Novi Sad J. Math.
Vol. 39, No. 2, 2009, 7-19

UNIQUENESS OF MEROMORPHIC FUNCTIONS WHOSE n-TH DERIVATIVE SHARE ONE OR TWO VALUES

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Abstract. We deal with the problem of uniqueness of meromorphic functions when their n-th derivatives share one or two values and improve all the results recently obtained by Jun and Mori [15]. We also provide an answer to the question of Yang [9].

AMS Mathematics Subject Classification (2000): 30D35

Key words and phrases: Uniqueness, meromorphic functions, weighted sharing, shared values

1. Introduction and Definitions.

In the paper by meromorphic function we always mean a function which is meromorphic in the open complex plane $\mathbb{C}$. We use the standard notations and definitions of the value distribution theory available in [2]. We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$. The notation $S(r)$ stands for any quantity satisfying $S(r) = o(T(r))$ as $r \longrightarrow \infty$, outside of a possible exceptional set of finite linear measure. If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a-points with the same multiplicities, we say that f and g share the value a CM (counting multiplicities), and if we do not consider the multiplicities then f, g are said to share the value a IM (ignoring multiplicities).

In [9], C. C. Yang asked the following:
What can be said about two entire functions f, g share the value 0 CM and their first derivatives share the value 1 CM ?
A substantial amount of work has already been done on the question of Yang or its related topics and continuous efforts are being put on to relax the hypothesis {c.f. [3], [9]-[15]}.
As an attempt to answer the question of Yang, improving the result of K. Shibazaki, Yi [12] obtained the following theorem.

Theorem A. *Let f and g be two entire functions such that $f^{(n)}$ and $g^{(n)}$ share the value 1 CM. If $\delta(0; f) + \delta(0; g) > 1$ then either $f \equiv g$ or $f^{(n)}g^{(n)} \equiv 1$.*

Considering meromorphic functions Yi and Yang [13] proved the following result.

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Theorem B. *Let f and g be two meromorphic functions satisfying $\delta(\infty; f) = \delta(\infty; g) = 1$. If $f^{'}$ and $g^{'}$ share the value 1 CM with $\delta(0; f) + \delta(0; g) > 1$ then either $f \equiv g$ or $f^{'}g^{'} \equiv 1$.*

In [13], the following question was asked:
Whether it is possible to replace the first derivatives $f^{'}$ and $g^{'}$ in Theorem B *by n-th derivatives $f^{(n)}$ and $g^{(n)}$?*
In this direction the following result was proved in [14].

Theorem C. *Let f and g be two meromorphic functions such that $\Theta(\infty; f) = \Theta(\infty; g) = 1$. If $f^{(n)}$ and $g^{(n)}$ share the value 1 CM with $\delta(0; f) + \delta(0; g) > 1$ then either $f \equiv g$ or $f^{(n)}g^{(n)} \equiv 1$.*

Recently, to deal with the question of Yang [9], Jun and Mori [15] obtained the following results which are different from *Theorem B.*

Theorem D. *Let f and g be two meromorphic functions such that $f^{(n)}$ and $g^{(n)}$ share 1 CM. If $\Delta := \Theta(\infty; f) + \Theta(\infty; g) + \Theta(0; f) + \Theta(0; g) > 4 - \frac{2}{5n+4}$ then either $f \equiv g$ or $f^{(n)}g^{(n)} \equiv 1$.*

Jun and Mori [15] also investigated the situation where two derivatives of the meromorphic functions share two values namely 1 and ∞ as follows.

Theorem E. *Let f and g be two meromorphic functions such that $f^{(n)}$ and $g^{(n)}$ share 1 and ∞ CM. If $\Delta_1 := \Theta(\infty; f) + \Theta(0; f) + \Theta(0; g) > 3 - \frac{1}{4n+3}$ then either $f \equiv g$ or $f^{(n)}g^{(n)} \equiv 1$.*

Theorem F. *Let f and g be two meromorphic functions such that $f^{(n)}$ and $g^{(n)}$ share 1 CM and ∞ IM. If $\Delta_1 > 3 - \frac{1}{4n+4}$ then either $f \equiv g$ or $f^{(n)}g^{(n)} \equiv 1$.*

We now introduce the notion of gradation of sharing known as weighted sharing which measures how close a shared value is to being shared CM or to being shared IM.

