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# A COMMON FIXED POINT THEOREM IN NON-ARCHIMEDEAN MENGER PM-SPACE

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**Abstract.** In the present paper we define the concept of R-weakly commuting mappings in non-Archimedean Menger PM-space and obtain a common fixed point theorem which unifies and generalizes the results of Pant [3] and Vasuki [4].

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## 1. Introduction

In 1994, Pant [3] introduced the concept of R-weakly commuting maps in metric spaces. Later on Pathak et al. [2] generalized this idea and gave the concept of R-weakly commuting maps of type (Ag). Vasuki [4] proved some common fixed point theorems for R-weakly commuting maps in fuzzy metric spaces.

The aim of this paper is to define the concept of R-weakly commuting maps and prove a common fixed point theorem in non-Archimedean Menger PM-space.

Hereby we give some preliminary definitions and notations.

## 2. Preliminaries

**Definition 1.** Let X be any non-empty set and D be the set of all left continuous distribution functions. An ordered pair (X, F) is said to be non-Archimedean probabilistic metric space (briefly N.A. PM-space) if F is a mapping from $X \times X$ *into* $D$ satisfying the following conditions, where the value of F at $(x, y) \in X \times X$ is represented by $F_{x,y}$ or $F(x, y)$ for all $x, y \in X$ such that

i) $F(x, y; t) = 1$ for all $t > 0$ if and only if $x = y$;

ii) $F(x, y; t) = F(y, x; t)$;

iii) $F(x, y; 0) = 0$;

iv) If $F(x, y; t_1) = F(y, z; t_2) = 1$ then $F(x, z; \max\{t_1, t_2\}) = 1$ for all $x, y, z \in X$.

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**Definition 2.** A t-norm is a function $\Delta : [0,1] \times [0,1] \to [0,1]$ which is associative, commutative, non-decreasing in each coordinate and $\Delta(a,1) = a$ for all $a \in [0,1]$

**Definition 3.** A non-Archimedean Menger PM-space is an ordered triplet $(X, F, \Delta)$, where $\Delta$ is a t-norm and $(X, F)$ is a N.A. PM-space satisfying the following condition:

$$F(x, z; \max\{t_1, t_2\}) \geq \Delta(F(x, y; t_1),\ F(y, z; t_2)) \text{ for all } x, y, z \in X,\ t_1, t_2 \geq 0.$$

For details of topological preliminaries on non-Archimedean Menger PM-spaces we refer to Cho, Ha and S.S. Chang [1].

**Definition 4.** An N. A. Menger PM-space $(X, F, \Delta)$ is said to be of type $(C)_g$ if there exists a $g \in \Omega$ such that $g(F(x, z; t)) \leq g(F(x, y; t)) + g(F(y, z; t))$ for all $x, y, z \in X,\ t \geq 0$, where $\Omega = \{g | g : [0,1] \to [0, \infty)$ is continuous, strictly decreasing $g(1) = 0$ and $g(0) < \infty\}$.

**Definition 5.** An N. A. Menger PM-space $(X, F, \Delta)$ is said to be of type $(D)_g$ if there exists a $g \in \Omega$ such that $g(\Delta(t_1, t_2)) \leq g(t_1) + g(t_2)$ for all $t_1, t_2 \in [0,1]$.

**Remark 1.**
i) If N. A. Menger PM-space is of type $(D)_g$ then $(X, F, \Delta)$ is of type $(C)_g$.
ii) If $(X, F, \Delta)$ is an N. A. Menger PM-space and $\Delta \geq \Delta(r, s) = \max(r + s - 1, 1)$, then $(X, F, \Delta)$ is of type $(D)_g$ for $g \in \Omega$ and $g(t) = 1 - t$.

Throughout this paper let $(X, F, \Delta)$ be a complete N.A. Menger PM-space with a continuous strictly increasing t-norm $\Delta$.

