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ON OPTIMAL MULTIPOINT METHODS FOR SOLVING NONLINEAR EQUATIONS[1]

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Abstract. A general class of three-point iterative methods for solving nonlinear equations is constructed. Its order of convergence reaches *eight* with only $four$ function evaluations per iteration, which means that the proposed methods possess as high as possible computational efficiency in the sense of the Kung-Traub hypothesis (1974). Numerical examples are included to demonstrate a spectacular convergence speed with only few function evaluations.

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During the last decade, multipoint iterative methods for solving nonlinear equations have been presented in many papers. The main goal and motivation of these papers were the construction of new methods with computational efficiency as high as possible, which assumes the design of iterative methods having the convergence as fast as possible under the condition that the number of function evaluations per iteration is fixed. This demand is close to the Kung-Traub conjecture [4] from 1974 that multipoint methods without memory, requiring $n+1$ function evaluations per iteration, have the order of convergence at most $r_n = 2^n$. Multipoint methods which satisfy the Kung-Traub conjecture are often called *optimal methods*; consequently, $r_n = 2^n$ is the *optimal order*. The class of optimal n-point methods of the order 2^n will be denoted with Ψ_{2^n}.

The aim of this paper is to construct a general family of three-point iterative methods for solving nonlinear equations, which requires $four$ function evaluations per iteration and has the convergence order $r_3 = 2^3 = 8$. The presented approach can be applied for developing optimal methods of higher order (precisely, of the order 2^n $(n \geq 3)$), as demonstrated in [8].

In this paper we will use the following assertion (see [10, Theorem 2.4]):

Theorem 1. *Let $\phi_1(x), \phi_2(x), \ldots, \phi_s(x)$ be iterative functions with the orders $r_1, r_2, \ldots, r_s$, respectively. Then the iterative function*

$$\phi(x) = \phi_1\big(\phi_2\big(\cdots\big(\phi_s(x)\big)\cdots\big)\big)$$

defines the composite iterative method of the order $r_1 r_2 \cdots r_s$.

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Let α be a simple zero of a real, sufficiently smooth, function f and let $u(x) = f(x)/f'(x)$. Assume that p is a real function defined in the neighborhood of 0. To construct optimal three-point methods, we start with a three-step iterative scheme

$$(1)\qquad \begin{cases} 1^\circ & y = N(x) = x - \dfrac{f(x)}{f'(x)}, \\ 2^\circ & z = y - p(t)\dfrac{f(y)}{f'(x)}, \quad t = \dfrac{f(y)}{f(x)}, \\ 3^\circ & \hat{x} = N(z) = z - \dfrac{f(z)}{f'(z)}, \end{cases}$$

where $x = x_m$ is a current approximation to α, $\hat{x} = x_{m+1}$ is a new approximation $(m = 0, 1, \ldots)$ and N denotes the Newton operator. Regarding this scheme, the first task in constructing optimal three-point methods is

(1) the determination of the form of a real function p to ensure the fourth order of convergence of two-step methods consisting of the steps 1° and 2° of (1). Since the third step defines Newton's method of the second order, according to Theorem 1 we would obtain the order $4 \cdot 2 = 8$. However, such a method *is not* optimal since it requires five function evaluations instead of four. For this reason, the second task is

2) the reduction of function evaluations by approximating the derivative $f'(z)$ in the third step of (1) in such a way that quadratic convergence of the modified Newton method is preserved. Using the idea presented in [8], we approximate $f'(z)$ by the derivative of the Hermite interpolation polynomial of the third degree which fits f.

Task 1: The choice of $p(t)$

We state the following assertion.

Theorem 2. *Let p be any real function satisfying $p(0) = 1,\ p'(0) = 2$ and $|p''(0)| < \infty$. If an initial approximation $x = x_0$ is sufficiently close to α, then the order of convergence of the family of two-step methods*

$$(2)\qquad \begin{cases} 1^\circ & y = N(x) = x - \dfrac{f(x)}{f'(x)}, \\ 2^\circ & \hat{x} = y - p(t)\dfrac{f(y)}{f(x)} \end{cases}$$

is four.