Definition 1.1. *[5, 6] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $1 + k$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.*

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly, if f, g share (a, k) then f, g share (a, p) for all integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

In the paper we employ the above notions to improve all the results of Jun and Mori [15]. We also investigate the situation when the sharing of zeros of $f^{(n)}$

and $g^{(n)}$ is taken into account as these types of problems are seldom studied. Lastly, we obtain a result which will provide a specific answer corresponding to the question of Yang [9] as mentioned above. The following theorems are the main results of the paper.

Theorem 1.1. *Let f and g be two meromorphic functions such that $f^{(n)}$ and $g^{(n)}$ share $(1,2)$. If $\Delta > 4 - \frac{1}{n+2}$ then either $f \equiv g$ or $f^{(n)}g^{(n)} \equiv 1$.*

Theorem 1.2. *Let f and g be two meromorphic functions such that $f^{(n)}$ and $g^{(n)}$ share $(1,2)$ and $(\infty;\infty)$. If $\Delta_1 > 3 - \frac{1}{n+2}$ then either $f \equiv g$ or $f^{(n)}g^{(n)} \equiv 1$.*

Theorem 1.3. *Let f and g be two meromorphic functions such that $f^{(n)}$ and $g^{(n)}$ share $(1,2)$ and $(\infty;0)$. If $\Delta_1 > 3 - \frac{1}{n+3}$ then either $f \equiv g$ or $f^{(n)}g^{(n)} \equiv 1$.*

Theorem 1.4. *Let f and g be two meromorphic functions such that $f^{(n)}$ and $g^{(n)}$ share $(1,2)$ and $(0;\infty)$. If $\Delta > 4 - \frac{1}{n+2}$ then either $f \equiv g$ or $f^{(n)}g^{(n)} \equiv 1$.*

Theorem 1.5. *Let f and g be two meromorphic functions such that $f^{(n)}$ and $g^{(n)}$ share $(1,2)$ and $(0;0)$. If $\Delta > 4 - \frac{2}{3n+4}$ then either $f \equiv g$ or $f^{(n)}g^{(n)} \equiv 1$.*

Theorem 1.6. *Let f and g be two meromorphic functions such that $f^{(n)}$ and $g^{(n)}$ share $(1,2)$ and f, g share $(0;0)$. If $\Delta_2 := 2\Theta(0;f) + \Theta(\infty;f) + \Theta(\infty;g) > 4 - \frac{1}{2n+2}$ then either $f \equiv g$ or $f^{(n)}g^{(n)} \equiv 1$.*

Definition 1.2. *[5] Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$.*

(i) $N(r,a;f \mid\geq p)$ ($\overline{N}(r,a;f \mid\geq p)$) denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than p.

(ii) $N(r,a;f \mid\leq p)$ ($\overline{N}(r,a;f \mid\leq p)$) denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than p.

Definition 1.3. *[7] Let $a, b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r,a;f \mid\ g = b)$ the counting function of those a-points of f, counted according to multiplicity, which are b-points of g.*

Definition 1.4. *[7] Let $a, b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r,a;f \mid\ g \neq b)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b-points of g.*

Definition 1.5. *Let $a \in \mathbb{C} \cup \{\infty\}$ and m, n and p be three positive integers. We denote by $N_p(r,a;g \mid\ m \leq f \leq n)$ ($N_p(r,a;g \mid\ f \geq m)$) the counting function of those a-points of g which are also the a-points of f, with multiplicities lying between m and n (not less than m), where an a-point of g with multiplicity t is counted t times if $t \leq p$ and p times if $t > p$. In a similar way we can define $N(r,a;g \mid\ m \leq f \leq n)$.*

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F, G be two non-constant meromorphic functions. Henceforth we shall denote by H the following function.

$$H=(\frac{F^{''}}{F^{'}}-\frac{2F^{'}}{F-1})-(\frac{G^{''}}{G^{'}}-\frac{2G^{'}}{G-1}). \tag{2.1}$$