Let $\phi : [0, \infty) \to [0, \infty)$ be a function satisfying the condition $(\Phi)$;

$(\Phi)$ $\phi$ is semi-upper continuous from the right and $\phi(t) < t$ for $t > 0$.

**Definition 6.** A sequence $\{x_n\}$ in the N. A Menger PM-space $(X, F, \Delta)$ converges to $x$ if and only if for each $\varepsilon > 0$, $\lambda > 0$ there exists $M(\varepsilon, \lambda)$ such that $g(F(x_n, x; \varepsilon)) < g(1 - \lambda)$ for all $n > M$.

**Definition 7.** A sequence $\{x_n\}$ in the N. A Menger PM-space is a Cauchy sequence if and only if for each $\varepsilon > 0$, $\lambda > 0$ there exists an integer $M(\varepsilon, \lambda)$ such that

$$g(F(x_n, x_{n+p}; \varepsilon)) < g(1 - \lambda) \text{ for all } n \geq M \text{ and } p \geq 1.$$

**Example 1.** Let $X$ be any set with at least two elements. If we define

$$F(x, x; t) = 1 \text{ for all } x \in X, t > 0$$

and

$$F(x, y; t) = \left\{ \begin{array}{l} 0, t \leq 1 \\ 1, t > 1 \end{array} \right\}$$

when $x, y \in X,\ x \neq y$, then $(X, F, \Delta)$ is the N. A.Menger PM-space with $\Delta(a, b) = \min(a, b)$ or $(a.b)$.

*Proof.* Conditions (i), (ii) and (iii) are trivial.

Let us go for (iv) condition. For this let $F(x, y; t_1) = 1 = F(y, z; t_2)$, $x \neq y$ $y \neq z$, then $t_1, t_2 > 1 \Rightarrow \max(t_1, t_2) > 1 \Rightarrow F(x, z; \max(t_1, t_2)) = 1$, $x \neq z$.

Also, Menger inequality $F(x, z; \max(t_1, t_2)) \geq \Delta(F(x, y; t_1), F(y, z; t_2))$ is obvious. Thus $(X, F, \Delta)$ is an N.A. Menger PM-space. □

**Example 2.** Let $X = R$ be the set of real numbers equipped with metric defined as

$$d(x, y) = |x - y|$$

Set $F(x, y; t) = \frac{t}{t+d(x,y)}$.

Then $(X, F, \Delta)$ is the N.A. Menger PM-space with $\Delta$ as continuous t-norm satisfying $\Delta(r, s) = \min(r, s)$ or $prod(r, s)$.

**Lemma 1.** *If a function $\phi : [0, \infty) \to [0, \infty)$ satisfies the condition $(\Phi)$ then we get*

1. *For all $t \geq 0$, $\lim_{n\to\infty} \phi^n(t) = 0$, where $\phi^n(t)$ is the $n^{th}$ iteration of $\phi(t)$.*
2. *If $\{t_n\}$ is a non decreasing sequence of real numbers and $t_{n+1} \leq \phi(t_n)$, $n = 1, 2, \ldots$, then $\lim_{n\to\infty} t_n = 0$. In particular, if $t \leq \phi(t)$, for each $t \geq 0$, then $t = 0$.*

**Lemma 2.** *([1]) Let $\{y_n\}$ be a sequence in $X$ such that $\lim_{n\to\infty} F(y_n, y_{n+1}; t) = 1$ for each $t > 0$. If the sequence $\{y_n\}$ is not a Cauchy sequence in X, then there exist $\varepsilon_0 > 0$, $t_0 > 0$, and two sequences $\{m_i\}$ and $\{n_i\}$ of positive integers such that*

1. *$m_i > n_i + 1$ and $n_i \to \infty$ as $i \to \infty$.*
2. *$F(y_{m_i}, y_{n_i}; t_0) < 1 - \varepsilon_0$ and $F(y_{m_i-1}, y_{n_i}; t_0) \geq 1 - \varepsilon_0, i = 1, 2, \ldots$*

**Definition 8.** Two maps $A$ and $S$ of an N.A. Menger PM-space $(X, F, \Delta)$ into itself are said to be $R$-weakly commuting if there exists some $R > 0$ such that $g(F(ASx, SAx; t)) \leq g(F(Ax, Sx; t/R))$ for every $x \in X$ and $t > 0$.