Proof. Let $c_k = f^{(k)}(\alpha)/(k!f'(\alpha))$ $(k = 2, 3, \ldots)$ and let us introduce the errors

$$\varepsilon = x - \alpha, \quad \eta = y - \alpha.$$

Using the Taylor series we find

$$(3)\qquad f(x) = f'(\alpha)\Big(\varepsilon + c_2\varepsilon^2 + c_3\varepsilon^3 + c_4\varepsilon^4 + O(\varepsilon^5)\Big)$$

and

$$f'(x) = f'(\alpha)\Big(1 + 2c_2\varepsilon + 3c_3\varepsilon^2 + 4c_4\varepsilon^3 + O(\varepsilon^4)\Big). \tag{4}$$

Hence, in view of (3) and (4), we obtain

$$\eta = \varepsilon - \frac{f(x)}{f'(x)} = c_2\varepsilon^2 + (2c_3 - 2c_2^2)\varepsilon^3 + (4c_2^3 - 7c_2c_3 + 3c_4)\varepsilon^4 + O(\varepsilon^5). \tag{5}$$

Furthermore, we have

$$f(y) = f'(\alpha)\Big(\eta + c_2\eta^2 + c_3\eta^3 + c_4\eta^4 + O(\eta^5)\Big). \tag{6}$$

Let us represent the function p by its Taylor's polynomial of the second order at the point $t = 0$,

$$p(t) = p(0) + p'(0)t + \frac{p''(0)}{2}t^2, \quad t = f(y)/f(x). \tag{7}$$

Now, using (3)–(7) we obtain

$$\begin{aligned}
\hat{x} - \alpha &= \eta - p(t)\frac{f(y)}{f'(x)} \\
&= \Big[-2c_3(p(0) - 1) + c_2^2(4p(0) - p'(0) - 2)\Big]\varepsilon^3 + \\
&\quad \Big[-3c_4(p(0) - 1) + c_2c_3(-7 + 14p(0) - 4p'(0)) \\
&\quad + c_2^3(4 - 13p(0) + 7p'(0) - p''(0)/2)\Big]\varepsilon^4 + O(\varepsilon^5).
\end{aligned}$$

Hence, taking into account the conditions $p(0) = 1$ and $p'(0) = 2$, we find

$$\hat{x} - \alpha = \Big[c_2^3(5 - p''(0)/2) - c_2c_3\Big]\varepsilon^4 + O(\varepsilon^5). \tag{8}$$

Therefore, the order of convergence of the considered family of two-step methods (2) is four and the theorem is proved. $\square$.

Remark 1. In regard to (5) and (1), setting $\delta = z - \alpha = \hat{x} - \alpha$ $(= O(\varepsilon^4))$, we find

$$N(z) - \alpha = c_2\delta^2 + (2c_3 - 2c_2^2)\delta^3 + (4c_2^3 - 7c_2c_3 + 3c_4)\delta^4 + O(\delta^5). \tag{9}$$

The iterative formula (2) generates a wide class of optimal two-point methods; for example,

$$p(t) = \frac{1 + \beta t}{1 + (\beta - 2)t}$$

leads to *King's family* [2]

$$K_f(\beta; x) = x - u(x) - \frac{f(x - u(x))}{f'(x)} \cdot \frac{f(x) + \beta f(x - u(x))}{f(x) + (\beta - 2)f(x - u(x))},$$

where β is a real parameter. Let us note that King's family gives the well-known *Ostrowski's method* [6] as a special case when $\beta = 0$,

$$O_f(x) = K_f(0; x) = x - u(x) - \frac{u(x)f(x - u(x))}{f(x) - 2f(x - u(x))}. \tag{10}$$

The choice $\beta = 1$ generates Kou's method [3], while $\beta = 2$ gives Chun's method [1].