Lemma 2.1. *If for two positive integers p, and k, $N_p\left(r,0;f^{(k)}\mid f\neq 0\right)$ denotes the counting function of those zeros of of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ with multiplicity m is counted m times if $m\leq p$ and p times if $m>p$ then*

$$\begin{aligned}N_p\left(r,0;f^{(k)}\mid f\neq 0\right)\leq & N_k(r,0;f)+k\overline{N}(r,\infty;f)\\ & -\sum_{m=p+1}^{\infty}\overline{N}\left(r,0;\frac{f^{(k)}}{f}\mid\geq m\right)+S(r,f).\end{aligned}$$

Proof. By the first fundamental theorem and Milloux theorem {p. 55 [2]} we get

$$\begin{aligned}& N(r,0;f^{(k)}\mid f\neq 0)\\ \leq\ & N(r,0;\frac{f^{(k)}}{f})\\ \leq\ & N(r,\infty;\frac{f^{(k)}}{f})+m(r,\frac{f^{(k)}}{f})+O(1)\\ =\ & N(r,0;f\mid<k)+k\overline{N}(r,0;f\mid\geq k)+k\overline{N}(r,\infty;f)+S(r,f)\\ =\ & N_k(r,0;f)+k\overline{N}(r,\infty;f)+S(r,f).\end{aligned}$$

Now

$$\begin{aligned}& N_p\left(r,0;\frac{f^{(k)}}{f}\right)+\sum_{m=p+1}^{\infty}\overline{N}\left(r,0;\frac{f^{(k)}}{f}\mid\geq m\right)=N\left(r,0;\frac{f^{(k)}}{f}\right)\\ \leq\ & N_k(r,0;f)+k\overline{N}(r,\infty;f)+S(r,f).\end{aligned}$$

Since $N_p\left(r,0;f^{(k)}\mid f\neq 0\right)\leq N_p\left(r,0;\frac{f^{(k)}}{f}\right)$, the lemma follows from the above. □

Lemma 2.2. *For two positive integers p and k*

$$N_p(r,0;f^{(k)})\leq N_{p+k}(r,0;f)+k\overline{N}(r,\infty;f)-\sum_{m=p+1}^{\infty}\overline{N}\left(r,0;\frac{f^{(k)}}{f}\mid\geq m\right)+S(r,f).$$

Proof. We note that

$$\begin{aligned} & N_p\left(r,0;f^{(k)} \mid f\geq k+1\right) \\ = \;& N(r,0;f \mid k\leq f\leq k+p-1)-k\overline{N}(r,0;f \mid k\leq f\leq k+p-1) \\ & +p\overline{N}(r,0;f \mid\geq k+p). \end{aligned}$$

Using *Lemma 2.1* we get

$$\begin{aligned} & N_p\left(r,0;f^{(k)}\right) \\ = \;& N_p\left(r,0;f^{(k)} \mid f\neq 0\right)+N_p\left(r,0;f^{(k)} \mid f\geq k+1\right) \\ & +N_p\left(r,0;f^{(k)} \mid 1\leq f\leq k\right) \\ \leq \;& N_k(r,0;f)+k\overline{N}(r,\infty;f)-\sum_{m=p+1}^{\infty}\overline{N}\left(r,0;\frac{f^{(k)}}{f} \mid\geq m\right) \\ & +N_p\left(r,0;f^{(k)} \mid f\geq k+1\right)+N_p\left(r,0;f^{(k)} \mid 1\leq f\leq k\right)+S(r,f) \\ \leq \;& N_{p+k}(r,0;f)+k\overline{N}(r,\infty;f)-\sum_{m=p+1}^{\infty}\overline{N}\left(r,0;\frac{f^{(k)}}{f} \mid\geq m\right)+S(r,f). \end{aligned}$$

□

Lemma 2.3. *For two positive integers p and k*

$$\begin{aligned} & N_p\left(r,0;f^{(k)}\right) \\ \leq \;& T\left(r,f^{(k)}\right)-T(r,f)+N_{p+k}(r,0;f)-\sum_{m=p+1}^{\infty}\overline{N}\left(r,0;f^{(k)} \mid f\neq 0 \mid\geq m\right), \end{aligned}$$

where by $\overline{N}\left(r,0;f^{(k)} \mid f\neq 0 \mid\geq m\right)$ *we mean the reduced counting function of those zeros of* $f^{(k)}$ *with multiplicities not less than m which are not the zeros of* f.