Weak commutativity implies $R$-weak commutativity and the converse is true for $R \leq 1$. Using $R$-weak commutativity Vasuki [4] proved the following result, generalizing the result of Pant [4].

**Theorem 1.** *([4]). Let $(X, M, *)$ be a complete fuzzy metric space and let $f$ and $g$ be $R$-weakly commuting self mappings of $X$ satisfying the conditions:*

*$M(fx, fy, t) \geq r(M(gx, gy, t))$ where $r : [0, 1] \to [0, 1]$ is a continuous function such that $r(t) > t$ for each $0 \leq t < 1$ and $r(1) = 1$ and the sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that $\{x_n\} \to x, \{y_n\} \to y$ implies $M(x_n, y_n, t) \to M(x, y, t)$.*

*If the range of $g$ contains the range of $f$ and either $f$ or $g$ is continuous, then $f$ and $g$ have a unique common fixed point.*

Now, we extend and generalize the above result.

## 3. Main result

**Theorem 2.** *Let $S$ and $T$ be two continuous self-maps of a complete N. A. Menger PM-space $(X, F, \Delta)$. Let $A$ be self-map of $X$ satisfying*

*(i) $\{A, S\}$ and $\{A, T\}$ are point wise R-weakly commuting and $A(X) \subseteq S(X) \cap T(X)$*

*(ii)*

$$g(F(Ax, Ay; t)) \leq \phi\left[\max\left\{\begin{array}{l} g(F(Sx, Ty; t)), g(F(Sx, Ax; t)), \\ g(F(Sx, Ay; t)), g(F(Ty, Ay; t)) \end{array}\right\}\right], \quad \text{for every } x, y \in X,$$

*where $\phi$ satisfies the condition $(\Phi)$. Then $A$, $S$ and $T$ have a unique common fixed point in $X$.*

*Proof.* Let $x_0 \in X$. Since $A(X) \subseteq S(X)$, there exists $x_1 \in X$ such that $Ax_0 = Sx_1$. Again as $A(X) \subseteq T(X)$, there is another point $x_2 \in X$ such that $Ax_1 = Tx_2$. Inductively we can choose $x_{2n+1}$ and $x_{2n+2}$ in $X$ such that

$$y_{2n} = Sx_{2n+1} = Ax_{2n}, Tx_{2n+2} = Ax_{2n+1} = y_{2n+1} \text{ for } n = 0, 1, 2, \ldots \tag{3.1}$$

Let $M_n = g(F(Ax_n, Ax_{n+1}; t))$, $n = 0, 1, 2, \ldots$ then

$$\begin{aligned} M_{2n} &= g(F(Ax_{2n+1}, Ax_{2n}; t)) \\ &\leq \phi\left[\max\left\{\begin{array}{l} g(F(Sx_{2n+1}, Tx_{2n}; t)), g(F(Sx_{2n+1}, Ax_{2n}; t)), \\ g(F(Sx_{2n+1}, Ax_{2n}; t)), g(F(Tx_{2n}, Ax_{2n}; t)) \end{array}\right\}\right] \\ &= \phi\left[\max\left\{\begin{array}{l} g(F(Ax_{2n}, Ax_{2n-1}; t)), g(F(Ax_{2n}, Ax_{2n+1}; t)), \\ g(F(Ax_{2n}, Ax_{2n}; t)), g(F(Ax_{2n-1}, Ax_{2n}; t)) \end{array}\right\}\right]. \end{aligned} \tag{3.2}$$

$$M_{2n} = \phi\left[(\max\{M_{2n-1}, M_{2n}, 0, M_{2n-1}\}\right]. \tag{3.3}$$

If $M_{2n} > M_{2n-1}$ then by (1.3) $M_{2n} \geq \phi(M_{2n})$, a contradiction.