Taking $p(t) = \frac{1}{t}\Big(\frac{2}{1 + \sqrt{1 - 4t}} - 1\Big)$ in (2) we obtain *Euler-like method*

$$E_f(x) = x - \frac{2u(x)}{1 + \sqrt{1 - \dfrac{4f(x - u(x))}{f(x)}}}, \tag{11}$$

proposed in [7] (see, also, [9]), while the choice $p(t) = \dfrac{t^2 - t - 1}{t - 1}$ in (2) yields *Maheshwari's method* [5]

$$M_f(x) = x - u(x)\left\{\frac{\big[f\big(x - u(x)\big)\big]^2}{f(x)^2} - \frac{f(x)}{f\big(x - u(x)\big) - f(x)}\right\}. \tag{12}$$

The order of convergence of these two-point methods is 4 and they require 3 function evaluations per iteration. Therefore, $K_f, E_f, M_f \in \Psi_4$.

Task 2: Reduction of function evaluations

As already mentioned, the use of Newton's method

$$N(z) = z - \frac{f(z)}{f'(z)} \tag{13}$$

in the third step of the three-step scheme (1) is rather inefficient since two new function evaluations are needed. For this reason, we will replace the derivative $f'(z)$ appearing in (13) by the derivative $h'(z)$ of the Hermite interpolation polynomial $h(z)$, constructed for the nodes $x,\ y$ and z to fit f.

Let us form the Hermite interpolation polynomial of the third order for the nodes $x,\ y$ and z,

$$h(t) = a_1 + a_2(t - x) + a_3(t - x)^2 + a_4(t - x)^3, \tag{14}$$

which satisfies the following conditions

$$h(x) = f(x), \tag{15}$$

$$h(y) = f(y), \tag{16}$$

$$h(z) = f(z), \tag{17}$$

and

$$h'(x) = f'(x). \tag{18}$$

We have exactly four conditions for determining four unknown coefficients $a_1,\ a_2,$ $a_3,\ a_4$ in (14). From the conditions (15) and (18) we immediately find

$$a_1 = f(x), \quad a_2 = f'(x).$$

The remaining two coefficients a_3 and a_4 are found from the system of two linear equations which is formed putting y and then z in (14) and using the conditions (16) and (17). We get

$$a_3 = \frac{(z-x)f[y,x]}{(z-y)(y-x)} - \frac{(y-x)f[z,x]}{(z-y)(z-x)} - f'(x)\Big(\frac{1}{z-x} + \frac{1}{y-x}\Big),$$

$$a_4 = \frac{f[z,x]}{(z-y)(z-x)} - \frac{f[y,x]}{(z-y)(y-x)} + \frac{f'(x)}{(z-x)(y-x)},$$

where $f[x,y] = \dfrac{f(x)-f(y)}{x-y}$. Differentiating (14) yields

$$h'(t) = a_2 + 2a_3(t-x) + 3a_4(t-x)^2. \tag{19}$$

Putting $t=z$ and substituting the coefficients $a_2,\ a_3$ and a_4 in (19), we obtain

$$h'(z) = 2\big(f[z,x] - f[y,x]\big) + f[z,y] + \frac{y-z}{y-x}\big(f[y,x] - f'(x)\big). \tag{20}$$

Now we replace the derivative $f'(z)$ appearing in the Newton method (13) by $h'(z)$, calculated by (20), to construct the modified Newton method:

$$\widetilde{N}_2(z) = z - \frac{f(z)}{h'(z)}. \tag{21}$$

Employing the modified Newton method (21) to (1), we construct the following family of three-point methods

$$\begin{cases} 1^\circ \qquad y = N(x) = x - \dfrac{f(x)}{f'(x)}, \\ 2^\circ \qquad z = y - p(t)\dfrac{f(y)}{f'(x)}, \\ 3^\circ \qquad \hat{x} = \widetilde{N}_2(z) = z - \dfrac{f(z)}{h'(z)}, \end{cases} \tag{22}$$

assuming that the function p satisfies $p(0)=1,\ p'(0)=2,\ |p''(0)|<\infty$. In what follows we will prove that the family of three-point methods (22) is *optimal*, that is, its order of convergence reaches *eight* using only *four* function evaluations.