Proof. Since

$$N\left(r,0;f^{(k)}\right)\leq T\left(r,f^{(k)}\right)-T(r,f)+N(r,0;f)+S(r,f),$$

it follows that

$$\begin{aligned} N_p\left(r,0;f^{(k)}\right) \leq\; & T\left(r,f^{(k)}\right)-T(r,f)+N(r,0;f) \\ & -\sum_{m=p+1}^{\infty}\overline{N}\left(r,0;f^{(k)} \mid\geq m\right)+S(r,f). \end{aligned}$$

But

$$\sum_{m=p+1}^{\infty} \overline{N}\left(r,0;f^{(k)} \mid\geq m\right) = \sum_{m=p+1}^{\infty} \overline{N}\left(r,0;f^{(k)} \mid f\geq m\right) + \sum_{m=p+1}^{\infty} \overline{N}\left(r,0;f^{(k)} \mid f\neq 0 \mid\geq m\right).$$

Also, since

$$N(r,0;f) - \sum_{m=p+1}^{\infty} \overline{N}\left(r,0;f^{(k)} \mid f\geq m\right) = N_{p+k}(r,0;f),$$

the lemma follows from the above. □

Lemma 2.4. *[6] If F, G share $(1,2)$ then one of the following cases holds.*
(i)

$$max\{\,T(r,F),T(r,G)\} \leq N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F) +N_2(r,\infty;G)+S(r,F)+S(r,G)$$

(ii) $F\equiv G$
(iii) $FG\equiv 1$.

Lemma 2.5. *If F, G share $(1,2)$ and (∞,k), where $0\leq k\leq\infty$ then one of the following cases holds.*
(i)

$$max\{\,T(r,F),T(r,G)\} \leq N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G) +\overline{N}_*(r,\infty;F,G)+S(r,F)+S(r,G)$$

(ii) $F\equiv G$
(iii) $FG\equiv 1$.

Proof. We omit the proof since the proof since it can be carried out in the line of the proof of *Lemma 2.13* [1]. □

Lemma 2.6. *If F, G share $(1,2)$ and $(0,k)$, where $0\leq k\leq\infty$ then one of the following cases holds.*
(i)

$$max\{\,T(r,F),T(r,G)\} \leq N_2(r,\infty;F)+N_2(r,\infty;G)+\overline{N}(r,0;F)+\overline{N}(r,0;G) +\overline{N}_*(r,0;F,G)+S(r,F)+S(r,G)$$

(ii) $F\equiv G$
(iii) $FG\equiv 1$.

Proof. Noting that F, G share $(1,2)$ and $(0,k)$ implies $\frac{1}{F}$ and $\frac{1}{G}$ share $(1,2)$ and (∞,k) and also from the first fundamental theorem we have $T(r,F)=T(r,\frac{1}{F})+O(1)$, the lemma can be proved in the line of the proof of *Lemma 2.5.* □

3. Proofs of the theorems

Proof of Theorem 1.1. Let $F = f^{(n)}$ and $G = g^{(n)}$. Since

$$\begin{aligned} T(r, f^{(n)}) &= N(r,\infty; f^{(n)}) + m(r, f^{(n)}) \\ &\le N(r,\infty; f) + n\overline{N}(r,\infty; f) + m(r,f) + S(r,f) \\ &\le T(r,f) + n\overline{N}(r,\infty; f) + S(r,f) \\ &\le (n+1)T(r,f) + S(r,f) \end{aligned}$$

and

$$T(r, g^{(n)}) \le (n+1)T(r,g) + S(r,g),$$

it follows that $S(r,F)$ and $S(r,G)$ can be replaced by $S(r,f)$ and $S(r,g)$ respectively. If possible, we suppose that *Case (i)* of *Lemma 2.4* holds. Then we have from *Lemmas 2.2-2.3*

