i_1}(x), \alpha)\,, \text{for every } x \in V.$$

Proof. Suppose that $(f_i)_{i\in I}$ converges quasi-uniformly to a continuous function f on X.

Let $(x_0, \alpha) \in X \times \Sigma$ and let $\beta \in \Sigma$, $\beta \prec \alpha$, such that there exists $A_0 \in \alpha$ with the property $St^2(f(x_0), \beta) \subset A_0$. From the quasi-uniform convergence of $(f_i)_{i\in I}$ it comes that there exist a directed set $(J, \geq')$, a "K-application" $h : J \to I$ and a family $\{G_j\}_{j\in J} \subset \tau$ such that $\bigcup_{j\in J} G_j = X$ and $f_{h(j)}(x) \in St(f(x), \beta)$ for every $x \in G_j$.

Let $j_0 \in J$ with $x_0 \in G_{j_0}$. Then $f_{h(j_0)}(x) \in St(f(x), \beta)$ for every $x_0 \in G_{j_0}$. Hence, taking $i_1 = h(j_0)$, we have:

(22) $$f_{i_1}(x) \in St(f(x), \beta)\,, \text{ for every } x \in G_{j_0}.$$

Also, from the quasi-uniform convergence of $(f_i)_{i\in I}$, it comes out that the net converges pointwise to f on X. Therefore, there exists $i_0 \in I$ such that

$$f_{\bar{\imath}}(x_0) \in St(f(x_0), \beta)\,, \text{ for every } \bar{\imath} \in I,\ \bar{\imath} \geq i_0. \tag{23}$$

But, since f was supposed to be continuous on X, there exists a neighbourhood V_1 of x_0 such that

$$f(x) \in St(f(x_0), \beta)\,, \text{ for every } x \in V_1. \tag{24}$$

Now, let $V = V_1 \cap G_{j_0}$ and let $i \in I$ be chosen arbitrarily. Then, according to relation (23), for an $i_2 \in I$, $i_2 \geq i_0$, $i_2 \geq i$, we have:

$$f_{i_2}(x_0) \in St(f(x_0), \beta). \tag{25}$$

From relations (22), (24), (25), we obtain:

$$f_{i_1}(x) \in St(f(x), \beta) \subset St^2(f(x_0), \beta) \subset A_0 \text{ for every } x \in V$$

and

$$f_{i_2}(x_0) \in St(f(x_0), \beta) \subset St^2(f(x_0), \beta) \subset A_0.$$

Therefore $f_{i_2}(x_0) \in St(f_{i_1}(x), \alpha)$ and also, since $\beta \prec \alpha$, from relation (22), it results $f_{i_1}(x) \in St(f(x), \beta) \subset St(f(x), \alpha)$ for every $x \in V$. Hence the necessity of theorem is proved.

Conversely, suppose that $(f_i)_{i\in I}$ converges quasi-uniformly to f and that relations (21) hold.

Let $(x_0, \alpha) \in X \times \Sigma$. From the quasi-uniform convergence of $(f_i)_{i\in I}$ to f it comes out that there exists $i_0 \in I$ such that

$$f_{\bar{\imath}}(x_0) \in St(f(x_0), \alpha) \text{ for every } \bar{\imath} \in I,\ \bar{\imath} \geq i_0. \tag{26}$$

Also, from relations (21), there exist $i_1 \in I$, a neighbourhood V of x_0 and $i_2 \in I$, $i_2 \geq i_0$, such that

$$f_{i_1}(x) \in St(f(x), \alpha) \text{ and } f_{i_2}(x_0) \in St(f_{i_1}(x), \alpha) \text{ for every } x \in V. \tag{27}$$

From relations (26) and (27) we obtain:

$$f(x) \in St(f_{i_1}(x), \alpha) \subset St^2(f_{i_2}(x_0), \alpha) \subset St^3(f(x_0), \alpha)$$

for every $x \in V$, that is

$$f(x) \in St^3(f(x_0), \alpha) \text{ for every } x \in V. \tag{28}$$

According to Remark 2.1, it comes out that f is continuous at x_0, hence the sufficiency of theorem is proved. □

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