Theorem 3. *If a real function p satisfies the conditions of Theorem* 2*, then the order of convergence of the family of three-point methods* (22) *is eight.*

Proof. Using the expression of the error of the Hermite interpolation (see, e,g., [10]) for the interpolation nodes $x,\ y,\ z$ of the multiplicities 2, 1 and 1 (according to the conditions (15)–(18)), we can write for the Hermite polynomial (14)

$$f(t)-h(t)=\frac{f^{(4)}(\xi)}{4!}(t-x)^2(t-y)(t-z), \tag{23}$$

where ξ belongs to the interval determined by the nodes $x,\ y$ and z. By the logarithmic differentiation, from (23) it follows

$$f'(t)-h'(t)=\big[f(t)-h(t)\big]\Big(\frac{2}{t-x}+\frac{1}{t-y}+\frac{1}{t-z}\Big),$$

whence, for $t=z$, we find

$$f'(z)-h'(z)=\frac{f^{(4)}(\xi)}{4!}(z-x)^2(z-y). \tag{24}$$

Since $z-\alpha=O(\varepsilon^4)$ (see Remark 1) and $y-\alpha=O(\varepsilon^2)$ (see (5)), we have

$$z-x=(z-\alpha)-(x-\alpha)=O(\varepsilon^4)-O(\varepsilon)=O(\varepsilon),\ \ z-y=(z-\alpha)-(y-\alpha)=O(\varepsilon^2).$$

According to this, we get from (24)

$$f'(z)-h'(z)=O(\varepsilon^4),\ \ \text{that is,}\ \ h'(z)=f'(z)\big(1+O(\varepsilon^4)\big). \tag{25}$$

Now we determine the order of convergence of the three-point method (22). Let $\hat\varepsilon=\hat x-\alpha$. In regard to (8), Remark 1 and (25) we obtain

$$\hat\varepsilon=z-\alpha-\frac{f(z)}{h'(z)}=\delta-\frac{f(z)}{f'(z)\big(1+O(\varepsilon^4)\big)}=\delta-\frac{f(z)}{f'(z)}\big(1+O(\varepsilon^4)\big). \tag{26}$$

Since $\delta=O(\varepsilon^4)$ (in regard to (8) and Remark 1) we have $f(z)=(z-\alpha)g(z)=O(\delta)=O(\varepsilon^4)$ $(g(\alpha)\neq 0)$. According to this and (9), from (26) we get

$$\hat\varepsilon=c_2\delta^2+(2c_3-2c_2^2)\delta^3+(4c_2^3-7c_2c_3+3c_4)\delta^4+O(\varepsilon^8)=O(\varepsilon^8). \tag{27}$$

This completes the proof of Theorem 3. $\square$

We note that the established family of three-point methods (22) *supports* the Kung-Traub conjecture for $n=3$, that is, IM(22) $\in\Psi_8$, where IM stands for iterative method. Following the presented approach based on the use of the Hermite interpolation polynomial, we can develop optimal n-point methods, see [8]. In the k-th step we approximate f' by the derivative $h'_{(k-1)}$ of the Hermitian polynomial $h_{(k-1)}$ of the third order constructed at the nodes represented by the *three last approximations* to the desired zero α. In the case of the iterative scheme (22) these approximations (nodes) are $z,\ y$ and x. In this way we obtain modified Newton's methods $\widetilde N_{k-1}$ $(k=3,\dots,n)$ of the second order. For example, the family of optimal four-point methods of the order $2^4 = 16$, requiring 5 function evaluations, has the form

$$
(28) \qquad \begin{cases} 1^\circ & y = N(x) = x - \dfrac{f(x)}{f'(x)}, \\ 2^\circ & z = y - p(t)\dfrac{f(y)}{f'(x)}, \\ 3^\circ & w = \widetilde{N}_2(z) = z - \dfrac{f(z)}{h'_{(2)}(z)}, \\ 4^\circ & \hat{x} = \widetilde{N}_3(w) = w - \dfrac{f(w)}{h'_{(3)}(w)}, \end{cases}
$$

where $h'_{(3)}(w)$ is calculated using the nodes $w,\ z$ and y. More details on n-point methods for arbitrary n may be found in [8].