$$\begin{aligned} T(r, f^{(n)}) &= T(r,F) \\ &\le N_2(r,0;F) + N_2(r,\infty;F) + N_2(r,0;G) + N_2(r,\infty;G) \\ &\quad + S(r,F) + S(r,G) \\ &\le N_2(r,0;f^{(n)}) + N_2(r,0;g^{(n)}) + 2\left(\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g)\right) \\ &\quad + S(r,f) + S(r,g) \\ &\le T(r,f^{(n)}) - T(r,f) + N_{n+2}(r,0;f) + N_{n+2}(r,0;g) \\ &\quad + (n+2)\left(\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g)\right) + S(r), \end{aligned}$$

that is,

$$(3.1)\quad T(r,f) \le (n+2)\{\overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g)\} + S(r)$$

In a similar way we can obtain

$$(3.2)\quad T(r,g) \le (n+2)\{\overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g)\} + S(r)$$

Combining (3.1) and (3.2) we get for $\varepsilon > 0$ that

$$\left(\Delta - 4 + \frac{1}{n+2} - \varepsilon\right) T(r) \le S(r).$$

Since $\Delta > 4 - \frac{1}{n+2}$ we can choose a δ such that $\Delta > 4 - \frac{1}{n+2} + \delta$ and so for $0 < \varepsilon < \delta$ we obtain a contradiction. Hence by *Lemma 2.4* we have either $f^{(n)} \equiv g^{(n)}$ or $f^{(n)} g^{(n)} \equiv 1$. If $f^{(n)} \equiv g^{(n)}$, then $f(z) = g(z) + p(z)$ where $p(z)$ is a polynomial of degree at most $n-1$. We claim that $p(z) \equiv 0$. Otherwise we have

$$\begin{aligned} T(r,f) &\le \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}(r,p;f) + S(r,f) \\ &\le \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}(r,0;g) + S(r) \\ &\le (4 - \Delta + \varepsilon)\, T(r) + S(r). \end{aligned}$$

Similarly we get

$$T(r,g) \le (4-\Delta+\varepsilon)\,T(r)+S(r).$$

So we have

$$(\Delta-3-\varepsilon)\,T(r) \le S(r).$$

Noting that $\Delta > 4-\frac{1}{n+2}$ and choosing $0<\varepsilon<\Delta-3$ we get a contradiction. So $f(z)\equiv g(z)$ and the proof is complete. $\square$

Proof of Theorem 1.2. Let F and G be defined as in the proof of *Theorem 1.1*. According to the statement of the theorem $\overline{N}_*(r,\infty,F,G)\equiv 0$. If possible, we suppose that *Case (i)* of *Lemma 2.5* holds. Then we have from *Lemmas 2.2-2.3*

$$\begin{aligned} T(r,f^{(n)}) = T(r,F) &\le N_2(r,0;F)+N_2(r,0;G)+2\overline{N}(r,\infty;F) \\ &\quad +S(r,F)+S(r,G) \\ &\le T(r,f^{(n)})-T(r,f)+N_{n+2}(r,0;f)+N_{n+2}(r,0;g) \\ &\quad +(n+2)\overline{N}(r,\infty;f)+S(r), \end{aligned}$$

Now, proceeding in the same way as in *Theorem 1.1* we get

$$\left(\Delta_1-3+\frac{1}{n+2}-\varepsilon\right)T(r)\le S(r),$$

from which we can deduce a contradiction. Hence by *Lemma 2.5* we have either $f^{(n)}\equiv g^{(n)}$ or $f^{(n)}g^{(n)}\equiv 1$. Now again, following the same method as in the proof of *Theorem 1.2* we can prove the theorem. $\square$

Proof of Theorem 1.3. Let F and G be defined as in the proof of *Theorem 1.1*. We note that here $\overline{N}_*(r,\infty,F,G)\le\overline{N}(r,\infty;f)$. Now we can prove the theorem in the line of proof of *Theorem 1.2*. $\square$