If $M_{2n-1} > M_{2n}$ then by (1.3) $M_{2n} \leq \phi(M_{2n-1})$ then by Lemma 1, we get $\lim_n M_{2n} = 0$, i.e.,

$$\lim_n g\left(F\left(Ax_{2n+1}, Ax_{2n}; t\right)\right) = 0.$$

Similarly, we can show that $\lim_n g\left(F\left(Ax_{2n+2}, Ax_{2n+1};\, t\right)\right) = 0$.

Thus we have

$$\begin{array}{l} \lim_n g\left(F\left(Ax_{2n}, Ax_{2n+1}; t\right)\right) = 0 \text{ for all } t > 0. \\ \lim_n g\left(F\left(y_n, y_{n+1}; t\right)\right) = 0 \text{ for all } t > 0. \end{array} \tag{3.4}$$

Before proceeding the proof of the theorem, we first prove a claim.

*Claim.* Let $A, S, T : X \to X$ be maps satisfying (i) and (ii) and $\{y_n\}$ defined by (1.1) such that

$$(3.5) \qquad \lim_n g\left(F\left(y_n, y_{n+1};\, t\right)\right) = 0$$

for all $n$ is a Cauchy sequence in $X$.

*Proof of Claim.* Since $g \in \Omega$ it follows that

$$\lim_{n\to\infty} F(y_n, y_{n+1}; t) = 1 \text{ for each } t > 0$$
$$\text{if and only if } \lim_{n\to\infty} g(F(y_n, y_{n+1}; t)) = 0 \text{ for each } t > 0$$

By Lemma 2 if $\{y_n\}$ is not a Cauchy sequence in $X$, there exist $\varepsilon_0 > 0$, $t_0 > 0$ and two sequences $\{m_i\}$ and $\{n_i\}$ of positive integers such that

A) $m_i > n_i + 1$ *and* $n_i \to \infty$ as $i \to \infty$;

B) $g(F(y_{m_i}, y_{n_i}; t_0)) > g(1 - \varepsilon_0\ )$ and $g(F(y_{m_i-1}, y_{n_i}; t_0)) \le g(1 - \varepsilon_0)$, $i = 1, 2, \ldots$

Since $g(t) = 1 - t$, we have

$$\begin{aligned} g(1-\varepsilon_0) &< g(F(y_{m_i}, y_{n_i}; t_0)) \\ &\le g(F(y_{m_i}, y_{m_i-1}; t_0)) + g(F(y_{m_{i-1}}, y_{n_i}; t_0)) \\ (3.6) \qquad &\le g(F(y_{m_i}, y_{m_i-1}; t_0)) + g(1-\varepsilon_0). \end{aligned}$$

As $i \to \infty$ in (1.6) we get

$$(3.7) \qquad \lim_{n\to\infty} g(F(y_{m_i}, y_{n_i}; t_0)) = g(1-\varepsilon_0).$$

On the other hand, we have

$$\begin{aligned} g(1-\varepsilon_0) &< g(F(y_{m_i}, y_{n_i}, a; t_0)) \\ (3.8) \qquad &\le g(F(y_{n_i}, y_{n_i+1}; t_0)) + g(F(y_{m_i}, y_{n_i+1}; t_0)) \end{aligned}$$