Example 1. We applied the three-point methods (22) of the order $r_3 = 2^3 = 8$ to the function $f(x) = (x-1)(x^{12} + x^2 + 1)\sin(5x)$ to find sufficiently close approximations to the zero $\alpha = 1$ of f. We chose $x_0 = 1.1$ as the initial approximation. The absolute values $|x_k - \alpha|$ in the first three iterations are given in Table 1, where $A(-q)$ means $A \times 10^{-q}$. The package *Mathematica* 6 with multiprecision arithmetic was used.

Three-point methods	$\lvert x_1-\alpha\rvert$	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$
$\{$Ostrowski's IM (10), $\widetilde{N}_2\}$	7.89(−6)	1.24(−74)	6.23(−1186)
$\{$Euler-like IM (11), $\widetilde{N}_2\}$	7.88(−6)	6.34(−75)	1.61(−1179)
$\{$Maheshvari's IM (12), $\widetilde{N}_2\}$	5.36(−6)	6.38(−76)	1.33(−1179)

Table 1 Results obtained by the three-point methods (22)

Example 2. The four-point methods (28) were applied to the function from Example 1 using the same initial approximation $x_0 = 1.1$. The obtained results are displayed in Table 2.

Four-point methods	$\lvert x_1-\alpha\rvert$	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$
$\{$Ostrowski's IM (10), $\widetilde{N}_2, \widetilde{N}_3\}$	2.50(−10)	3.06(−149)	7.68(−2372)
$\{$Euler-like IM (11), $\widetilde{N}_2, \widetilde{N}_3\}$	2.37(−10)	1.99(−143)	1.29(−2357)
$\{$Maheshvari's IM (12), $\widetilde{N}_2, \widetilde{N}_3\}$	1.41(−10)	1.16(−148)	5.01(−2358)

Table 2 Results obtained by the four-point methods (28)

From Tables 1 and 2 we observe extraordinary accuracy of the produced approximations, obtained using only few function evaluations. Such an accuracy is not needed in practice at present, but has a theoretical importance. We emphasize that our primary aim was to construct a general class of very efficient multipoint methods that supports the Kung-Traub conjecture.

References

[1] C. Chun, Some fourth-order iterative methods for solving nonlinear equations, Appl. Math. Comput. 195 (2008), 454–459.

[2] R. King, A family of fourth order methods for nonlinear equations, SIAM J. Numer. Anal. 10 (1973), 876–879.

[3] J. Kou, Y. Li, X. Wang, A composite fourth-order iterative method, Appl. Math. Comput. 184 (2007), 471–475.

[4] H. T. Kung, J. F. Traub, Optimal order of one-point and multipoint iteration, Journal of the ACM 21 (1974), 643–651.

[5] A. K. Maheshwari, A fourth-order iterative method for solving nonlinear equations, Appl. Math. Comput. 211 (2009), 383–391.

[6] A. M. Ostrowski, Solution of Equations and Systems of Equations, New York-London: Academic Press, 1966.

[7] Lj. D. Petković, M. S. Petković, On the fourth order root finding methods of Euler's type, Novi Sad J. Math. 36 (2006), 155–163.

[8] M. S. Petković, On a general class of multipoint root-finding methods of high computational efficiency, SIAM. J. Numer. Anal. (to appear).

[9] M. S. Petković, Lj. D. Petković, A one parameter square root family of two-step root-finders, Appl. Math. Comput. 188 (2007), 339–344.

[10] J. F. Traub, Iterative Methods for the Solution of Equations, New York: Prentice Hall, 1964.

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