Proof of Theorem 1.5. Let F and G be defined as in the proof of *Theorem 1.1*. According to the statement of the theorem $\overline{N}_*(r,0,F,G)\le\overline{N}(r,0;F)=\frac{1}{2}\overline{N}(r,0;F)+\frac{1}{2}\overline{N}(r,0;G)$. If possible, let us suppose that *Case (i)* of *Lemma 2.6* holds. Then we have from *Lemmas 2.2-2.3*

$$\begin{aligned} T(r,f^{(n)}) = T(r,F) &\le N_2(r,\infty;F)+N_2(r,\infty;G)+\frac{3}{2}\overline{N}(r,0;F)+\frac{3}{2}\overline{N}(r,0;G) \\ &\quad +S(r,F)+S(r,G) \\ &\le T(r,f^{(n)})-T(r,f)+\frac{3}{2}N_{n+1}(r,0;f)+\frac{3}{2}N_{n+1}(r,0;g) \\ &\quad +\left(\frac{n}{2}+2\right)\overline{N}(r,\infty;f)+\left(\frac{3n}{2}+2\right)\overline{N}(r,\infty;g)+S(r), \end{aligned}$$

that is

(3.3)

$$T(r,f)\le\left(\frac{3n}{2}+2\right)\left(\overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right)+S(r)$$

In a similar manner we can obtain
(3.4)
$$T(r,g) \le \left(\frac{3n}{2}+2\right)\left(\overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right)+S(r)$$

Combining (3.3) and (3.4) we get for $\varepsilon > 0$ that
$$\left(\Delta - 4 + \frac{2}{3n+4} - \varepsilon\right) T(r) \le S(r),$$
Since $\varepsilon > 0$ be arbitrary we obtain a contradiction. Hence by *Lemma 2.6* we have either $f^{(n)} \equiv g^{(n)}$ or $f^{(n)} g^{(n)} \equiv 1$. If $f^{(n)} \equiv g^{(n)}$, then $f(z) = g(z) + p(z)$ where $p(z)$ is a polynomial of degree at most $n-1$. We claim that $p(z) \equiv 0$. Otherwise we have
$$\begin{aligned} T(r,f) &\le \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,p;f)+S(r,f) \\ &\le \overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;f)+S(r) \\ &\le (4-\Delta+\varepsilon)\,T(r)+S(r) \end{aligned}$$
Similarly we get
$$T(r,g) \le (4-\Delta+\varepsilon)\,T(r)+S(r).$$
So we obtain
$$(\Delta - 3 - \varepsilon)\,T(r) \le S(r).$$
Since $\Delta > 4 - \frac{2}{3n+4}$ and $\varepsilon > 0$ be arbitrary we get a contradiction. So $f(z) \equiv g(z)$ and the proof is complete. □

Proof of Theorem 1.4. Let F and G be defined as in the proof of *Theorem 1.1.* We note that here $\overline{N}_*(r,0,F,G) \equiv 0$. Now, proceeding in the same way as in the proof of *Theorem 1.5* we can prove the theorem. □

Proof of Theorem 1.6. Let F and G be defined as in the proof of *Theorem 1.1.*
Case 1 Let $H \not\equiv 0$.
From (2.1) it can be easily calculated that the possible poles of H occur at (i) common zeros of F and G with different multiplicities, (ii) zeros of F (G) which are not zeros of G (F), (iii) those 1 points of F and G whose multiplicities are different (iv) those poles of F and G whose multiplicities are different, (v) zeros of $F^{'}$ $(G^{'})$ which are not the zeros of $F(F-1)$ $(G(G-1))$.
Since H has only simple poles we get

(3.5) $$\begin{aligned} N(r,\infty;H) \le\ & \overline{N}(r,\infty;F \mid\ge 2)+\overline{N}(r,\infty;G \mid\ge 2)+\overline{N}_*(r,1;F,G) \\ &+\overline{N}(r,0;f^{(n)} \mid f\ne 0)+\overline{N}(r,0;g^{(n)} \mid g\ne 0) \\ &+\overline{N}_*(r,0;f,g)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'}), \end{aligned}$$

where $\overline{N}_0(r,0;F^{'})$ is the reduced counting function of those zeros of $F^{'}$ which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G^{'})$ is similarly defined. Let z_0 be a simple zero of $F-1$. Then by a simple calculation we see that z_0 is a zero of H and hence

$$N(r,1;F\mid=1)\le N(r,0;H)\le N(r,\infty;H)+S(r,F) \tag{3.6}$$

By the second fundamental theorem we get

$$T(r,F)\le \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,1;F)-N_0(r,0;F^{'})+S(r,F) \tag{3.7}$$