Now, consider $g(F(y_{m_i}, y_{n_i+1}; t_0))$ in (1.8) and assume that both $m_i$ and $n_i$ are even. Then, by (ii), we have

$$\begin{aligned} & g(F(y_{m_i}, y_{n_i+1}, a; t_0)) \\ &\quad = g(F(Ax_{m_i}, Ax_{n_i+1}; t_0)) \\ &\quad \le \phi[\max\{g(F(Sx_{m_i}, Tx_{n_i+1}; t_0)), g(F(Sx_{m_i}, Ax_{m_i}; t_0)), \\ &\qquad g(F(Sx_{m_i}, Ax_{n_i+1}; t_0)), g(F(Tx_{n_i+1}, Ax_{n_i+1}; t_0))\}] \\ (3.9) \qquad &\quad \le \phi[\max\{g(F(y_{m_i}, y_{n_i}; t_0)), g(F(y_{m_i-1}, y_{m_i}; t_0)), \\ &\qquad g(F(y_{m_i-1}, y_{n_i+1}; t_0)), g(F(y_{n_i}, y_{n_i+1}; t_0))\}] \end{aligned}$$

Now, consider $g(F(y_{m_i-1}, y_{n_i+1}; t_0))$ from (1.9).

$$(3.10) \qquad g(F(y_{m_i-1}, y_{n_i+1}; t_0)) \le g(F(y_{m_i-1}, y_{n_i}; t_0)) + g(F(y_{n_i}, y_{n_i+1}; t_0)).$$

Using (1.10) in (1.9) and letting $i \to \infty$.

$$g(1-\varepsilon_0) \le \phi\,[\max\{g(1-\varepsilon_0), 0, g(1-\varepsilon_0), 0\}] \text{ i.e., } g(1-\varepsilon_0) \le \phi\,(g(1-\varepsilon_0).$$

Which is a contradiction. Hence the sequence $\{y_n = Ax_n\}$ defined by (1.1) is a Cauchy sequence, which concludes the proof of Claim.

By the completeness of $X$, $\{Ax_n\}$ converges to a point $z \in X$. Consequently, the subsequences $\{Sx_{2n+1}\}$ and $\{Tx_{2n}\}$ of $\{Ax_n\}$ also converge to $z \in X$. Since $A$ and $S$ are $R$-weakly commuting, so $g(F(ASx_{2n+1}, SAx_{2n+1}; t)) \le g(F(Ax_{2n+1}, Sx_{2n+1}; t/R))$, which gives $\lim_{n\to\infty} ASx_{2n+1} = \lim_{n\to\infty} SAx_{2n+1} = Sz$ (as $S$ is continuous).

Now, we claim that $Sz = z$.

Suppose that $Sz \neq z$. Then, using (ii), we get

$$\begin{aligned}
&g(F(ASx_{2n+1}, Ax_{2n}; t)) \\
&\quad \le \phi[\max\{g(F(SSx_{2n+1}, Tx_{2n}\,; t)), g(F(SSx_{2n+1}, ASx_{2n+1}; t)), \\
&\qquad g(F(SSx_{2n+1}, Ax_{2n}; t))\,,\, g(F(Ax_{2n}, Tx_{2n}; t))\}].
\end{aligned}$$

Taking $n \to \infty$ we get,

$$\begin{aligned}
&g(F(Sz, z\,; t)) \\
&\quad \le \phi[\max\{g(F(Sz, z\,; t)), g(F(Sz, Sz\,; t)), g(F(Sz, z\,; t)), g(F(z, z\,; t))\}] \\
&\quad = \phi(g(F(Sz, z, a; t))) < g(F(Sz, z, a; t)),
\end{aligned}$$

which is a contradiction.

Thus $z$ is a fixed point of $S$. Similarly, we can show that $z$ is a fixed point of $A$.

Now, the pair $\{A, T\}$ is $R$-weakly commuting so

$$g(F(ATx_{2n+1}, TAx_{2n+1}; t)) \le g(F(Ax_{2n+1}, Tx_{2n+1}; t/R))$$

which gives

$$\lim_{n\to\infty} ATx_{2n+1} = \lim_{n\to\infty} TAx_{2n+1} = Tz \text{ (as T is continuous).}$$

Now, we claim that $z$ is also a fixed point of $T$.