So, from (3.5), (3.6) and (3.7) we get

$$\begin{aligned} T(r,F) & \\ \le\ & \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+N(r,1;F\mid=1)+\overline{N}(r,1;F\mid\ge 2)\\ & -N_0(r,0;F^{'})+S(r,F)\\ \le\ & \overline{N}(r,0;F)+N_2(r,\infty;F)+\overline{N}(r,\infty;G\mid\ge 2)\\ & +\overline{N}(r,0;f^{(n)}\mid f\ne 0)+\overline{N}(r,0;g^{(n)}\mid g\ne 0)\\ & +\overline{N}(r,0;f)+\overline{N}(r,1;F\mid\ge 3)\\ & +\overline{N}(r,1;F\mid\ge 2)+\overline{N}_0(r,0;G^{'})+S(r,f) \end{aligned} \tag{3.8}$$

Since F, G share $(1,2)$, using *Lemma 2.1* we obtain

$$\begin{aligned} &\overline{N}(r,1;F\mid\ge 2)+\overline{N}(r,1;F\mid\ge 3)+\overline{N}_0(r,0;G^{'})\le\\ &\qquad \overline{N}(r,0;G^{'}\mid G\ne 0)\le \overline{N}(r,0;G)+\overline{N}(r;\infty;G) \end{aligned}$$

So, using *Lemma 2.3* and *Lemma 2.1* from (3.8) we get

$$\begin{aligned} T(r,f)\ \le\ & N_{n+1}(r,0;f)+N_{n+1}(r,0;g)+N_n(r,0;f)+N_n(r,0;g)+\overline{N}(r,0;f)\\ & +(n+2)\overline{N}(r,\infty;f)+(2n+2)\overline{N}(r,\infty;g)+S(r,f)\\ \le\ & (2n+2)\left(2\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right)+S(r). \end{aligned}$$

So we get

$$\left(\Delta_2-4+\frac{1}{2n+2}-\varepsilon\right)T(r)\le S(r),$$

which is a contradiction for arbitrary $\varepsilon>0$.
Case 2 Next we suppose that $H\equiv 0$. Then by integration we get from (2.1)

$$\frac{1}{F-1}\equiv\frac{bG+a-b}{G-1}, \tag{3.9}$$

where a, b are constants and $a\ne 0$. From (3.9) it is clear that $F=f^{(n)}$ and $G=g^{(n)}$ share $(1,\infty)$. Also

$$T(r,F)=T(r,G)+O(1). \tag{3.10}$$

We now consider the following cases.
Subcase 2.1 Let $b = 0$. From (3.9) we obtain

$$f = \frac{1}{a} g + p(z),$$

where $p(z)$ is a polynomial of degree at most $n - 1$. We claim that $p(z) \equiv 0$. Otherwise we have

$$\begin{aligned} T(r, f) &\leq \overline{N}(r, 0; f) + \overline{N}(r, \infty; f) + \overline{N}(r, p; f) + S(r, f) \\ &\leq 2\overline{N}(r, 0; f) + \overline{N}(r, \infty; f) + S(r) \\ &\leq (4 - \Delta_2 + \varepsilon)\, T(r) + S(r), \end{aligned}$$

that is

$$(\Delta_2 - 3 - \varepsilon)\, T(r) \leq S(r),$$

which is a contradiction for arbitrary $\varepsilon > 0$. So

$$f = \frac{1}{a} g. \tag{3.11}$$

Differentiating (3.11) n times we get

$$f^{(n)} = \frac{1}{a} g^{(n)}.$$

The above equation together with the fact that $f^{(n)}$ and $g^{(n)}$ share $(1, \infty)$ yields $a = 1$.
Subcase 2.2 Let $b \neq 0$ and $a \neq b$.
If $b = -1$, then from (3.9) we have

$$F = \frac{-a}{G - a - 1}.$$

Therefore

$$\overline{N}(r, a + 1; G) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f).$$

Since from *Lemma 2.3* we have

$$\begin{aligned} T(r, g) &\leq T(r, g^{(n)}) + N_{p+n}(r, 0; g) - N_p(r, 0; g^{(n)}) + S(r) \\ &\leq T(r, g^{(n)}) + N_{p+n}(r, 0; g) - N_p(r, 0; g^{(n)} \mid g = 0) + S(r) \\ &\leq T(r, g^{(n)}) + N_n(r, 0; g) + S(r), \end{aligned}$$

by the second fundamental theorem we get

$$\begin{aligned} T(r, g) &\leq T(r, G) + N_n(r, 0; g) \\ &\leq \overline{N}(r, 0; G) + \overline{N}(r, \infty; G) + \overline{N}(r, a + 1; G) + N_n(r, 0; g) + S(r, g) \\ &\leq N_n(r, 0; g) + N_{n+1}(r, 0; g) + (n + 1)\overline{N}(r, \infty; g) + \overline{N}(r, \infty; f) + S(r, g) \\ &\leq (n + 1)\left(2\overline{N}(r, 0; f) + \overline{N}(r, \infty; g) + \overline{N}(r, \infty; f)\right) + S(r, g). \end{aligned}$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f) \leq T(r,g)$ for $r \in I$.
So, for $r \in I$ we have

$$\left(\Delta_2 - 4 + \frac{1}{n+1} - \varepsilon\right) T(r,g) \leq S(r,g).$$

Since $\Delta_2 > 4 - \frac{1}{2n+2}$ and $\varepsilon > 0$ we get a contradiction from the above.
If $b \neq -1$, from (3.9) we obtain that

$$F - \left(1 + \frac{1}{b}\right) = \frac{-a}{b^2[G + (a-b)/b]}.$$

Therefore

$$\overline{N}(r,(b-a)/b;G) = \overline{N}\,(r,\infty;F-(1+1/b)) = \overline{N}(r,\infty;f)$$

Using the second fundamental theorem and the same argument as used in the case when $b = -1$ we can get a contradiction.
Subcase 2.3 Let $b \neq 0$ and $a = b$.
If $b = -1$, then from (3.9) we have

$$FG = 1.$$

that is

$$f^{(n)}g^{(n)} = 1.$$

If $b \neq -1$, from (3.9) we have

$$\frac{1}{F} = \frac{bG}{(1+b)G - 1}.$$

Hence from *Lemma 2.2* we have

$$\begin{aligned}\overline{N}\,(r,1/(1+b);G) &= \overline{N}(r,0;f^{(n)})\\ &\leq N_{n+1}(r,0;f) + n\overline{N}(r,\infty;f).\end{aligned}$$

So, by the second fundamental theorem and *Lemma 2.2* we get

$$\begin{aligned}T(r,g) &\leq T(r,G) + N_n(r,0;g)\\ &\leq \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + \overline{N}\,(r,1/(1+b);G) + N_n(r,0;g) + S(r,g)\\ &\leq N_n(r,0;g) + N_{n+1}(r,0;g) + N_{n+1}(r,0;f) + n\overline{N}(r,\infty;f)\\ &\quad +(n+1)\overline{N}(r,\infty;g) + S(r,g)\\ &\leq (2n+1)\left(2\overline{N}(r,0;f) + \overline{N}(r,\infty;g) + \overline{N}(r,\infty;f)\right) + S(r,g).\end{aligned}$$

So for $r \in I$ we have

$$\left(\Delta_2 - 4 + \frac{1}{2n+1} - \varepsilon\right) T(r,g) \leq S(r,g),$$

which is a contradiction for arbitrary $\varepsilon > 0$. □

Acknowledgement

The first author is grateful to Prof. W.R. Lü for supplying him the electronic file of the paper [15].

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Received by the editors June 20, 2007