Suppose that $Tz \neq z$, then using (ii) we have

$$\begin{aligned}
g(F(Az, ATx_{2n}\,; t)) \le\ & \phi[\max\{g(F(Sz, T^2x_{2n}\,; t)), g(F(Sz, Az\,; t)), \\
& g(F(Sz, ATx_{2n}\,; t)), g(F(T^2x_{2n}, ATx_{2n}\,; t))\}].
\end{aligned}$$

On taking limit as $n \to \infty$, it yields

$$\begin{aligned}
g(F(z, Tz\,; t)) \le\ & \phi[\max\{g(F(z, Tz\,; t)), g(F(z, z\,; t)), \\
& g(F(z, Tz\,; t)), g(F(Tz, Tz\,; t))\}].
\end{aligned}$$

This gives that $z = Tz$. Thus $z$ is a common fixed point of $A$, $S$ and $T$.

Uniqueness can be proved by using condition (ii). □

Taking $T = S$ in the above theorem we get the following corollary unifying Vasuki's theorem [4], which in turn also generalizes the result of Pant [3].

**Colorallary 1.** *Let $(X, F, \Delta)$ be a complete N.A. Menger PM-space and $S$ be a continuous self-mapping of $X$. Let $A$ be another self-mapping of $X$ satisfying that $\{A, S\}$ is R-weakly commuting of type with $A(X) \subseteq S(X)$ and*

$$\begin{aligned} g(F(Ax, Ay, a; t)) \quad &\leq \quad \phi[\max\{g(F(Sx, Ty\,; t)), g(F(Sx, Ax\,; t)), \\ &\qquad g(F(Sx, Ay\,; t)),\ g(F(Sy, Ay\,; t))\}] \end{aligned}$$

*for each $x, y \in X$ and $\phi$ satisfies the condition $(\Phi)$. Then the maps $A$ and $S$ have a unique common fixed point.*

**Remark 2.** In our generalization the inequality condition (ii) satisfied by the mappings $A$ , $S$ and $T$ is stronger than that of Theorem 1.9 of Vasuki [4].

**Example 3.** Let $X = R$ and $A, S, T : X \to X$ be mappings such that

$$S\,(x) \;=\; 2x - 1,\; T\,(x) \;=\; \begin{cases} -1 - x, & x < 0 \\ 2x - 1, & 0 \leq x < 1 \\ \dfrac{x+1}{2}, & x \geq 1 \end{cases} \text{ and } A\,(x) \;=\; \begin{cases} 0, & x = -1 \\ x^2, & x \neq -1 \end{cases}.$$

Then we see that

(i) $(A, S)$ and $(A, T)$ are point-wise $R$-weakly commuting.

(ii) $A\,(X)\; \subseteq\; S\,(X) \cap T\,(X)$

(iii) '1' is the unique common fixed point of $A$, $S$ and $T$.

(iv) $g(F(Ax, Ay\,; t)) \;\leq\; \phi\left[\max\left\{\begin{array}{c} (Sx, Ty\,; t), g(F(Sx, Ax; t)), \\ g(F(Sx, Ay\,; t)), g(F(Ty, Ay; t)) \end{array}\right\}\right],$

for every $x, y \in X$ is also true.

Thus all the conditions of our Theorem 2 are satisfied.

## References


[1] Cho, Y. J., Ha, K. S., Chang, S. S., Common fixed point theorems for compatible mappings of type (A) in non-Archimedean Menger PM-space. Math Japonica (46)(1)(1997), 169-179, CMP 1466 131. zbl 883.47038.

[2] Cho, Y. J., Patak, H. K., Kang, S. M., Remarks on R- weakly commuting maps and common fixed point theorems, Bull. Korean Math. Soc. (34) (1997), 247-257.

[3] Pant, R. P., Common fixed points of non-commuting mappings. J. Math. Anal. App. 188 (2)(1994), 436-440.

[4] Vasuki, R., Common fixed points for R-weakly commuting maps in fuzzy metric spaces. Indian J. Pure and Applied Math. (30) (1999), 419-423.


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